# Georgian Mathematical Journal 

Volume 14 (2007), Number 1, 169-193

# AN EXAMPLE OF FRACTAL SINGULAR HOMOGENIZATION 

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In Memoriam Gaetano Fichera


#### Abstract

We construct a sequence of quadratic weighted energy forms in an open domain of the plane, that $M$-converges to an energy form with a singular fractal term. The weights belong to the Muckenoupt class $A_{2}$ and have pointwise singularities. The result implies the spectral convergence of a sequence of second-order weighted elliptic operators in divergence form in the plane to a singular elliptic operator with a second order fractal term.


2000 Mathematics Subject Classification: 35J20, 35J70; 35B20, 35B27; 74K05, 74K15.
Key words and phrases: Fractal singular homogenization, weighted elliptic operators.

## Introduction

The motivation for this paper comes from the theory of so-called highly conductive thin layers, revisited from the point of view of fractals.

Homogenization problems involving highly conductive thin layers have been studied since the 70 s of the 20th century, in connection with various applications, see, e.g., [3], [8], [1].

Transmission problems with infinitely conductive fractal layers have been recently studied by Lancia [15] and Lancia-Vivaldi [18]. In this paper, we perform the asymptotic homogenization analysis of such fractal problems.

In the homogenization approach, a (lower dimensional) infinitely conductive thin layer is approximated by a (full dimensional) thin layer of vanishing thickness, $\varepsilon$, and increasingly high conductivity, $a$. This approach provides better physical grounds to the model.

In the case of a fractal layer, however, the asymptotic thin-layer approach meets with a basic difficulty in the very definition of what a thin set surrounding the given fractal, $K$, should be. The natural definition of thin layer, that applies to flat or smooth layer, that is, the set of points whose (Euclidean) distance to the layer is, say, less than a small $\varepsilon$, may not be an appropriate notion for a thin neighborhood of the fractal $K$. The reason is that, contrary to the flat case, the parameter $\varepsilon$ cannot be interpreted as a small ratio between transversal and tangential dimensions, the latter being infinite (in the Euclidean sense) for a fractal layer, at every small scale.

We overcome this difficulty by adding a third asymptotic parameter, in addition to $\varepsilon$ and $a$. The new parameter is the index $n$ of the iteration process of the
fractal construction in terms of a given family of (Euclidean) contractive similarities. For every given non-negative integer $n$ and every (small) $\varepsilon$, we define conveniently - a neighborhood $\sum_{\varepsilon}^{n}$ of the pre-fractal set $K^{n}$ obtained at the $n$-th iteration in the construction of $K$. Inside the set $\Sigma_{\varepsilon}^{n}$, we define - conveniently the conductivity coefficient, $a$. We consider the simultaneous limit of the three parameters, $n \rightarrow \infty, \varepsilon \rightarrow 0, a \rightarrow \infty$. A nontrivial outcome of the process is assured only by enforcing a correct balance among the parameters.

The crucial point in this construction is the choice of the geometry of $\Sigma_{\varepsilon}^{n}$, as well as of the coefficient $a$ within $\Sigma_{\varepsilon}^{n}$. We incorporate both geometry and conductivity of $\sum_{\varepsilon}^{n}$ into a suitable weight function, $w_{\varepsilon}^{n}$, which occurs as a singular "Riemannian" metric in the approximating energy functionals. The weights display pointwise singularities, which, in the limit, give rise to the fractal energy contribution.

We confine ourselves here to an example - $K$ being the Koch curve in a plane domain - as an illustration of the general variational setting outlined before. We do not describe the boundary value transmission problems underlying our variational setting. However, we establish some convergence and regularity properties of variational solutions.

As a Euclidean representative domain we consider a polygon $\Omega$ of the plane and as a thin layer we take a Koch curve $K$ inside $\Omega$, whose end-points meet the boundary of $\Omega$. We construct the family of weights $w_{\varepsilon}^{n}$ and we consider the related energy functionals $F_{\varepsilon}^{n}$ in $\Omega$. The weights $w_{\varepsilon}^{n}$ belong to the class of Muckenhoupt $A_{2}$.

Our main result is the proof of the $M$-convergence of the functionals to a limit energy functional, which contains a singular term supported on the fractal $K$. We point out that the $M$-convergence of the functionals implies the spectral convergence of the self-adjoint operators associated with the forms. Therefore our result assures the spectral convergence of the boundary transmission problems mentioned before. In this paper we do not develop these spectral aspects. For the spectral properties of $M$-convergence we refer to [20].

In Section 1 we set our notation and state the main result, Theorem 1.1. The proof is divided into two parts developed in Section 2 and Section 3. In the last section, Section 4, we describe some properties of the minimizers.

## 1. Main Result

Let $\Omega$ be the polygonal domain with vertices $A=(0,0), B=(1,0), C=$ $(1 / 2, \sqrt{3} / 2)$ and $D=(1 / 2,-\sqrt{3} / 2)$. We consider the following 4 contractive similarities $\left\{\psi_{1}, \psi_{2}, \psi_{3}, \psi_{4}\right\}$ in $\mathbb{R}^{2}$ :

$$
\begin{gathered}
\psi_{1}(z)=\frac{z}{3}, \quad \psi_{2}(z)=\frac{z}{3} e^{i \frac{\pi}{3}}+\frac{1}{3} \\
\psi_{3}(z)=\frac{z}{3} e^{-i \frac{\pi}{3}}+\frac{1}{2}+i \frac{\sqrt{3}}{2}, \quad \psi_{4}(z)=\frac{z+2}{3},
\end{gathered}
$$

where $z=(x, y) \in \mathbb{R}^{2}$.

For each integer $n>0$, we consider arbitrary $n$-tuples of the indices $i \mid n=$ $\left(i_{1}, i_{2}, \ldots, i_{n}\right) \in\{1,2,3,4\}^{n}$. We then define $\psi_{i \mid n}=\psi_{i_{1}} \circ \psi_{i_{2}} \circ \cdots \circ \psi_{i_{n}}$ and for every set $\mathcal{G}\left(\subseteq \mathbb{R}^{2}\right)$ we set $\mathcal{G}^{i \mid n}=\psi_{i \mid n}(\mathcal{G})$. Occasionally, the set of indexes $i \mid n$ will be referred to as the $n$-address of the set $\mathcal{G}^{i \mid n}$.

Let $V_{0}=\{A, B\}$. For every integer $n>0$, let

$$
V^{n}=\bigcup_{i \mid n} V_{0}^{i \mid n}
$$

where $V_{0}^{i \mid n}=\left(V_{0}\right)^{i \mid n}$ in the preceding notation. We put

$$
V^{\infty}=\bigcup_{n=1}^{+\infty} V^{n}
$$

and

$$
K=\bar{V}^{\infty}
$$

the closure being in $\mathbb{R}^{2}$. The set $K$ is the Koch curve in $\mathbb{R}^{2}$, with end-points $A, B$.

Now let $K_{0}$ be the interval with end-points $A$ and $B$. For every $0<\varepsilon \leq \varepsilon_{0} \leq$ $c_{1} / 2$, where $c_{1}=\tan (\pi / 12)$, we define the " $\varepsilon$-neighborhood" of $K_{0}$, denoted by $\Sigma_{\varepsilon}$, to be the polygon whose vertices are the points $A, P_{1}, P_{2}, B, P_{3}, P_{4}$, where

$$
P_{1}=\left(\frac{\varepsilon}{c_{1}}, \frac{\varepsilon}{2}\right), \quad P_{2}=\left(1-\frac{\varepsilon}{c_{1}}, \frac{\varepsilon}{2}\right), \quad P_{3}=\left(1-\frac{\varepsilon}{c_{1}},-\frac{\varepsilon}{2}\right), \quad P_{4}=\left(\frac{\varepsilon}{c_{1}},-\frac{\varepsilon}{2}\right)
$$

We then subdivide $\Sigma_{\varepsilon}$ into the rectangle $\mathcal{R}_{\varepsilon}$ and two triangles $\mathcal{T}_{j, \varepsilon}, j=1,2$. Here, $\mathcal{R}_{\varepsilon}$ is the rectangle with vertices $P_{1}, P_{2}, P_{3}, P_{4} ; \mathcal{T}_{1, \varepsilon}$ is the triangle with vertices $A, P_{1}, P_{4}$ and $\mathcal{T}_{2, \varepsilon}$ is the triangle with vertices $P_{2}, B, P_{3}$.

