

ON CUSP FORMS OF WEIGHT $9/2$

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Abstract. The modular properties of general theta-functions with characteristics are used to build a cusp form of weight $9/2$.

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1. INTRODUCTION

It is well known that $r(n; f)$, the number of representations of a positive integer n by a positive definite quadratic form f , can be expressed as the sum

$$r(n; f) = \rho(n; f) + \nu(n; f), \quad (1)$$

where $\rho(n; f)$ is a singular series. This series has been exhaustively studied and formulas for computing it are known ([1]; [2]). The second summand $\nu(n; f)$ is a Fourier coefficient of a cusp form. It can be expressed in terms of modular forms as follows:

$$\vartheta(\tau; f) = E(\tau; f) + X(\tau), \quad (2)$$

$$\vartheta(\tau; f) = 1 + \sum_{n=1}^{\infty} r(n; f)Q^n, \quad (3)$$

where $\tau \in H = \{\tau : \text{Im } \tau > 0\}$, $Q = e^{2\pi i\tau}$, $X(\tau)$ is a cusp form, and

$$E(\tau; f) = 1 + \sum_{n=1}^{\infty} \rho(n; f)Q^n \quad (4)$$

is the Eisenstein series corresponding to f . If the genus of the quadratic form f contains one class, then, according to Siegel's theorem ([3]–[5]), $\vartheta(\tau; f) = E(\tau; f)$ and therefore the problem of obtaining “exact” formulas for $r(n; f)$ is solved completely. If the genus contains more than one class, then it is necessary to determine the cusp form $X(\tau)$. Many papers are devoted to the problem of finding $X(\tau)$. The cusp forms in these works are constructed in the form of linear combinations of product of simple theta-functions with characteristics or their derivatives (see, e.g., [6]–[8]), products of Jacobi theta-functions or their derivatives (see, e.g., [9]), and theta-functions with spherical polynomials (see, e.g., [10]). All these functions are special cases of linear combinations of the so-called *generalized theta-functions* with characteristics defined below by (7).

In the present paper, using modular properties of these functions, a cusp form of weight $9/2$ is constructed which belongs to the space of entire modular forms of type $(-9/2, 48, \nu(M))$. Here

$$M \in \Gamma_0(48) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(Z) \mid c \equiv 0 \pmod{48} \right\}$$

and $\nu(M)$ is a system of multipliers with respect to the quadratic form of level 16:

$$f^{(k)} = 4 \sum_{j=1}^k x_j^2 + \sum_{j=k+1}^9 x_j^2 \quad (k = 1, 2, \dots, 8). \tag{5}$$

(All of these forms have same multiplier system.)

This cusp form can be used to obtain exact formulas for the number of representations of positive integers by the forms $f^{(k)}$. Using modular properties of simple theta-functions with characteristics, Lomadze obtained such formulas for $k = 1, 6, 8$ ([7], [8]).

2. PRELIMINARIES

Lemma 1 ([11], p. 444). *Let $f = f(x)$ be a positive definite quadratic form in s variables, N be the level of f . Then function (3) belongs to the space of entire modular forms of type $(-s/2, N, \nu(M))$, where $\nu(M)$ is a system of multipliers with respect to f and*

$$M \in \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(Z) \mid c \equiv 0 \pmod{N} \right\}. \tag{6}$$

Definition ([11], p. 438). Let $f = \frac{1}{2} x'Ax$ be a positive definite quadratic form, A be an integral matrix with even diagonal elements, and $x \in Z^s$, $s \geq 2$, be a column vector. Moreover, let g and h be special vectors with respect to A , $P_\nu = P_\nu(x)$ be a spherical function of order ν with respect to A , and N be the level of f . Then we define the corresponding generalized theta-function with characteristics as follows:

$$\vartheta_{gh}(\tau; P_\nu, f) = \sum_{x \equiv g \pmod{N}} (-1)^{\frac{h'A(x-g)}{N^2}} P_\nu(x) e^{\frac{\pi i \tau x'Ax}{N^2}}. \tag{7}$$

