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SOME PROPERTIES OF THE INVERSION COUNTING FUNCTION

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Abstract. Let h, k be integers such that $0 < h < k$ and $(h, k) = 1$. If $1 \leq i \leq k-1$, let r_i be the least positive residue $\pmod{k}$ of hi. Let the permutation

$$\sigma_{h,k} = \begin{pmatrix} 1 & 2 & 3 & \cdots & k-1 \\ r_1 & r_2 & r_3 & \cdots & r_{k-1} \end{pmatrix}$$

For $1 \leq i < j \leq k-1$, if $r_i > r_j$, this is called an *inversion* of $\sigma_{h,k}$. Let $I(h, k)$ denote the total number of inversions of $\sigma_{h,k}$. In this note, we prove several identities concerning $I(h, k)$.

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1. Introduction. Let h, k be integers such that $0 < h < k$ and $(h, k) = 1$. Consider the permutation

$$\sigma_{h,k} = \begin{pmatrix} 1 & 2 & 3 & \cdots & k-1 \\ r_1 & r_2 & r_3 & \cdots & r_{k-1} \end{pmatrix}$$

where r_i is the least positive residue $\pmod{k}$ of hi for $1 \leq i \leq k-1$. In particular, if p is an odd prime and if $0 < h < p$, then the Legendre symbol $(\frac{h}{p}) = 1$ if and only if $\sigma_{h,p}$ is the product of an even number of transpositions. (See [2], p. 280.) Let $I_i(h, k)$ denote the number of elements in the sequence $\{r_1, r_2, r_3, \ldots, r_{i-1}\}$ that exceed r_i. Let

$$I(h, k) = \sum_{i=1}^{k-1} I_i(h, k)\,.$$

Note that $I(h, k)$ counts the number of so-called *inversions* in $\sigma_{h,k}$. (The term *inversion* is due to C. Meyer [3].) In [4], several identities concerning $I(h, k)$ were presented. In this note, we develop further properties of $I(h, k)$.

2. Preliminaries. Let $s(h, k)$ denote the usual Dedekind sum.

$$I(h, k) = -3ks(k, h) + \frac{1}{4}\,(k-1)(k-2). \tag{1}$$

$$hI(h, k) + kI(k, h) = \frac{1}{4}\,(h-1)(k-1)(h+k-1). \tag{2}$$

$$\text{If} \quad h' \equiv \pm h \pmod{k}, \quad \text{then} \quad s(h', k) = \pm s(h, k). \tag{3}$$

$$\text{If} \quad h' \equiv h \pmod{k}, \quad \text{then} \quad I(h', k) = I(h, k). \tag{4}$$

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Remarks. (1), (2) are (48), (50) respectively in [4]. (See p. 37, 39.) (2) is attributed by Meyer to H. Salie. (See [3], p. 163.) (3) is Theorem 3.6, part (a) on p. 62 of [1]. (4) follows from (1) and (3). An additional identity ((49) in [4]) states that

$$(-1)^{I(h,k)} = \left(\frac{h}{k}\right) .$$

This last identity, sometimes called Zolotarev's Theorem, yields an alternate proof of the quadratic reciprocity law for the Jacobi symbol.

3. The Main Results.

Theorem 1. *Let h, k be integers such that $0 < h < k$ and $(h, k) = 1$. Then*

$$\text{(a)} \qquad I(k-h,k) + I(h,k) = \frac{(k-1)(k-2)}{2} ,$$

$$\text{(b)} \qquad I(k-h,k) - I(h,k) = 6ks(h,k).$$

Proof. Since (3) implies $s(k-h,k) = s(-h,k) = -s(h,k)$, it follows from (1) that

$$I(k-h,k) = 3ks(h,k) + \frac{1}{4}(k-1)(k-2). \tag{5}$$

The conclusion now follows from (1) and (7). □

Theorem 2. *If k is odd, then*

$$\sum_{h=1}^{k-1} I(h,k) = \frac{1}{4}(k-1)^2(k-2). \tag{6}$$

Proof.

$$\sum_{h=1}^{k-1} I(h,k) = \sum_{h=1}^{\frac{k-1}{2}} I(h,k) + \sum_{h=\frac{k+1}{2}}^{k-1} I(h,k)$$

but

$$\sum_{h=\frac{k+1}{2}}^{k-1} I(h,k) = \sum_{h=1}^{\frac{k-1}{2}} I(k-h,k)$$

so that, using Theorem 1(a), we get:

$$\sum_{h=1}^{k-1} I(h,k) = \sum_{h=1}^{\frac{k-1}{2}} (I(h,k) + I(k-h,k))$$

$$= \sum_{h=1}^{\frac{k-1}{2}} \frac{1}{2}(k-1)(k-2) = \frac{1}{4}(k-1)^2(k-2). \qquad \square$$

Theorem 3. *If $0 < h < k$, $(h,k) = 1$ and $k \equiv 0 \pmod 3$, then $I(h,k) \equiv h - 1 \pmod 3$.*

Proof. By hypothesis, we must have $h \not\equiv 0 \pmod 3$. Therefore, by (2) and the hypothesis, we have

$$hI(h,k) \equiv -(h-1)^2 \pmod 3$$

hence

$$I(h,k) \equiv -h + 2 - h^{-1} \pmod 3$$

from which the conclusion follows. □

Theorem 4. *If* $0 < h < k$*,* $(h,k) = 1$*, and* $k \not\equiv 0 \pmod 3$*, then* $I(h,k) \equiv 0 \pmod 3$*.*

Proof (Induction on k). Note that $I(1,2) = 0 \equiv 0 \pmod 3$. First assume that
(a) $h \not\equiv 0 \pmod 3$.
Now (2) implies

$$hI(h,k) + kI(k,h) \equiv 0 \pmod 3.$$

Let $k = qh + r$, where $0 < r < h$, then $I(k,h) = I(r,h)$, so that

$$hI(h,k) + kI(r,h) \equiv 0 \pmod 3.$$

By the induction hypothesis, we have $I(r,h) \equiv 0 \pmod 3$. Therefore $I(h,k) \equiv 0 \pmod 3$. Now assume that
(b) $h \equiv 0 \pmod 3$, so that $k - h \not\equiv 0 \pmod 3$.
Since, by hypothesis, $k \not\equiv 0 \pmod 3$ it follows that $3 | (k-1)(k-2)$. Therefore (2) implies

$$I(h,k) + I(k-h,k) \equiv 0 \pmod 3. \tag{7}$$

Our result from part (a) implies $I(k-h,k) \equiv 0 \pmod 3$ so that (9) implies $I(h,k) \equiv 0 \pmod 3$. □

Theorem 5.

(a) $$I(m, mn \pm 1) = \frac{m(m-1)n(n \pm 1)}{4}.$$

(b) *If m is odd, then* $$I(m, mn \pm 2) = \frac{n(m-1)(2mn \pm m \pm 3)}{8}.$$

Proof. These results are obtained via (2) and (4). □

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