For every integer $n$, let $K^{n}$ be the polygonal curve

$$
K^{n}=\bigcup_{i \mid n} K_{0}^{i \mid n}
$$

where $K_{0}^{i \mid n}=\left(K_{0}\right)^{i \mid n}$. For every $n$ and $\varepsilon$ as above, we define the " $\varepsilon$-neighborhood", $\Sigma_{\varepsilon}^{n}$, of $K^{n}$ to be the (open) set

$$
\Sigma_{\varepsilon}^{n}=\bigcup_{i \mid n} \Sigma_{\varepsilon}^{i \mid n}
$$

where $\Sigma_{\varepsilon}^{i \mid n}=\left(\Sigma_{\varepsilon}\right)^{i \mid n}$. Note that $\Sigma_{\varepsilon}^{n}$ is a topological neighborhood of $K^{n} \backslash V^{n}$.
In the domain $\Omega$, taken together with the embedded layer $\Sigma_{\varepsilon}^{n}$ for given $n$ and $\varepsilon$, we now define a weight, $w_{\varepsilon}^{n}$, as follows. Let $P$ - for some $i \mid n$ - belong to the boundary $\partial\left(\Sigma_{\varepsilon}^{i \mid n}\right)$ of $\Sigma_{\varepsilon}^{i \mid n}$ and let $P^{\perp}$ be the orthogonal projection of $P$ on $K_{0}^{i \mid n}$. If $(\xi, \eta)$ belongs to the segment with end-points $P$ and $P^{\perp}$, we set, in our current notation,

$$
w_{\varepsilon}^{n}(\xi, \eta)=\left\{\begin{array}{lll}
\frac{2+c_{1}^{2}}{4|P-P \perp|} c_{0} & \text { if } & (\xi, \eta) \in \mathcal{T}_{j, \varepsilon}^{i \mid n}, \quad j=1,2,  \tag{1.1}\\
\frac{1}{2\left|P-P^{\perp}\right|} c_{0} & \text { if } \quad(\xi, \eta) \in \mathcal{R}_{\varepsilon}^{i \mid n},
\end{array}\right.
$$



Figure 1. Geometry of the layer
where $c_{0}$ is a fixed positive constant, $\left|P-P^{\perp}\right|$ is the (Euclidean) distance between $P$ and $P^{\perp}$ in $\mathbb{R}^{2}, \mathcal{T}_{j, \varepsilon}^{i \mid n}=\left(\mathcal{T}_{j, \varepsilon}\right)^{i \mid n}, \mathcal{R}_{\varepsilon}^{i \mid n}=\left(\mathcal{R}_{\varepsilon}\right)^{i \mid n}$. Moreover, we set

$$
\begin{equation*}
w_{\varepsilon}^{n}(\xi, \eta)=1 \quad \text { if } \quad(\xi, \eta) \notin \Sigma_{\varepsilon}^{n} . \tag{1.2}
\end{equation*}
$$

Associated with the weight $w_{\varepsilon}^{n}$, are the Sobolev spaces

$$
\begin{equation*}
H^{1}\left(\Omega ; w_{\varepsilon}^{n}\right)=\left\{u \in L^{2}(\Omega): \int_{\Omega}|\nabla u|^{2} w_{\varepsilon}^{n} d \xi d \eta<+\infty\right\} \tag{1.3}
\end{equation*}
$$

and $H_{0}^{1}\left(\Omega ; w_{\varepsilon}^{n}\right)$, the latter being the completion of $C_{0}^{\infty}(\Omega)$ in the norm

$$
\left.\|u\|_{H^{1}\left(\Omega ; w_{\varepsilon}^{n}\right)}=\left\{\int_{\Omega}|u|^{2} d \xi d \eta+\int_{\Omega}|\nabla u|^{2} w_{\varepsilon}^{n} d \xi d \eta\right\}^{\frac{1}{2}}\right\}
$$

and the "weighted" energy functionals in $L^{2}(\Omega)$

$$
F_{\varepsilon}^{n}([u])=\left\{\begin{array}{lll}
\int_{\Omega} a_{\varepsilon}^{n}(\xi, \eta)|\nabla u|^{2} d \xi d \eta & \text { if } & u \in H_{0}^{1}\left(\Omega, w_{\varepsilon}^{n}\right)  \tag{1.4}\\
+\infty & \text { if } & u \in L^{2}(\Omega) \backslash H_{0}^{1}\left(\Omega, w_{\varepsilon}^{n}\right)
\end{array}\right.
$$

where the unbounded conductivity coefficient is

$$
a_{\varepsilon}^{n}(\xi, \eta)= \begin{cases}\rho_{n} w_{\varepsilon}^{n}(\xi, \eta) & \text { if } \quad(\xi, \eta) \in \Sigma_{\varepsilon}^{n}  \tag{1.5}\\ 1 & \text { if } \quad(\xi, \eta) \notin \Sigma_{\varepsilon}^{n}\end{cases}
$$

with some given positive constant $\rho_{n}$.
We recall that the set $K$ has Hausdorff dimension $d=\ln 4 / \ln 3$ and that it supports the (invariant) Hausdorff measure $\mathcal{H}^{d}$. Moreover, an energy form $E[u]$ is also defined on $K$, which is the limit of an increasing sequence of quadratic forms constructed by finite difference schemes, namely,

$$
\left\{\begin{array}{l}
E[u]=\lim _{n \rightarrow+\infty} E^{(n)}[u]  \tag{1.6}\\
E^{(n)}[u]=4^{n} \sum_{i \mid n}\left(u\left(\psi_{i \mid n}(A)\right)-u\left(\psi_{i \mid n}(B)\right)\right)^{2}
\end{array}\right.
$$

The form $E[u]$ is a regular Dirichlet form on $L^{2}\left(K, \mathcal{H}^{d}\right)$, with a domain $D[E]$ dense in $L^{2}\left(K, \mathcal{H}^{d}\right)$. The functions $u \in D[E]$ turn out to be continuous functions on $K$, which are indeed Hölder continuous with exponent $\delta=d / 2$. The subspace of $D[E]$ of all functions $u \in D[E]$ that vanish at the end-points $A$ and $B$ of $K$ will be denoted by $D_{0}[E]$. In the following, we consider the form $E$ always on its domain $D_{0}[E]$. For definitions and more details on these properties, we refer to [6], [14] and [15]. In order to state our main result, we also need to recall the notion of $M$-convergence of functionals, introduced in [22], see also [20].

Definition 1.1. A sequence of functionals $F^{n}: H \rightarrow(\infty,+\infty]$ is said to $M$-converge to a functional $F: H \rightarrow(\infty,+\infty]$ in a Hilbert space $H$, if
(a) For every $u \in H$ there exists $u_{n}$ converging strongly in $H$ such that

$$
\begin{equation*}
\overline{\lim } F^{n}\left[u_{n}\right] \leq F[u], \quad \text { as } \quad n \rightarrow+\infty . \tag{1.7}
\end{equation*}
$$

(b) For every $v_{n}$ converging weakly to $u$ in $H$

$$
\begin{equation*}
\underline{\varliminf} F^{n}\left[v_{n}\right] \geq F[u], \quad \text { as } \quad n \rightarrow+\infty \tag{1.8}
\end{equation*}
$$

The result is the following

Theorem 1.1. Let $n \rightarrow \infty$, let $\varepsilon=\varepsilon(n)$ be an arbitrary sequence such that $\varepsilon(n) \rightarrow 0$ as $n \rightarrow \infty$ and let $\rho_{n}=\left(3^{d-1}\right)^{n}$ for every $n$. Then the sequence of functionals $F_{\varepsilon(n)}^{n}$, defined in (1.4), $M$-converges to the following functional $F$ as $n \rightarrow \infty$ :

$$
F[u]= \begin{cases}\int_{\Omega}|\nabla u|^{2} d \xi d \eta+c_{0} E[u] & \text { if } u \in D_{0}[F],  \tag{1.9}\\ +\infty & \text { if } u \in L^{2}(\Omega) \backslash D_{0}[F]\end{cases}
$$

where

$$
\begin{equation*}
\left.D_{0}[F]=\left\{v \in H_{0}^{1}(\Omega): v_{\left.\right|_{K}} \in D_{0}[E)\right]\right\} \tag{1.10}
\end{equation*}
$$

and $v_{\left.\right|_{K}}$ denotes the trace of $v$ on $K$.
The idea of the proof is to use the self similarity of the Koch curve. We introduce an "interpolation operator" $G_{\varepsilon}$ and a "mean value" operator $\mathcal{M}_{\varepsilon}$ on a reference domain $\mathcal{D}$ and then we perform local changes of variable to bring all calculations on the set $\mathcal{D}$.

Remark 1.1. $M$-convergence of the functionals implies, in particular, strong convergence of the resolvent operators, semigroups and spectral families, associated with the forms (see Theorem 2.4, Corollaries 2.6 and 2.7 of [20]). However, in this paper, we will not deal with these consequences of Theorem 1.1.

We split the proof of Theorem 1.1 in two parts and each part in a few steps.

## 2. Proof of (a): the "limsup" Condition

We start by constructing our reference domain $\mathcal{D}$, which is a subset of $\Omega$, and a larger layer $\Sigma_{2 \varepsilon}$, which contains $\Sigma_{\varepsilon}$ and is in turn contained in $\mathcal{D}$, see Figure 1.