Lemma 2 ([11], Lemma 4). *Let k be an arbitrary vector and l be a special vector with respect to the form f . Then the following equalities hold:*

$$\begin{aligned} \vartheta_{g+Nk,h}(\tau; P_\nu, f) &= (-1)^{\frac{h'Ah}{N}} \vartheta_{gh}(\tau; P_\nu, f), \\ \vartheta_{g,h+2l}(\tau; P_\nu, f) &= \vartheta_{gh}(\tau; P_\nu, f). \end{aligned}$$

Lemma 3 ([11], p. 444). *Let $f_k = f_k(x)$ be a positive definite quadratic form in s variables, A_k be the matrix of f_k , Δ_k be the determinant of A_k , N_k be the level of f_k , $g^{(k)}$ and $h^{(k)}$ be special vectors with respect to A_k (if $2 \nmid \frac{N}{N_k}$, then $h^{(k)}$ is a vector with even components), B_k be arbitrary complex numbers ($k = 1, 2, \dots, j$). Let $P_\nu^{(k)} = P_\nu^{(k)}(x)$ be a spherical function of order ν with*

respect to f_k ($k = 1, 2, \dots, j$), and Δ be the determinant of a positive definite quadratic form f in $s + 2\nu$ variables.

Then the function

$$X(\tau; f) = \sum_{k=1}^j B_k \vartheta_{g^{(k)}h^{(k)}}(\tau; P_\nu^{(k)}, f_k) \tag{8}$$

is an entire modular form of type $(-(s/2 + \nu), N, \nu(M))$ if and only if the following conditions are met

$$N_k \mid N, \quad N_k^2 \mid f_k(g^{(k)}), \quad 4N_k \mid \frac{N}{N_k} f_k(h^{(k)}). \tag{9}$$

Furthermore, for all α and δ such that $\alpha\delta \equiv 1 \pmod{N}$ we have

$$\begin{aligned} & \sum_{k=1}^j B_k \vartheta_{\alpha g^{(k)}, h^{(k)}}(\tau; P_\nu^{(k)}, f_k) (\text{sgn } \delta)^\nu \left(\frac{(-1)^{\lfloor s/2 \rfloor} \Delta_k}{|\delta|} \right) \\ &= \left(\frac{(-1)^{\lfloor (s+2\nu)/2 \rfloor} \Delta}{|\delta|} \right) \sum_{k=1}^j B_k \vartheta_{g^{(k)}h^{(k)}}(\tau; P_\nu^{(k)}, f_k). \end{aligned}$$

Lemma 4 ([11], p. 446). *If all conditions of Lemma 3 are fulfilled and $\nu > 0$, then the function $X(\tau)$ is a cusp form of type $(-(s/2 + \nu), N, \nu(M))$.*

3. BASIC RESULT

Theorem. *Assume $k = 1, 2, \dots, 8$,*

$$\begin{aligned} f^{(k)} &= 4 \sum_{j=1}^k x_j^2 + \sum_{j=k+1}^9 x_j^2, \quad P_3 = x_1 x_2 x_3, \quad h' = (0, 0, 0), \\ (g^{(1)})' &= (8, 8, 8), \quad (g^{(2)})' = (16, 16, 16), \quad (g^{(3)})' = (8, 8, 16), \\ f_1 &= 12x_1^2 + 12x_2^2 + 12x_3^2. \end{aligned}$$

Let B_1, B_2, B_3 be arbitrary complex numbers. Then the function

$$X(\tau) = B_1 \vartheta_{g^{(1)}, h}(\tau; P_3, f_1) + B_2 \vartheta_{g^{(2)}, h}(\tau; P_3, f_1) + B_3 \vartheta_{g^{(3)}, h}(\tau; P_3, f_1)$$

is a cusp form of type $(-9/2; 48, \nu(M))$, where $\nu(M)$ is a system of multipliers with respect to $f^{(k)}$ (all of these forms have same system of multipliers).

