Georgian Mathematical Journal
Volume 13 (2006), Number 4, 807–812

COPRIMENESS AMONG ELEMENT ORDERS OF FINITE GROUPS

XINGZHONG YOU

Abstract. In this paper we classify the finite groups satisfying the following property P_3: every three distinct orders of elements are setwise relatively prime.

2000 Mathematics Subject Classification: 20D60, 20E34.
Key words and phrases: Finite group, element order, prime graph, simple group.

1. Introduction. Let G be a finite group and $Ch(G)$ be one of the following sets:

$cd(G)=\{\chi(1)|\chi\in\mathrm{Irr}(G)\}$, i.e. the set of irreducible character degrees of G;
$con(G)=\{g^G|g\in G\}$, i.e., the set of conjugacy classes of G; and
$\pi_e(G)=\{o(g)|g\in G\}$, i.e., the set of element orders of G.

We say that G has the following property P_n: if $Ch(G)=cd(G)$ or $\pi_e(G)$ then every set of n distinct elements of $Ch(G)$ is setwise coprime; if $Ch(G)=con(G)$ then their sizes are setwise coprime for any n distinct elements of $Ch(G)$.

In [1], the author studied the structure of the solvable group G with P_n when $Ch(G)=cd(G)$, where a quadratic bound that depends only on n for $|cd(G)|$ was obtained. It was conjectured in [2] that if any prime integer p divides at most m elements in $cd(G)$, then $|cd(G)|\leq 3m$. In [3], the authors studied the structure of the group G with P_n when $Ch(G)=con(G)$. In particular, they classified the finite groups when $n=5$, extending the result in [4].

In this paper, we study the structure G with P_n in the case of $Ch(G)=\pi_e(G)$. We classify the finite groups that satisfy property P_3, extending the results in [5] and [6]. We also obtain a bound that depends only on n for $|\pi(G)|$, the number of distinct prime divisors of the order of G.

2. Preliminaries. In the following we always assume that property P_n takes place in the case $Ch(G)=\pi_e(G)$. First, we present some preliminary results we need.

Lemma 2.1. *Let G be a finite group that satisfies property P_n. Then property P_n is inherited by subgroups and quotient groups of G.*

Proof. The proof is straightforward. □

Lemma 2.2. *Let r be a prime and n a positive integer such that $\frac{q+1}{d}$ and $\frac{q-1}{d}$ are products of at most two primes, where $q=r^n$ and $d=(r-1,2)$.*

(a) *If $r>2$ and $q\equiv 3$ or $5 \pmod 8$, then $n=1$ or an odd prime.*

ISSN 1072-947X / $8.00 / © Heldermann Verlag *www.heldermann.de*

(b) *If $r = 2$, then $n = 1, 2$ or an odd prime.*

Proof. (a) Suppose that $r > 2$. If $q \equiv 3 \pmod 8$, then $4 \nmid q-1$. If $n = 2m$, then $q - 1 = (r^m + 1)(r^m - 1)$ and so that $4|q-1$, a contradiction; if $n = km$ is odd with $k, m > 1$, then $q^n - 1 = (q-1)(q^{m-1} + q^{m-2} \cdots + q + 1)(q^{m(k-1)} + q^{m(k-2)} + \cdots + q^m + 1)$, a contradiction. Using a similar argument, we also conclude that if $q \equiv 5 \pmod 8$, then $n = 1$ or an odd prime.

(b) Suppose that $r = 2$. If $n = 4m$, then $2^n - 1 = (2^{2m} + 1)(2^m + 1)(2^m - 1)$, a contradiction; if $n = 4m + 2$, then $2^n - 1 = (2^{2m+1} - 1)(2 + 1)(2^{2m} - 2^{2m-1} + 2^{2m-2} - \cdots + 2^2 - 2 + 1)$, a contradiction; if $n = km$ is odd with $k, m > 1$, then $2^n + 1 = (2+1)(2^{k-1} - 2^{k-2} + 2^{k-3} - \cdots + 2^2 - 2 + 1)(2^{k(m-1)} - 2^{k(m-2)} + 2^{k(m-3)} - \cdots + 2^{2k} - 2^k + 1)$, a contradiction. □

Recall that the prime graph $\Gamma(G)$ of G is defined as follows: its vertex set is $\pi(G)$, and two distinct vertices p, q in $\pi(G)$ are connected with an edge if $pq \in \pi_e(G)$. Denote by $t(G)$ the number of connected components of $\Gamma(G)$ and by $\pi_i = \pi_i(G), i = 1, 2, \ldots, t(G)$ the connected components of $\Gamma(G)$. If G is of even order, we always assume that $2 \in \pi_1$.