The set $\mathcal{D}$ is the polygonal domain with vertices $A, B, E=(1 / 2,1 / 2 \sqrt{3})$, and $F=(1 / 2,-1 / 2 \sqrt{3})$. For every $0<\varepsilon \leq \varepsilon_{0} \leq c_{1} / 2$, as in Section 1, we define the set $\Sigma_{2 \varepsilon}$ to be the polygonal domain with vertices $A, Q_{1}, Q_{2}, B, Q_{3}, Q_{4}$, where

$$
Q_{1}=\left(\frac{\varepsilon}{c_{1}}, \varepsilon\right), \quad Q_{2}=\left(1-\frac{\varepsilon}{c_{1}}, \varepsilon\right), \quad Q_{3}=\left(1-\frac{\varepsilon}{c_{1}},-\varepsilon\right), \quad Q_{4}=\left(\frac{\varepsilon}{c_{1}},-\varepsilon\right) .
$$

Clearly, $K_{0} \subset \Sigma_{\varepsilon} \subset \Sigma_{2 \varepsilon} \subset \mathcal{D} \subset \Omega$.
We now consider the space $C^{\delta}(\overline{\mathcal{D}}) \cap H^{1}(\mathcal{D})$, where $H^{1}(\mathcal{D})$ is the usual Sobolev space on $\mathcal{D}$ and $C^{\delta}(\overline{\mathcal{D}})$ is the space of Hölder continuous functions on $\overline{\mathcal{D}}$, with Hölder exponent $\delta>0$. In the following, we fix the exponent $\delta$ to be $\delta=d / 2=$ $\ln 4 / \ln 9$.

We define the operator $G_{\varepsilon}: C^{\delta}(\overline{\mathcal{D}}) \cap H^{1}(\mathcal{D}) \rightarrow C^{\delta}(\overline{\mathcal{D}}) \cap H^{1}(\mathcal{D})$, by setting, for a given function $g$ on $\mathcal{D}, g_{\varepsilon}=G_{\varepsilon}(g)$, where the function $g_{\varepsilon}(x, y)$ is defined for $(x, y) \in \mathcal{D}$ as follows.

For every $x \in(0,1)$, we define $P_{ \pm}=P_{ \pm}(x)=\left(x, \widehat{y}_{ \pm}(x)\right) \in \partial \Sigma_{\varepsilon}$ to be the intersections of $\partial \Sigma_{\varepsilon}$ with the vertical line through the point $(x, 0) \in K_{0}$, and $Q_{ \pm}=Q_{ \pm}(x)=\left(x, \widetilde{y}_{ \pm}(x)\right) \in \partial \Sigma_{2 \varepsilon}$ to be the intersections of $\partial \Sigma_{2 \varepsilon}$ with the
vertical line through the point $(x, 0) \in K_{0}$. Then, we put

$$
g_{\varepsilon}(x, y)= \begin{cases}g(x, y) & \text { if } \quad(x, y) \in \mathcal{D} \backslash \Sigma_{2 \varepsilon}  \tag{2.1}\\ g(x, 0) & \text { if } \quad(x, y) \in \bar{\Sigma}_{\varepsilon} \\ g(x, 0) \frac{\widetilde{\dddot{y}}_{ \pm}-y}{\tilde{y}_{ \pm}-\hat{\dddot{y}}_{ \pm}}+g\left(Q_{ \pm}\right) \frac{y-\widehat{y}_{ \pm}}{\tilde{y}_{ \pm}-\hat{\jmath}_{ \pm}} & \text {if } \quad(x, y) \in \bar{\Sigma}_{2 \varepsilon} \backslash \bar{\Sigma}_{\varepsilon}\end{cases}
$$

Thus, $g_{\varepsilon}$ is equal to $g$ in $\mathcal{D} \backslash \Sigma_{2 \varepsilon}$ and, on each segment $S$ obtained as the intersection of $\bar{\Sigma}_{2 \varepsilon}$ with the vertical line through the point $(x, 0) \in K_{0}, g_{\varepsilon}$ is the piecewise-affine function, which is (constant) and equal to $g(x, 0)$ on $S \cap \bar{\Sigma}_{\varepsilon}$ and equal to $g$ on $S \cap \partial \Sigma_{2 \varepsilon}$.

Before proceeding with the proof, we construct a nested sequence of regular triangulations $T_{n}$ of the polygonal region $\Omega$. For our purposes, it is crucial that the vertices of $K^{n}$ are, at the same time, the nodes of the triangulation $T_{n}$, at each $n$ th-level of iteration. We define the initial triangulation $T_{1}$. We start by constructing the equilateral triangle with vertices $(1 / 3,0),(2 / 3,0),(1 / 2, \sqrt{3} / 6)$ and we proceed by constructing other five equal triangles such that the union of six equilateral triangles gives the regular hexagon centered in $(1 / 2, \sqrt{3} / 6)$. The triangle $A B C$ is the union of 9 equal (equilateral) triangles. By proceeding in a symmetric way in the triangle $A B D$, we complete the triangulation of $\Omega$. We then construct a nested family by $T_{1}, T_{2}, T_{3}, \ldots$ of triangulations of $\Omega$, by subdividing any triangle of $T_{n}$ into nine congruent sub-triangles, as has been done previously (see Figure 2).

Let $N_{n}$ be the set of the vertices of the triangles of $T_{n}$ and let $\mathcal{S}_{n}$ be the space of functions which are continuous on $\bar{\Omega}$ and affine on each triangle of $T_{n}$. We call the points in $N_{n}$ nodes and the functions in $\mathcal{S}_{n}$ finite element functions of level $n$. We have $N_{n} \subset N_{n+1}$, and $\mathcal{S}_{n} \subset \mathcal{S}_{n+1}$.

For a given continuous function $u$, for each $n$ we denote by $I_{n} u$ the function of $\mathcal{S}_{n}$ which is the affine interpolate of $u$ at the nodes of $N_{n}$. We have

$$
\begin{equation*}
I_{n} u \in \mathcal{S}_{n}, \quad\left(I_{n} u\right)(P)=u(P), \quad P \in N_{n} \tag{2.2}
\end{equation*}
$$

We now proceed with the proof of condition (a) in Definition 1.1. We consider a given function $u$ as in condition (a) and we observe that, without loss of generality, we can assume that $u \in D_{0}[F]$, otherwise the inequality (1.7) becomes trivial. The proof will take place in three steps.

Step 1. In this first step, we assume, in addition, that $u \in D_{0}[F] \cap C^{\delta}(\bar{\Omega})$, $\delta=d / 2$. Let $\varepsilon=\varepsilon(n)$ be the sequence occurring in the statement of the Theorem, such that $\varepsilon(n) \rightarrow 0$ as $n \rightarrow \infty$. From now on, we simply denote $\varepsilon(n)$ by $\varepsilon$. Moreover, in order to further simplify the notation, whenever $\varepsilon=\varepsilon(n)$ we write $F^{n}$ in place of $F_{\varepsilon}^{n}(n)$, as well as $a^{n}$ in place of $a_{\varepsilon(n)}^{n}$ and other similar abbreviations. Below, by $c$ we denote possibly different positive constants, all independent of $n$.

In the notation of Section 1, for every $n$-tuples of indices $i \mid n$ we put $\mathcal{D}^{i \mid n}=$ $\psi_{i \mid n}(\mathcal{D})$ and

$$
\mathcal{D}^{n}=\bigcup_{i \mid n} \mathcal{D}^{i \mid n}
$$



Figure 2. Triangulation $T_{2}$ (in the triangle ABC ). Triangulation $T_{1}$ (in the triangle ABD)

For every $n$ and $\varepsilon$ as above, we define

$$
u_{n}(\xi, \eta)=\left\{\begin{array}{lll}
u_{I_{n}}(\xi, \eta) & \text { if } & (\xi, \eta) \in \Omega \backslash \mathcal{D}^{n}  \tag{2.3}\\
G_{\varepsilon}\left(u_{I_{n}} \circ \psi_{i \mid n}\right) \circ \psi_{i \mid n}^{-1}(\xi, \eta) & \text { if } & (\xi, \eta) \in \mathcal{D}^{i \mid n}
\end{array}\right.
$$

where $G_{\varepsilon}$ is the operator defined before.
Proposition 2.1. In the assumptions of Theorem 1.1, for every $u \in C^{\delta}(\bar{\Omega}) \cap$ $D_{0}[F], \delta=d / 2$, we have

$$
\begin{equation*}
\varlimsup_{n \rightarrow+\infty} F^{n}\left[u_{n}\right] \leq F[u], \tag{2.4}
\end{equation*}
$$

where $u_{n}$ is defined in (2.3).
Proof. We start by noticing that two contracted copies $K_{0}^{i \mid n}$ and $K_{0}^{j \mid n}$ of $K^{n}$, with different $n$-addresses $i \mid n$ and $j \mid n$, may intersect each other only at one of the vertices of the polygonal curve $K^{n}$. A similar remark holds for two distinct copies of $\Sigma_{\varepsilon}$ and of $\Sigma_{2 \varepsilon}$. Analogously, two distinct copies of the domain $\mathcal{D}$ may share only vertices or a whole side, but they never overlap in their interiors (see


Figure 3. Iteration procedure
also Figure 3). As a consequence of the preceding intersection properties and of our choice of the nodes of the triangulation $T_{n}$, the functions $u_{n}$ defined in (2.3) belong to $H^{1}(\Omega) \cap C^{\delta}(\bar{\Omega})$.