Proof. In Lemma 3 we assume

$$\begin{aligned} \Delta &= 2^9 \cdot 4^k \quad (k = 1, 2, \dots, 8), \\ f &= f^{(k)}, \quad N_1 = 48, \quad N_2 = 48, \quad N_3 = 48, \\ \nu &= 3, \quad P_3^{(1)} = P_3^{(2)} = P_3^{(3)} = P_3 = x_1 x_2 x_3, \\ N &= 48, \quad \Delta_1 = \Delta_2 = \Delta_3 = 6^3 \cdot 4^3, \\ (g^{(1)})' &= (8, 8, 8), \quad (g^{(2)})' = (16, 16, 16), \\ (g^{(3)})' &= (8, 8, 16), \quad j = 3, \quad s = 3. \end{aligned}$$

It is easy to verify that the function $X(\tau)$ satisfies the conditions (9) of Lemma 3.

We have

$$\begin{aligned} (\operatorname{sgn} \delta)^\nu \left(\frac{(-1)^{\lfloor s/2 \rfloor} \Delta_1}{|\delta|} \right) &= (\operatorname{sgn} \delta)^\nu \left(\frac{(-1)^{\lfloor s/2 \rfloor} \Delta_2}{|\delta|} \right) \\ &= \operatorname{sgn} \delta \left(\frac{-3}{|\delta|} \right) \left(\frac{2}{|\delta|} \right), \\ (\operatorname{sgn} \delta)^\nu \left(\frac{(-1)^{\lfloor s/2 \rfloor} \Delta_3}{|\delta|} \right) &= \operatorname{sgn} \delta \left(\frac{-3}{|\delta|} \right) \left(\frac{2}{|\delta|} \right), \\ \left(\frac{(-1)^{\lfloor (s+2\nu)/2 \rfloor} \Delta}{|\delta|} \right) &= \left(\frac{2}{|\delta|} \right). \end{aligned} \tag{10}$$

If $\alpha\delta \equiv 1 \pmod{48}$, then $\alpha\delta \equiv 1 \pmod{12}$. In particular,

$$\alpha \equiv \pm 1, \pm 5 \pmod{12} \text{ and respectively } \delta \equiv \pm 1, \pm 5 \pmod{12}. \tag{11}$$

If in (10) we take $\delta > 0$ and $\delta \equiv 1 \pmod{12}$, then

$$\operatorname{sgn} \delta \left(\frac{-3}{|\delta|} \right) = \left(\frac{-3}{\delta} \right) = 1.$$

In case $\delta > 0$ and $\delta \equiv -5 \pmod{12}$, we have

$$\operatorname{sgn} \delta \left(\frac{-3}{|\delta|} \right) = \left(\frac{-3}{\delta} \right) = -\left(\frac{3}{\delta} \right) = \left(\frac{\delta}{3} \right) = 1.$$

But if in (10) we take $\delta < 0$ and $\delta \equiv 1 \pmod{12}$, then

$$\operatorname{sgn} \delta \left(\frac{-3}{|\delta|} \right) = -\left(\frac{-3}{-\delta} \right) = \left(\frac{3}{-\delta} \right) = -\left(\frac{-\delta}{3} \right) = 1.$$

In case $\delta < 0$ and $\delta \equiv -5 \pmod{12}$, we have

$$\operatorname{sgn} \delta \left(\frac{-3}{|\delta|} \right) = -\left(\frac{-3}{-\delta} \right) = -\left(\frac{-\delta}{3} \right) = 1.$$

Thus

$$\operatorname{sgn} \delta \left(\frac{-3}{|\delta|} \right) = 1 \text{ for } \delta \equiv 1 \pmod{6}. \tag{12}$$