A group G is 2-Frobenius if there exists a normal series $1 \lhd N \lhd K \lhd G$ such that G/N and K are Frobenius groups with kernels K/N and N respectively. For any 2-Frobenius G, we have $t(G) = 2$ and $\pi_1 = \pi(N) \cup \pi(G/K)$, $\pi_2 = \pi(K/N)$. In particular, G is solvable.

In relation to the number of connected components of $\Gamma(G)$ we have the following Lemma due to Gruenberg and Kegel (see Theorem A [7]).

Lemma 2.3. *Let G be a finite group with disconnected prime graph, then G has one of the following structures:*
(a) *Frobenius or 2-Frobenius;*
(b) *simple;*
(c) *an extension of a π_1-group by a simple group;*
(d) *simple by π_1; or*
(e) *π_1 by simple by π_1.*

From this Lemma we can deduce that if G is neither Frobenius nor 2-Frobenius, then G has a normal series $1 \lhd N \lhd M \lhd G$ such that N is a nilpotent π_1-group, M/N is a simple group and G/M is a solvable π_1-group.

Lemma 2.4. *Let G be a finite group, N a normal subgroup of G, and G/N a Frobenius group with kernel F and cyclic complement C. If $(|F|, |N|) = 1$ and F is not contained in $NC_G(N)/N$, then $p|C| \in \pi_e(G)$ for some prime divisor p of $|N|$.*

Proof. This follows from [8]. □

Lemma 2.5. *Let G be a simple group that satisfies property P_3. Then G has one of the following structures:*

(a) *$L_2(q)$, $q = 2^f$ or $q = r^f \equiv 3, 5 \pmod 8$, where $\frac{q+1}{d}$ and $\frac{q-1}{d}$ are products of at most two primes, $d = (2, q-1)$;*

(b) *$L_2(9)$, $L_2(7)$ or $L_3(4)$; or*

(c) *$Sz(q)$, where $q = 2^{2n+1}, n \geq 1, q-1, q-\sqrt{2q}+1$ and $q+\sqrt{2q}+1$ are products of at most two primes.*

Proof. Let $P \in Syl_2(G)$. If P is non-abelian, then $4 \in \pi_e(G)$ and hence $\pi_1 = \{2\}$. It follows that G is a C_{22}-group and therefore is isomorphic to $L_2(q)$, $L_3(4)$ or $Sz(2^{2n+1})(n \geq 1)$ by [9]. If $G \cong L_2(q)$, then $q \equiv 1$ or $7 \pmod 8$. Observing that $(q+1)/2, (q-1)/2 \in \pi_e(L_2(q))$, we conclude that if $q \equiv 1 \pmod 8$, then $(q-1)/2 = 4$ and $q = 9$, so $G \cong L_2(9)$; if $q \equiv 7 \pmod 8$, then $(q+1)/2 = 4$ and $q = 7$, so $G \cong L_2(7)$.

If P is abelian, we conclude from [10, Ch. XI, Theorem 13.7] that $G \cong L_2(q)$, $q = 2^f$ or $q \equiv 3$ or $5 \pmod 8$, J_1, or $R(q)$. Since $|\pi_1(J_1)| \geq 3$ (see [11]) and $|\pi_1(R(q))| \geq 3$ (see [10, Ch. XI, Theorem 13.4]), we have $G \cong L_2(q)$.

Considering the cyclic subgroups in $L_2(q)$ and $Sz(q)$, we can obtain other properties of G. □

Lemma 2.6. *Suppose that G is a finite group with property P_n. If there exists an element $x \in G$ whose order is a product of m distinct primes, then $2^{m-1} + 1 \leq n$.*

Proof. Let $x_1 \in \langle x \rangle$ be of order a prime p_1. Then there are 2^{m-1} elements $x_1 y_i$ of distinct orders where $y_i \in \langle x^{p_1} \rangle$. For those 2^{m-1} elements, their orders have the common divisor p_1. This implies that $2^{m-1} + 1 \leq n$. □

3. Main results. Now, we are ready to formulate our results.

Theorem 3.1. *Let G be a finite solvable group that satisfies P_3 and $G_p \in Syl_p(G)$. Then one of the following statements holds.*

(a) *G is a p-group with $\exp(G) \leq p^2$.*

(b) *$G = G_p G_q$ with $\exp(G_p) = p$ and $\exp(G_q) = q$.*

(c) *G is a Frobenius group with kernel N and complement H, where $N = G_p$ with $\exp(N) \leq p^2$ or $N = G_p \times G_q$ with $\exp(G_p) = p$ and $\exp(G_q) = q$; while H is isomorphic to Q_8 or a subgroup of Z_{rs}, r, s primes.*

(d) *G is 2-Frobenius, that is G has a normal series $1 \lhd N \lhd K \lhd G$ such that G/N and K are Frobenius groups with kernels K/N and N respectively, where K/N is isomorphic to a subgroup of Z_{rs}, r, s primes and G/K is of order p; while N is a q-group with $\exp(N) = q$ or a p-group with $\exp(G_p) \leq p^2$.*

Proof. If $t(G) = 1$, then $|\pi(G)| \leq 2$. We conclude that G is the group in (a) or (b). If $t(G) \geq 2$, since G is solvable, we conclude from Lemma 2.3 that $t(G) = 2$ and G is Frobenius or 2-Frobenius.