For each $n$, we split the integral $F^{n}\left[u_{n}\right]$ in three terms, according to the definitions of $a^{n}$ (see (1.5)) and of $u_{n}$.

$$
F^{n}\left[u_{n}\right]=\int_{\Omega \backslash \mathcal{D}^{n}}\left|\nabla u_{I_{n}}\right|^{2} d \xi d \eta+\frac{4^{n}}{3^{n}} \int_{\Sigma_{\varepsilon}^{n}}\left|\nabla u_{n}\right|^{2} w^{n} d \xi d \eta+\int_{\Sigma_{2 \varepsilon}^{n} \backslash \sum_{\varepsilon}^{n}}\left|\nabla u_{n}\right|^{2} d \xi d \eta
$$

By standard properties of the interpolate functions $u_{I_{n}}$, since the 2-dimensional Lebesgue measure of $\mathcal{D}^{n}$ tends to zero as $n \rightarrow+\infty$, we get

$$
\begin{equation*}
\lim _{n \rightarrow+\infty} \int_{\Omega \backslash \mathcal{D}^{n}}\left|\nabla u_{I_{n}}\right|^{2} d \xi d \eta=\int_{\Omega}|\nabla u|^{2} d \xi d \eta \tag{2.5}
\end{equation*}
$$

Then, in order to achieve the proof of Proposition 2.1, we must only prove that

$$
\begin{equation*}
\lim _{n \rightarrow+\infty} \int_{\Sigma_{2 \varepsilon}^{n} \backslash \Sigma_{\varepsilon}^{n}}\left|\nabla u_{n}\right|^{2} d \xi d \eta=0 \tag{2.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\varlimsup_{n \rightarrow+\infty} \frac{4^{n}}{3^{n}} \int_{\Sigma_{\varepsilon}^{n}}\left|\nabla u_{n}\right|^{2} w^{n} d \xi d \eta \leq c_{0} E[u] \tag{2.7}
\end{equation*}
$$

Indeed, at the end of this Step 1, we shall prove, in addition, that $u_{n} \in$ $H^{1}\left(\Omega ; w^{n}\right)$ for every $n$.

We have

$$
\int_{\Sigma_{2 \varepsilon}^{n} \backslash \Sigma_{\varepsilon}^{n}}=\bigcup_{i \mid n}^{\Sigma_{2 \varepsilon}^{i n} \backslash \sum_{\varepsilon}^{i \mid n}} \int .
$$

For a fixed $n$-address $i \mid n$, the set $\Sigma_{2 \varepsilon}^{i \mid n} \backslash \Sigma_{\varepsilon}^{i \mid n}$ can be seen as the union of two rectangles and four triangles, and we split the corresponding integral according to this decomposition. More precisely, we write

$$
\begin{equation*}
\int_{\Sigma_{2 \varepsilon}^{i \mid n} \backslash \sum_{\varepsilon}^{i \mid n}}\left|\nabla u_{n}\right|^{2} d \xi d \eta \equiv R_{1}+R_{2}+\sum_{j=3}^{6} X_{j} \tag{2.8}
\end{equation*}
$$

where

$$
\begin{gathered}
R_{1}=\int_{\psi_{i \mid n}\left(\mathcal{R}_{\varepsilon}^{+}\right)}\left|\nabla u_{n}\right|^{2} d \xi d \eta, \quad R_{2}=\int_{\psi_{i \mid n}\left(\mathcal{R}_{\varepsilon}^{-}\right)}\left|\nabla u_{n}\right|^{2} d \xi d \eta \\
X_{j}=\int_{\psi_{i \mid n}\left(\mathcal{T}_{j, \varepsilon}\right)}\left|\nabla u_{n}\right|^{2} d \xi d \eta, \quad j=3,4,5,6
\end{gathered}
$$

where $\mathcal{R}_{\varepsilon}^{+}$is the rectangle of vertices $P_{1}, P_{2}, Q_{2}, Q_{1}, \mathcal{R}_{\varepsilon}^{-}$is the rectangle of vertices $Q_{4}, P_{4}, P_{3}, Q_{3}, \mathcal{T}_{j, \varepsilon}$ is the triangle of vertices $A, P_{j}, Q_{j}$ if $j=3,6$, or the triangle $P_{j}, Q_{j}, B$ if $j=4,5$.

We prove that

$$
\begin{equation*}
R_{i} \leq c c_{H}^{2} 4^{-n} \varepsilon, \quad i=1,2 \tag{2.9}
\end{equation*}
$$

where $c_{H}$ is the Hölder constant of $u$ in $\bar{\Omega}$, We do the proof for $R_{1}$, the proof for $R_{2}$ being analogous. By the "change of coordinates" $(\xi, \eta)=\psi_{i \mid n}(x, y)$, we get

$$
\begin{align*}
g(x, y):= & \left(u_{I_{n}} \circ \psi_{i \mid n}\right)(x, y)=u\left(\psi_{i \mid n}(A)\right)\left(1-x-\frac{y}{\sqrt{3}}\right) \\
& +u\left(\psi_{i \mid n}(B)\right)\left(x-\frac{y}{\sqrt{3}}\right)+u\left(\psi_{i \mid n}(C)\right) \frac{2}{\sqrt{3}} y \tag{2.10}
\end{align*}
$$

for all $(x, y) \in \mathcal{R}_{\varepsilon}^{+}$. As on $\mathcal{R}_{\varepsilon}^{+}$, we have $\widehat{y}_{+}(x)=\varepsilon / 2, \widetilde{y}_{+}(x)=\varepsilon, P_{+}=$ $(x, \varepsilon / 2), Q_{+}=(x, \varepsilon)$, by applying (2.1) to the function $g$, we obtain

$$
\begin{aligned}
g_{\varepsilon}(x, y)= & -\frac{2 y}{\varepsilon} \frac{\varepsilon}{\sqrt{3}}\left\{u\left(\psi_{i \mid n}(A)\right)-2 u\left(\psi_{i \mid n}(C)\right)+u\left(\psi_{i \mid n}(B)\right)\right\} \\
& +2\left\{u\left(\psi_{i \mid n}(A)\right)(1-x)+u\left(\psi_{i \mid n}(B)\right) x\right\}-\left\{u\left(\psi_{i \mid n}(A)\right)\left(1-x-\frac{\varepsilon}{\sqrt{3}}\right)\right.
\end{aligned}
$$

$$
\left.+u\left(\psi_{i \mid n}(B)\right)\left(x-\frac{\varepsilon}{\sqrt{3}}\right)+u\left(\psi_{i \mid n}(C)\right) \frac{2 \varepsilon}{\sqrt{3}}\right\} .
$$

Since $u \in C^{\delta}(\bar{\Omega})$,

$$
R_{1}=\int_{\mathcal{R}_{\varepsilon}^{+}}\left|\nabla_{x, y} g\right|^{2} d x d y \leq c c_{H}^{2}\left(1-\frac{2 \varepsilon}{c_{1}}\right) \varepsilon,
$$

and the proof of (2.9) is completed. Now we prove that

$$
\begin{equation*}
X_{j} \leq c c_{H}^{2} 4^{-n} \varepsilon, \quad j=3,4,5,6 . \tag{2.11}
\end{equation*}
$$

We only consider $X_{3}$, since the proof for the other terms $X_{j}, j=4,5,6$, is analogous. As in $\mathcal{T}_{3, \varepsilon}, \widehat{y}_{+}(x)=c_{1} x / 2, \widetilde{y}_{+}(x)=c_{1} x, P_{+}=\left(x, c_{1} x / 2\right), Q_{+}=$ $\left(x, c_{1} x\right)$, we have

$$
\begin{aligned}
g_{\varepsilon}(x, y)= & -\left\{u\left(\psi_{i \mid n}(A)\right)\left(1-x-\frac{c_{1} x}{\sqrt{3}}\right)+u\left(\psi_{i \mid n}(B)\right)\left(x-\frac{c_{1} x}{\sqrt{3}}\right)\right. \\
& \left.-u\left(\psi_{i \mid n}(C)\right) \frac{2 c_{1} x}{\sqrt{3}}\right\}-\frac{2 y}{c_{1} x \sqrt{3}}\left\{u\left(\psi_{i \mid n}(A)\right)-u\left(\psi_{i \mid n}(C)\right)+u\left(\psi_{i \mid n}(B)\right)\right. \\
& \left.-u\left(\psi_{i \mid n}(C)\right)\right\} c_{1} x+2\left\{u\left(\psi_{i \mid n}(A)\right)(1-x)+u\left(\psi_{i \mid n}(B)\right) x\right\} .
\end{aligned}
$$