Now suppose that in (10) $\delta > 0$ and $\delta \equiv 5 \pmod{12}$. Then

$$\operatorname{sgn} \delta \left(\frac{-3}{|\delta|} \right) = \left(\frac{-3}{\delta} \right) = \left(\frac{3}{\delta} \right) = \left(\frac{\delta}{3} \right) = -1.$$

In case $\delta > 0$ and $\delta \equiv -1 \pmod{12}$, we have

$$\operatorname{sgn} \delta \left(\frac{-3}{|\delta|} \right) = \left(\frac{-3}{\delta} \right) = -\left(\frac{3}{\delta} \right) = \left(\frac{\delta}{3} \right) = -1.$$

But if in (10) we take $\delta < 0$ and $\delta \equiv 5 \pmod{12}$, then

$$\operatorname{sgn} \delta \left(\frac{-3}{|\delta|} \right) = -\left(\frac{-3}{-\delta} \right) = \left(\frac{3}{-\delta} \right) = -\left(\frac{-\delta}{3} \right) = -1.$$

In case $\delta < 0$ and $\delta \equiv -1 \pmod{12}$ we get

$$\operatorname{sgn} \delta \left(\frac{-3}{|\delta|} \right) = - \left(\frac{-3}{-\delta} \right) = -1.$$

Thus

$$\operatorname{sgn} \delta \left(\frac{-3}{|\delta|} \right) = -1 \text{ for } \delta \equiv -1 \pmod{6}. \tag{13}$$

Hence, by (10), (12) and (13), for all α and δ with $\alpha\delta \equiv 1 \pmod{48}$ we have

$$\begin{aligned} & (\operatorname{sgn} \delta)^\nu \left(\frac{(-1)^{\lfloor s/2 \rfloor} \Delta_k}{|\delta|} \right) \\ &= \left(\frac{(-1)^{\lfloor (s+2\nu)/2 \rfloor} \Delta}{|\delta|} \right) \text{ for } \alpha \equiv 1 \pmod{6}, \\ &= - \left(\frac{(-1)^{\lfloor (s+2\nu)/2 \rfloor} \Delta}{|\delta|} \right) \text{ for } \alpha \equiv -1 \pmod{6}. \end{aligned} \tag{14}$$

Due to Lemma 2

$$\begin{aligned} \vartheta_{\alpha g^{(1)}h}(\tau; P_3, f_1) &= \vartheta_{g^{(1)}h}(\tau; P_3, f_1) \text{ for } \alpha \equiv 1 \pmod{6}, \\ &= -\vartheta_{g^{(1)}h}(\tau; P_3, f_1) \text{ for } \alpha \equiv -1 \pmod{6}, \end{aligned} \tag{15}$$

$$\begin{aligned} \vartheta_{\alpha g^{(2)}h}(\tau; P_3, f_1) &= \vartheta_{g^{(2)}h}(\tau; P_3, f_1) \text{ for } \alpha \equiv 1 \pmod{6}, \\ &= -\vartheta_{g^{(2)}h}(\tau; P_3, f_1) \text{ for } \alpha \equiv -1 \pmod{6}, \end{aligned} \tag{16}$$

$$\begin{aligned} \vartheta_{\alpha g^{(3)}h}(\tau; P_3, f_1) &= \vartheta_{g^{(3)}h}(\tau; P_3, f_1) \text{ for } \alpha \equiv 1 \pmod{6}, \\ &= -\vartheta_{g^{(3)}h}(\tau; P_3, f_1) \text{ for } \alpha \equiv -1 \pmod{6}. \end{aligned} \tag{17}$$

By (14)–(17) it is easy to verify that all conditions of Lemma 3 are satisfied and therefore the function $X(\tau)$ belongs to the space of entire modular forms of type $(-9/2; 48, \nu(M))$. By Lemma 4 our theorem is proved. \square

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