Suppose first that G is a Frobenius group with kernel N and complement H. We have $\pi_1 = \pi(H)$ and $\pi_2 = \pi(N)$ and thus $|\pi_i| \leq 2$ for $i = 1, 2$. Since N is nilpotent, we conclude that $N = G_p$ with $\exp(N) \leq p^2$ or $N = G_p \times G_q$ with $\exp(P) = p$ and $\exp(Q) = q$. If $\pi(H) = \{r\}$, then H is isomorphic to Q_8 or a subgroup of Z_{r^2}; if $|\pi(H)| = 2$, then H is a product of two Sylow subgroups of prime orders of G. The statement (c) holds.

Suppose now that G is 2-Frobenius. Then G has a normal series $1 \lhd N \lhd K \lhd G$ such that G/N and K are Frobenius groups with kernels K/N and N

respectively. It follows that $\pi_1 = \pi(G/K) \cup \pi(N)$ and K/N is a cyclic π_2-group with $|\pi_i| \leq 2$ for $i = 1, 2$. If $\pi(G/K) = \{p, q\}$, then p or q belongs to $\pi(N)$. We may assume without loss of generality that $q \in \pi(N)$. If $p = 2$ and the Sylow 2-subgroups of G/K are generalized quaternion groups, we conclude that $2, 4$ and $2q \in \pi_e(G)$, a contradiction. It follows that G/K is a cyclic group of order pq. Since N is nilpotent, by induction we may assume that N is an elementary abelian q-group. We conclude from Lemma 2.4 that $pq^2 \in \pi_e(G)$, a contradiction. So G/K is a p-group and $|\pi(N)| = 1$. We have that if $\pi(N) = \{q\}$ and $q \neq p$, then N is a q-group with $exp(N) = q$; if $\pi(N) = \{p\}$, then $\exp(G_p) \leq p^2$. It is clear that K/N is isomorphic to a subgroup of Z_{rs} with $r, s \in \pi(G)$. The statement (d) holds. □

Theorem 3.2. *Let G be a finite non-solvable group that satisfies P_3. Then one of the following statements holds.*

(a) *$G/O_2(G) \cong L_2(q)$, where $q = 2^f \geq 4$ and $q+1, q-1$ are products of at most two primes.*

(b) *$G \cong L_2(q)$, where $q > 5$ and $q \equiv 3$ or $5 \pmod 8$, $\frac{q+1}{2}$ and $\frac{q-1}{2}$ are products of at most two primes.*

(c) *$G \cong L_2(7)$, $L_2(9)$, or $L_3(4)$.*

(d) *$G \cong Sz(q)$, where $q = 2^{2n+1}, n \geq 1$, $q-1$, $q - \sqrt{2q} + 1$ and $q + \sqrt{2q} + 1$ are products of at most two primes.*

Proof. Since G is unsolvable, we conclude that $|\pi(G)| \geq 3$ and $t(G) \geq 2$. By Lemma 2.3, G has a normal series $1 \lhd N \lhd M \lhd G$ such that N is a nilpotent π_1-group, M/N is one of the simple groups listed in Lemma 2.5 and G/M is a solvable π_1-group.

Claim 1. $G = M$.

Suppose first that $M/N \cong L_2(7)$, $L_2(9)$ or $L_3(4)$. We conclude from [11] that $G = M$.

Suppose now that $q = r^f \equiv 3$ or $5 \pmod 8$ or $q = 2^f$. Let $t \in \pi(G/M)$. Since $G/M \leq Out(M/N)$, we have $t||Out(M/N)|$. Observe that $|Out(M/N)| = fd$, where $d = (q-1, 2)$, it follows that if $t \not| f$ then $t = d = 2$ and $q \equiv 3$ or $5 \pmod 8$. This forces $G/N \cong PGL(2, q)$, which implies that G/N has at least three elements of distinct even orders, a contradiction. Thus $t|f$.