Again, since $u \in C^{\delta}(\bar{\Omega})$, we find

$$
X_{3}=\int_{0}^{\frac{\varepsilon}{c_{1}}} \frac{1}{x} d x \int_{\frac{c_{1} x}{2}}^{c_{1} x}\left|\nabla g_{\varepsilon}\right|^{2} d y \leq c c_{H}^{2} 4^{-n} \varepsilon
$$

Thus (2.11) has been proved. By taking estimates (2.9) and (2.11) into account we get from (2.8)

$$
\int_{\Sigma_{2 \varepsilon}^{n} \backslash \Sigma_{\varepsilon}^{n}}\left|\nabla u_{n}\right|^{2} d x d y=\sum_{i \mid n} \int_{\Sigma_{2 \varepsilon}^{i \mid n} \backslash \Sigma_{\varepsilon}^{i l n}}\left|\nabla u_{n}\right|^{2} d x d y \leq c c_{H}^{2} \sum_{i \mid n} 4^{-n} \varepsilon \leq c c_{H}^{2} \varepsilon .
$$

and this proves (2.6).
In order to conclude the proof of Proposition 2.1, we have only to show that (2.7) holds. As in our previous calculation, we split the integral on $\Sigma_{\varepsilon}^{n}$ as the sum of the $4^{n}$ integrals on the sets $\Sigma_{\varepsilon}^{i \mid n}$. We decompose each $\Sigma_{\varepsilon}^{i \mid n}$ as the union of one rectangle and two triangles and we evaluate the corresponding integrals by making use, as before, of the coordinates change provided by the map $\psi_{i \mid n}$. Thus we write

$$
\frac{4^{n}}{3^{n}} \int_{\Sigma_{\varepsilon}^{i \mid n}}\left|\nabla u_{n}\right|^{2} w^{n} d \xi d \eta \equiv R_{0}+\sum_{j=1}^{2} X_{j}
$$

where

$$
R_{0}=\frac{4^{n}}{3^{n}} \int_{\psi_{i \mid n}\left(\mathcal{R}_{\varepsilon}\right)}\left|\nabla u_{n}\right|^{2} w^{n} d \xi d \eta, \quad X_{j}=\frac{4^{n}}{3^{n}} \int_{\psi_{i \mid n}\left(\mathcal{T}_{j, \varepsilon}\right)}\left|\nabla u_{n}\right|^{2} w^{n} d \xi d \eta, \quad j=1,2 .
$$

Here, $\mathcal{R}_{\varepsilon}$ is the rectangle of vertices $P_{1}, P_{2}, P_{3}, P_{4} ; \mathcal{T}_{1, \varepsilon}$ is the triangle of vertices $A, P_{1}, P_{4}$ and $\mathcal{T}_{2, \varepsilon}$ the triangle with vertices $B, P_{2}, P_{3}$. We note that for $(\xi, \eta) \in$ $\sum_{\varepsilon}^{i \mid n},(\xi, \eta)=\psi_{i \mid n}(x, y)$ and $w_{n}(\xi, \eta)=3^{n} \ell_{\varepsilon}^{-1}(x) c_{0}$, where

$$
\ell_{\varepsilon}(x)= \begin{cases}\varepsilon, & \frac{\varepsilon}{c_{1}}<x<1-\frac{\varepsilon}{c_{1}}, \\ \frac{2\left(c_{1}-c_{1} x\right)}{\left(2+c_{1}^{2}\right)}, & 1-\frac{\varepsilon}{c_{1}}<x<1, \\ \frac{2 c_{1} x}{\left(2+c_{1}^{2}\right)}, & 0<x<\frac{\varepsilon}{c_{1}} .\end{cases}
$$

Therefore, by taking (2.3) and (2.10) into account, we compute

$$
R_{0}=4^{n}\left(u\left(\psi_{i \mid n}(A)\right)-u\left(\psi_{i \mid n}(B)\right)\right)^{2} c_{0}\left(1-\frac{2 \varepsilon}{c_{1}}\right)
$$

and

$$
X_{1}=4^{n}\left(u\left(\psi_{i \mid n}(A)\right)-u\left(\psi_{i \mid n}(B)\right)\right)^{2} c_{0} \varepsilon \frac{2+c_{1}^{2}}{2 c_{1}}
$$

As $X_{2}$ is analogous, we conclude that

$$
\begin{align*}
& \frac{4^{n}}{3^{n}} \int_{\Sigma_{\varepsilon}^{(n)}}\left|\nabla u_{n}\right|^{2} w^{n} d \xi d \eta \\
= & \sum_{i \mid n} 4^{n}\left(u\left(\psi_{i \mid n}(A)\right)-u\left(\psi_{i \mid n}(B)\right)\right)^{2} c_{0}\left(1-\frac{2 \varepsilon}{c_{1}}+2 \varepsilon \frac{2+c_{1}^{2}}{2 c_{1}}\right) \\
= & c_{0} E[u]\left(1+\varepsilon c_{1}\right), \tag{2.12}
\end{align*}
$$

where, in the last identity, we have taken the definition of $E[u]$ into account (see (1.6)). This complete the proof of estimate (2.7), hence also that of Step 1.

As observed at the beginning of the proof, the functions $u_{n}$ belong to $H^{1}(\Omega)$ for all $n$. Now, as a consequence of estimate (2.12), we see that $u_{n} \in H^{1}\left(\Omega ; w^{n}\right)$ for every $n$.

Step 2. With this step, we remove the additional assumption in Step 1, namely, that $u \in D_{0}[F]$ belongs to $C^{\delta}(\bar{\Omega})$. Our proof relies on trace, extension and density results for functions in Sobolev and Besov spaces on so-called $d$-sets. For these results we refer to Jonsson [10], Jonsson and Wallin [12], Triebel [24] and, in relation to the present setting, also to [15], [18].

The main point of this part of the proof consists in proving the following approximation property

Proposition 2.2. For any function $u \in D_{0}[F]$, there exists a sequence of functions $\widehat{u}_{m} \in H_{0}^{1}(\Omega) \cap C^{\delta}(\bar{\Omega})$, with $\delta=d / 2$, which converge strongly to $u \in$ $H_{0}^{1}(\Omega)$ and are such that

$$
\widehat{u}_{\left.m\right|_{K}}=u_{\left.\right|_{K}} .
$$

Proof. Let $u \in D_{0}[F]$. Then, the trace $u_{\left.\right|_{K}}$ of $u$ on $K$ - which is defined quasieverywhere (q.e.) in the capacity sense - belongs to the space $\operatorname{Lip}_{d, 2, \infty}(K)$ introduced by Jonsson, see [10] and [17]. Therefore $u_{\left.\right|_{K}}$ admits an extension $\check{u}$ to $\mathbb{R}^{2}$ such that $\check{u} \in B_{1+d / 2}^{2, \infty}\left(\mathbb{R}^{2}\right)$, where $B_{1+d / 2}^{2, \infty}\left(\mathbb{R}^{2}\right)$ is the fractional Besov space
defined by Jonsson and Wallin, see [12]. By applying the imbedding properties of Besov spaces (see [12]), we then find that $\check{u} \in H^{1+d / 2-\varepsilon}\left(\mathbb{R}^{2}\right) \cap C^{\delta}\left(\mathbb{R}^{2}\right)$, where $\delta=d / 2$ and where $H^{1+d / 2-\varepsilon}\left(\mathbb{R}^{2}\right)$ is the usual fractional Sobolev space on $\mathbb{R}^{2}$. We now modify $\check{u}$ in order to obtain a function $\hat{u}$ such that

$$
\begin{equation*}
\hat{u} \in C^{\delta}(\bar{\Omega}) \cap H_{0}^{1}(\Omega) \tag{2.13}
\end{equation*}
$$

Let us consider $u^{*}=\varphi \check{u}$, where $\varphi$ is a suitable (smooth) cut-off-function, for instance, $\varphi=1$ on $\Omega \cap\{-\sqrt{3} / 6 \leq y \leq \sqrt{3} / 6\}$ and $\varphi=0$ on $\Omega \cap(\{y \geq$ $\left.\left.\frac{\sqrt{3}}{3}\right\} \cup\{y \leq-\sqrt{3} / 3\}\right)$. The function $u^{*}$ belongs to $H^{1+d / 2-\varepsilon}(\Omega) \cap C^{\delta}(\bar{\Omega})$ and $\left\|u^{*}\right\|_{H_{0}^{1}(\Omega) \cap C^{\delta}(\bar{\Omega})} \leq c\|u\|_{\mathcal{D}_{0}(K)}$. Moreover, $u^{*} \equiv 0$ on $\Omega \cap(\{y \geq \sqrt{3} / 3\} \cup\{y \leq$ $-\sqrt{3} / 3\})$.

We now consider in $\Omega$ the four (right) triangles $\mathcal{T}_{7}, \mathcal{T}_{8}, \mathcal{T}_{9}, \mathcal{T}_{10}$, where $\mathcal{T}_{7}$ has vertices $A, G, H ; \mathcal{T}_{8}$ has vertices $B, G, L ; \mathcal{T}_{9}$ and $\mathcal{T}_{10}$ are symmetric triangles with respect to the $x$-axis. Here $G=(1 / 2, \sqrt{3} / 3) ; H=(3 / 8,3 \sqrt{3} / 8) ; L=$ ( $5 / 8,3 \sqrt{3} / 8$ ).

Let us focus our attention on the triangle $\mathcal{T}_{7}$. By making a suitable, regular change of coordinates, with origin in $A$, and by calling $(s, t)$ the new coordinates, we can suppose that $\mathcal{I}_{7}$ admits the representation

$$
\left\{\begin{array}{l}
0 \leq s \leq \frac{3}{4} \\
0 \leq t \leq s \frac{1}{3 \sqrt{3}}
\end{array}\right.
$$

In this representation, we define the function $g_{7}$ in $\mathcal{T}_{7}$ by interpolating the value


Figure 4
0 and the value $u^{*}(s, s /(3 \sqrt{3}))$ on each "vertical" segment of $\mathcal{I}_{7}$, that is, we put

$$
g_{7}(s, t)=\frac{t 3 \sqrt{3}}{s} u^{*}\left(s, \frac{s}{3 \sqrt{3}}\right) .
$$

Let us note that

$$
g_{\left.7\right|_{A H}}=0=g_{\left.7\right|_{G H}}=u^{*}(G)=\check{u}(G) \varphi(G)
$$

The function $g_{7}$ inherits the Hölder continuity of $u^{*}$ (with the same exponent $\delta$ ), moreover, $g_{7}$ belongs to $H^{1}\left(\stackrel{\circ}{\mathcal{T}}_{7}\right)$, in fact,

$$
\int_{\mathcal{T}_{7}} g_{t}^{2} d s d t=\frac{27}{3 \sqrt{3}} \int_{0}^{\frac{3}{4}} \frac{1}{s^{2}} s u^{* 2}\left(s, \frac{s}{3 \sqrt{3}}\right) d s \leq c
$$

(note that $\left|u^{*}(s, s /(3 \sqrt{3}))\right| \leq c_{H} \cdot s^{\delta}$ ) and

$$
\int_{\mathcal{T}_{7}} g_{s}^{2} d s d t \leq c \int_{0}^{\frac{3}{4}}\left\{u^{* 2}\left(s, \frac{s}{3 \sqrt{3}}\right) \frac{1}{s}+s\left(u_{s}^{* 2}+\frac{1}{27} u_{t}^{* 2}\right)\right\} d s \leq c
$$

Note that $\nabla u^{*}$ belongs to $H^{d / 2-\varepsilon}\left(\mathbb{R}^{2}\right)$, hence the trace of $\nabla u^{*}$ on $A G$ belongs to $L^{2}(A G)$. Similar arguments hold for the other triangles.

We now define the function $\hat{u}$ by putting

$$
\hat{u}=\left\{\begin{array}{lll}
g_{j} & \text { in } & \mathcal{T}_{j}, \quad j=7,8,9,10  \tag{2.14}\\
u^{*} & \text { in } & \Omega \backslash \bigcup_{j=7}^{10} \mathcal{T}_{j} .
\end{array}\right.
$$

Let us note that, in particular, $\hat{u}-u \in H_{0}^{1}\left(\Omega^{k}\right)$ for $k=1,2$, with $\Omega=\Omega^{1} \cup$ $\Omega^{2} \cup \stackrel{\circ}{K}$. As $C^{\delta}\left(\bar{\Omega}^{k}\right) \cap H_{0}^{1}\left(\Omega^{k}\right)$ is dense in $H_{0}^{1}\left(\Omega^{k}\right)$, there exists a sequence of functions $\hat{u}_{m}^{k} \in C^{\delta}\left(\bar{\Omega}^{k}\right) \cap H_{0}^{1}\left(\Omega^{k}\right)$ that converges strongly to $u-\hat{u}$ in $H^{1}\left(\Omega^{k}\right)$ as $m \rightarrow+\infty$. We then define

$$
\hat{u}_{m}= \begin{cases}u, & K,  \tag{2.15}\\ \hat{u}_{m}^{k}+\hat{u}, & \Omega^{k}, \quad k=1,2,\end{cases}
$$

and the thesis follows easily from (2.13), (2.14), (2.15). The proof of Proposition 2.2 is now complete.

Step 3. We complete the proof of part (a) of the Theorem by making use of the "diagonal" formula of Corollary 1.16 of [1].

Proposition 2.3. For any function $u \in D_{0}[F]$, there exists a sequence of functions $\bar{u}_{n} \in H_{0}^{1}\left(\Omega ; w^{n}\right)$ such that

$$
\begin{equation*}
\bar{u}_{n} \rightarrow u \quad L^{2}(\Omega) \quad \text { and } \quad F[u] \leq \varlimsup_{n \rightarrow+\infty} F^{n}\left[\bar{u}_{n}\right] \tag{2.16}
\end{equation*}
$$

Proof. By (2.15), we have

$$
\begin{equation*}
E[u]=E\left[\hat{u}_{m}\right] . \tag{2.17}
\end{equation*}
$$

As in Step 1 (see Proposition 2.1), for any (fixed) $m$, we can start with $\hat{u}_{m}$, then we consider the "interpolate" $\left(\hat{u}_{m}\right)_{I_{n}}$ and, by the same procedure followed in that Proposition (see (2.3)), we define the function $\hat{u}_{m, n}$. Then we get

$$
\begin{equation*}
\varlimsup_{n \rightarrow+\infty} F^{n}\left[\hat{u}_{m, n}\right] \leq F\left[\hat{u}_{m}\right] . \tag{2.18}
\end{equation*}
$$

From (2.17), by using the fact that the functions $\hat{u}_{m}$ converge to $u$, we find

$$
\begin{aligned}
F[u] & =\lim _{m \rightarrow+\infty}\left\{\int_{\Omega}\left|\nabla \hat{u}_{m}\right|^{2} d x d y+c_{0} E\left[\hat{u}_{m}\right]\right\} \\
& \left.=\lim _{m \rightarrow+\infty} F\left[\hat{u}_{m}\right)\right] \geq \lim _{m \rightarrow+\infty}\left\{\varlimsup_{n \rightarrow+\infty} F^{n}\left(\left[\hat{u}_{m, n}\right]\right)\right\}
\end{aligned}
$$

and

$$
\lim _{m \rightarrow+\infty}\left(\lim _{n \rightarrow+\infty}\left\|\hat{u}_{m, n}-u\right\|_{H^{1}(\Omega)}\right)=0
$$

We now proceed by applying the diagonal formula of Corollary 1.16 in [1]. This shows that there exists a strictly increasing mapping $n \rightarrow m(n)$, with $\lim _{n \rightarrow+\infty} m(n)=+\infty$, such that, by denoting $\bar{u}_{n}=u_{m(n), n}$, we have

$$
\varlimsup_{n \rightarrow+\infty} F^{n}\left[\bar{u}_{n}\right] \leq F[u] .
$$

The proof of part (a) of the Theorem is now complete.

## 3. Proof of (b): the "lim inf" Condition

In this part of the proof of Theorem 1.1, we make use of the operator $\mathcal{M}_{\varepsilon}$ : $C^{1}\left(\bar{\Sigma}_{\varepsilon}\right) \rightarrow C^{1}\left(K_{0}\right)$ defined for all $0<\varepsilon \leq c_{1} / 2$ as follows. If $h \in C^{1}\left(\bar{\Sigma}_{\varepsilon}\right)$, we define the function $h_{\varepsilon}=\mathcal{M}_{\varepsilon}(h)$ by putting, for every $\bar{x} \equiv(\bar{x}, 0) \in K_{0}$,

$$
\begin{equation*}
h_{\varepsilon}(\bar{x})=\frac{1}{2\left|\hat{y}_{+}(\bar{x})\right|} \int_{\bar{\Sigma}_{\varepsilon} \cap\{x=\bar{x}\}} h(\bar{x}, y) d y, \tag{3.1}
\end{equation*}
$$

where, in the same notation as used in definition (2.1) of the function $g_{\varepsilon}, P_{+}=$ $\left(\bar{x}, \hat{y}_{+}(\bar{x})\right) \in \partial \Sigma_{\varepsilon}$ is the "upper" intersection of $\partial \Sigma_{\varepsilon}$ with the vertical line through the point $(\bar{x}, 0) \in K_{0}$. Clearly, $h_{\varepsilon} \in C^{1}\left(K_{0}\right)$.