If $q \equiv 3 \pmod 8$, then $\frac{q+1}{2} = 2t_1$ and t_1 is an odd prime. We conclude from Lemma 2.2 that $f = t$ is an odd prime, which implies that $t = t_1$ and so that $\frac{q+1}{2} = \frac{1}{2}(r^t + 1) = 2t$, a contradiction. If $q \equiv 5 \pmod 8$, a similar argument yields a contradiction. If $q = 2^f$, we also conclude from Lemma 2.2 that $f = t$ is an odd prime (note that $L_2(4) \cong L_2(5)$, the case of $q = 5$ has been considered). Observe that $|G/M| = |Out(M/N)| = f$, we claim that $f \not| |M/N|$. If this is false, then $f|2^f + 1$ or $f|2^f - 1$. It follows that $f, 2f, 2^f + 1 \in \pi_e(G/N)$ or $f, 2f, 2^f - 1 \in \pi_e(G/N)$, a contradiction. Let $x \in G/N - M/N$ such that x is of order f. We conclude that x acts fixed-point freely on the subgroups of orders $2^f + 1$ and $2^f - 1$ of M/N and therefore $f|(2^f + 1) + 1$ and $f|(2^f - 1) + 1$. This forces $f|2$, a contradiction.

Suppose then that $M/N \cong Sz(2^{2n+1})$. We have $|Out(M/N)| = 2n+1$. If $G > M$, since $|G/M|||Out(M/N)|$ and $\pi(G/M) = \pi_1(G) = \{2\}$, we have $2|2n+1$, a contradiction.

Claim 2. N is a normal p-subgroup of G and $C_G(N) \leq N$. In particular, $N = O_p(G)$.

Since N is a nilpotent π_1-group, by Lemma 2.1 we can assume that N is a p-group. If $C_G(N) \not\leq N$, then $G = C_G(N)N$. We conclude that G has elements of distinct orders p, pl and pm respectively, where $l, m \in \pi(G)$, a contradiction.

Claim 3. If $G/N \cong L_2(q)$, $q = 2^f \geq 4$, then N is a 2-group. Otherwise N=1.

Suppose first that $G/N \cong L_2(q)$, $q = 2^f \geq 4$. Since N is a p-group, we may assume from Lemma 2.1 that N is a minimal normal subgroup of G. Let $P \in Sly_2(G/N)$, and $Q = N_{G/N}(P)$ and $W/N \cong Q$. If N is not a 2-group, then Q is a Frobenius group of order $(2^f-1)2^f$. From Claim 2, we conclude that Q acts faithfully on N. It follows from Lemma 2.4 that $(2^f-1)p \in \pi_e(W)$. Also, we have $2p \in \pi_e(W)$ since P is elementary abelian, a contradiction. Therefore N is a 2-group.

Suppose now that $G/N \cong L_2(q)$, where $q = r^f > 5$ and $q \equiv 3$ or $5 \pmod 8$. Arguing as in the preceding paragraph, we conclude that $N = 1$.

Suppose then that $G/N \cong L_2(9), L_2(7)$ or $L_3(4)$. In order to prove $N = 1$, let G be a minimal counterexample. We may assume that N is a minimal normal subgroup of G. We conclude from the above argument that $\pi_1(G) = \{2\}$ and so N is an elementary abelian 2-group. By claim 2, all $2'$-subgroups of G act fixed-point freely on N. If $G/N = L_2(9)$, since G/N has Frobenius subgroups of order 36, we conclude from Lemma 2.4 that $8 \in \pi_e(W)$, a contradiction. If $G/N \cong L_2(7)$ or $L_3(4)$, a similar argument yields a contradiction.

Suppose finally that $G/N = Sz(q)$, where $q = 2^{2n+1}, n \geq 1$. If $N > 1$, we may assume that N is an elementary abelian 2-group. Since G/N has Frobenius subgroups of order $4(q - \sqrt{2q} + 1)$ with cyclic complements of order four, we conclude from Lemma 2.4 that $8 \in \pi_e(G)$, a contradiction. □

Theorem 3.3. *If G is a finite group that satisfies P_n, then $|\pi(G)| \leq C(\log n)^4 \log\log n$, where C is a constant. In particular, $|\pi_e(G)| \leq C(n-1)(\log n)^4 \log\log n + 1$.*

Proof. Since G satisfies P_n, we conclude that $|\pi_e(G)| \leq (n-1)|\pi(G)| + 1$. By Lemma 2.6 we have that the number of different prime divisors of the order of an element of G is bounded by a logarithmic function of n. The result follows from Theorem A of [12]. □

Acknowledgement

This work has been supported by the National Natural Science Foundation of China (grant No. 10671026), and by the Scientific Research Fund of Hunan Provincial Education Department, and by the Scientific Research Fund of Changsha University of Science and Technology.

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(Received 22.03.2006)

Author's address:

College of Mathematics and Computing
Changsha University of Science and Technology
Changsha, Hunan, 410077
P. R. China
E-mail: xzyou2003@yahoo.com.cn