Now, let $v_{n}$ be a sequence as in (b) of Definition 1.1. In order to prove inequality (1.8), it is not restrictive to assume that $\underline{\lim } F^{n}\left[v_{n}\right]<\infty$. Since the functionals $F^{n}$ are equicoercive on $H^{1}(\Omega)$, it is also not restrictive - by possibly extracting a subsequence of $v_{n}$, still denoted by $v_{n}$ below - to assume further that

$$
\begin{equation*}
v_{n} \rightarrow u \quad \text { in } L^{2}(\Omega) \text { strongly } \tag{3.2}
\end{equation*}
$$

as $n \rightarrow \infty$ (in this context see also Lemma 2.3 in [20]). Moreover, again up to extraction of a subsequence, we can also suppose that

$$
\left\{\begin{array}{l}
\left\|v_{n}\right\|_{H_{0}^{1}(\Omega)}^{2} \leq c^{*},  \tag{3.3}\\
\lim _{n \rightarrow+\infty} F^{n}\left[v_{n}\right]=c^{*}
\end{array}\right.
$$

for all $n$, with a constant $c^{*}$ independent of $n$. The proof of condition (b) of Definition 1.1 will take place in two steps.

Step 1. We now assume, in addition, that $v_{n} \in C_{0}^{1}(\Omega)$ for every $n$.
Proposition 3.1. Let $v_{n}$ belong to $C_{0}^{1}(\Omega)$. Then

$$
\begin{equation*}
F[u] \leq \underline{\lim } F^{n}\left[v_{n}\right] \tag{3.4}
\end{equation*}
$$

as $n \rightarrow \infty$.
Proof. If $(\xi, \eta) \in K^{n}$ then $(\xi, \eta)=\psi_{i \mid n}(x, 0)$ for some $n$-address $i \mid n$. We set for $(\xi, \eta) \in K_{0}^{i \mid n}$

$$
\begin{equation*}
\tilde{v}_{n}(\xi, \eta)=\mathcal{M}_{\varepsilon}\left(v_{n} \circ \psi_{i \mid n}\right) \circ \psi_{i \mid n}^{-1}(\xi, \eta), \tag{3.5}
\end{equation*}
$$

where $\mathcal{M}_{\varepsilon}$ is the operator defined above. If $\nabla_{\tau} \tilde{v}_{n}$ is the tangential derivative on the polygonal curve $K^{n}$, we have

$$
\begin{equation*}
E^{(n)}\left[\tilde{v}_{n}\right]=\sum_{i \mid n} 4^{n}\left(\tilde{v}_{n}\left(\psi_{i \mid n}(A)\right)-\tilde{v}_{n}\left(\psi_{i \mid n}(B)\right)\right)^{2} \leq \frac{4^{n}}{3^{n}} \sum_{i \mid n} \int_{K_{i \mid n}^{0}}\left|\nabla_{\tau} \tilde{v}_{n}\right|^{2} d s, \tag{3.6}
\end{equation*}
$$

where $E^{(n)}$ is the form defined in (1.6). By putting $h(x, y)=\left(v_{n} \circ \psi_{i \mid n}\right)(x, y)$ and by making use of change of the coordinates provided by the map $\psi_{i \mid n}$, we now show that

$$
\begin{equation*}
\frac{4^{n}}{3^{n}} \int_{K_{0}^{i \mid n}}\left|\nabla_{\tau} \tilde{v}_{n}\right|^{2} d s=4^{n} \int_{K_{0}}\left|\nabla_{x} h_{\varepsilon}(x)\right|^{2} d x \leq \frac{4^{n}}{3^{n}} \frac{1}{c_{0}} \int_{\Sigma_{\varepsilon}^{i \mid n}}\left|\nabla_{(\xi, \eta)} v_{n}\right|^{2} w^{n} d \xi d \eta \tag{3.7}
\end{equation*}
$$

In fact,

$$
\begin{gathered}
\int_{K_{0}}\left|\nabla_{x} h_{\varepsilon}(x)\right|^{2} d x=\int_{0}^{\frac{\varepsilon}{c_{1}}}\left(\frac{1}{c_{1} x} \int_{-\frac{c_{1} x}{2}}^{\frac{c_{1} x}{2}} h(x, y) d y\right)_{x}^{2} d x \\
+\int_{\frac{\varepsilon}{c_{1}}}^{1-\frac{\varepsilon}{c_{1}}}\left(\frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} h(x, y) d y\right)_{x}^{2} d x+\int_{1-\frac{\varepsilon}{c_{1}}}^{1}\left(\frac{1}{c_{1}(1-x)} \int_{\frac{c_{1} x-c_{1}}{2}}^{\frac{c_{1}-c_{1} x}{2}} h(x, y) d y\right)_{x}^{2} d x \\
\equiv I_{1}+I_{2}+I_{3} .
\end{gathered}
$$

We start by evaluating $I_{2}$. For $x \in\left[\varepsilon / c_{1}, 1-\varepsilon / c_{1}\right], \hat{y}_{+}(x)=\varepsilon / 2$. Then

$$
\begin{aligned}
4^{n} I_{2} & \leq 4^{n} \int_{\frac{\varepsilon}{c_{1}}}^{1-\frac{\varepsilon}{c_{1}}} \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} h_{x}^{2}(x, y) d y d x=\frac{4^{n}}{\varepsilon} \int_{\mathcal{R}_{\varepsilon}} h_{x}^{2}(x, y) d x d y \\
& \leq \frac{4^{n}}{3^{n} c_{0}} \int_{\psi_{i \mid n}\left(\mathcal{R}_{\varepsilon}\right)}\left|\nabla_{(\xi \cdot \eta)} v_{n}\right|^{2} w^{n} d \xi d \eta
\end{aligned}
$$

since, on $\psi_{i \mid n}\left(\mathcal{R}_{\varepsilon}\right)$, we have $w^{n}(\xi, \eta) \equiv 3^{n} c_{0} / \varepsilon$. Now we evaluate $I_{1}$ ( $I_{3}$ can be dealt with similarly). If $x \in\left(0, \varepsilon / c_{1}\right)$, then $\hat{y}(x)=c_{1} x / 2$. Therefore we obtain

$$
\begin{aligned}
4^{n} I_{1} & \leq 4^{n} \int_{0}^{\frac{\varepsilon}{c_{1}}} \frac{1+\frac{c_{1}^{2}}{2}}{c_{1} x} d x \int_{-\frac{c_{1} x}{2}}^{\frac{c_{1} x}{2}}\left\{h_{x}^{2}(x, y)+h_{y}^{2}(x, y)\right\} d y \\
& \leq \frac{4^{n}}{3^{n} c_{0}} \int_{\psi_{i \mid n}\left(\mathcal{T}_{1, \varepsilon}\right)}\left|\nabla_{(\xi, \eta)} v_{n}\right|^{2} w^{n} d \xi d \eta
\end{aligned}
$$

thus completing the proof of (3.7).

We now use the harmonic extension of $\left.\tilde{v}_{n}\right|_{V^{n}}$ obtained by decimation (see [14]). Here we follow [18] closely. We define the function

$$
H_{n+1} \tilde{v}_{n}: V^{n+1} \rightarrow \mathbb{R}
$$

as the unique minimizer on $V^{n+1}$ of the following problem

$$
\begin{equation*}
\min _{\substack{v: V^{n+1} \rightarrow \mathbb{R} \\ v=\tilde{v}_{n} \text { on } V^{n}}} E^{(n+1)}[v]=E^{(n)}\left[\tilde{v}_{n}\right] . \tag{3.8}
\end{equation*}
$$

For $m>n$, the function

$$
H_{m} \tilde{v}_{n}: V^{n} \rightarrow \mathbb{R}
$$

is defined as

$$
H_{m} \tilde{v}_{n}=H_{m}\left(H_{m-1}\left(H_{m-2} \ldots\left(H_{n+1} \tilde{v}_{n}\right)\right) .\right.
$$

We have

$$
\begin{equation*}
\left.H_{m} \tilde{v}_{n}\right|_{V^{n}}=\left.\tilde{v}_{n}\right|_{V^{n}} \tag{3.9}
\end{equation*}
$$

and

$$
\begin{equation*}
E^{(m)}\left[H_{m} \tilde{v}_{n}\right]=E^{(n)}\left[\tilde{v}_{n}\right] \leq \frac{c^{*}}{c_{0}} \tag{3.10}
\end{equation*}
$$

where the last inequality follows from (3.6), (3.7) and (3.3).
For every $n \in \mathbb{N}$, we define the function $H \tilde{v}_{n}$ on $V^{\infty}$ as follows. For $P \in V^{\infty}$, we choose $m>n$ such that $P \in V^{m}$ and we set

$$
\begin{equation*}
H \tilde{v}_{n}(P):=H_{m} \tilde{v}_{n}(P) . \tag{3.11}
\end{equation*}
$$

The right-hand side is independent of the choice of $m$. By taking (3.10) into account, for all $m>n$ we obtain $E\left[H \tilde{v}_{n}\right]=E^{(n)}\left[\tilde{v}_{n}\right] \leq c^{*} / c_{0}$, where $E$ is the form (1.6). The function $H v_{n}$ has a continuous extension on $K$, still denoted by $H v_{n}$, and we find

$$
\begin{equation*}
E\left[H \tilde{v}_{n}\right]=E^{(n)}\left[\tilde{v}_{n}\right] \leq \frac{c^{*}}{c_{0}} \tag{3.12}
\end{equation*}
$$

Moreover, $H \tilde{v}_{n} \in D_{0}(E)$. By (3.12), $\left\{H \tilde{v}_{n}\right\}_{n}$ is a bounded sequence in the Hilbert space $D_{0}[E]$. Therefore, there exists a subsequence, still denoted by $H \tilde{v}_{n}$, which converges weakly to a function $u^{*}$ in $D_{0}[E]$ and we obtain

$$
\begin{equation*}
E\left[u^{*}\right] \leq \underline{\lim } E\left[H \tilde{v}_{n}\right]=\underline{\lim } E^{(n)}\left[\tilde{v}_{n}\right] \leq \frac{c^{*}}{c_{0}} \tag{3.13}
\end{equation*}
$$

We now prove that

$$
\begin{equation*}
u^{*}=\left.u\right|_{K} \text { in } L^{2}(K, \mu) . \tag{3.14}
\end{equation*}
$$

By (3.3), there exists a subsequence of $v_{n}$ weakly converging in $H_{0}^{1}(\Omega)$ and hence strongly converging in $L^{2}(\Omega)$. By (3.2), the whole sequence $v_{n}$ weakly converges to $u$ in $H_{0}^{1}(\Omega)$ and

$$
\begin{equation*}
\int_{\Omega}|\nabla u|^{2} \mathrm{~d} x \mathrm{~d} y \leq \underline{\lim } \int_{\Omega \backslash \sum_{\varepsilon}^{n}}\left|\nabla v_{n}\right|^{2} \mathrm{~d} x \mathrm{~d} y . \tag{3.15}
\end{equation*}
$$

On the other hand, as $v_{n}$ is weakly convergent to $u$ in $H_{0}^{1}(\Omega)$, so it strongly converges to $u$ in $H^{s}(\Omega)$ for every $1-d / 2<s<1$ and, by the trace results
already mentioned ( see [12]), $\left.v_{n}\right|_{K}$ converges to $\left.u\right|_{K}$ strongly in $B_{s-1+d / 2}^{2,2}(K)$, and hence also strongly in $L^{2}\left(K, \mathcal{H}^{d}\right)$.
In order to prove (3.14), we introduce an auxiliary function $\hat{\hat{v}}$ and we prove that

$$
\hat{\hat{v}}_{\left.\right|_{K}}=u_{\left.\right|_{K}} \quad \text { and } \quad \hat{\hat{v}}_{\left.\right|_{K}}=u^{*} .
$$

The polygonal curve $K^{n}$ divides the domain $\Omega$ into two adjacent subdomains $\Omega_{n}^{1}$ and $\Omega_{n}^{2}$ and in each subdomain we define $\hat{v}_{n}^{k}$ as the (unique) solution in $H^{1}\left(\Omega_{n}^{k}\right)$ of the problem

$$
\begin{cases}\Delta \hat{v}_{n}^{k}=\Delta v_{n} & \text { in } \quad \Omega_{n}^{k} \\ \hat{v}_{n}^{k}=0 & \text { on } \partial \Omega_{n}^{k} \backslash K^{n} \\ \hat{v}_{n}^{k}=\tilde{v}_{n} & \text { on } \quad K^{n}\end{cases}
$$

and

$$
\hat{\hat{v}}_{n}= \begin{cases}\hat{v}_{n}^{k} & \text { in } \Omega_{n}^{k} \\ \tilde{v}_{n} & \text { in } K^{n}\end{cases}
$$

For the definition of the Sobolev spaces $H^{1}\left(K^{n}\right), H^{1 / 2}\left(K^{n}\right)$ used below, we refer, e.g., to [23]. We have $\hat{\hat{v}}_{n}-v_{n} \in H_{0}^{1}(\Omega)$ and

$$
\left\|\hat{v}_{n}\right\|_{H^{1}(\Omega)}^{2}=\sum_{k=1}^{2}\left\|\hat{v}_{n}^{k}\right\|_{H^{1}\left(\Omega_{n}^{k}\right)}^{2} \leq c\left\{\sum_{k=1}^{2}\left\|v_{n}\right\|_{H^{1}\left(\Omega_{n}^{k}\right)}^{2}+\left\|\tilde{v}_{n}\right\|_{H^{\frac{1}{2}}\left(K^{n}\right)}^{2}\right\} \leq c .
$$

Note that $\tilde{v}_{n} \in H^{1}\left(K^{n}\right)$, hence $\tilde{v}_{n}(A)=\tilde{v}_{n}(B)=0$. Moreover,

$$
\left\|\tilde{v}_{n}\right\|_{H^{\frac{1}{2}\left(K^{n}\right)}}^{2} \leq c\left\|\tilde{v}_{n}\right\|_{H^{1}\left(K^{n}\right)}^{2} \leq c^{*} c
$$

see (3.7), and

$$
\left\|v_{n}\right\|_{H^{\frac{1}{2}\left(K^{n}\right)}}^{2} \leq c\left\|v_{n}\right\|_{H^{1}(\Omega)}^{2} \leq c^{*} c
$$

see (3.3). Therefore, $\hat{\hat{v}}_{n}$ converges weakly in $H_{0}^{1}(\Omega)$ to a function $\hat{\hat{v}}$. We prove that $\hat{\hat{v}}=u$. Indeed,

$$
\begin{align*}
\left\|v_{n}-\hat{\hat{v}}_{n}\right\|_{L^{2}(\Omega)}^{2} & =\sum_{k=1}^{2}\left\|v_{n}-\hat{v}_{n}^{k}\right\|_{L^{2}\left(\Omega_{n}^{k}\right)}^{2} \\
& \leq \sum_{k=1}^{2}\left\|v_{n}-\hat{v}_{n}^{k}\right\|_{H^{\frac{1}{2}\left(\Omega_{n}^{k}\right)}}^{2} \leq 2 c\left\|v_{n}-\tilde{v}_{n}\right\|_{L^{2}\left(K^{n}\right)}^{2} . \tag{3.16}
\end{align*}
$$

The last inequality has taken into account that the following Poisson problem

$$
\left\{\begin{array}{lll}
\Delta g=0 & \text { in } & H^{\frac{1}{2}}\left(\Omega_{n}^{k}\right), \\
g=h & \text { in } & L^{2}\left(\partial \Omega_{n}^{k}\right)
\end{array}\right.
$$

has a unique solution in the Sobolev space $H^{1 / 2}\left(\Omega_{n}^{k}\right), k=1,2$, see [9]. Now we prove that

$$
\begin{equation*}
\left\|v_{n}-\tilde{v}_{n}\right\|_{L^{2}\left(K^{n}\right)} \rightarrow 0 \quad \text { as } \quad n \rightarrow+\infty \tag{3.17}
\end{equation*}
$$

In fact, by performing the same change of the coordinates as before, we find

$$
\begin{aligned}
\| v_{n}- & \tilde{v}_{n} \|_{L^{2}\left(K^{n}\right)}^{2}=\sum_{i \mid n} 3^{-n}\left\{\int_{0}^{\frac{\varepsilon}{c_{1}}}\left[\frac{1}{c_{1} x} \int_{-\frac{c_{1} x}{2}}^{\frac{c_{1} x}{2}}(h(x, y)-h(x, 0)) d y\right]^{2} d x\right. \\
& +\int_{\frac{\varepsilon}{c_{1}}}^{1-\frac{\varepsilon}{c_{1}}}\left[\frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}}(h(x, y)-h(x, 0)) d y\right]^{2} d x \\
& \left.+\int_{1-\frac{\varepsilon}{c_{1}}}^{1}\left[\frac{1}{c_{1}(1-x)} \int_{\frac{c_{1} x-c_{1}}{2}}^{\frac{c_{1}-c_{1} x}{2}}(h(x, y)-h(x, 0)) d y\right]^{2} d x\right\} \\
\leq & \sum_{i \mid n} 3^{-n}\left\{\int_{0}^{\frac{\varepsilon}{c_{1}}} \frac{d x}{c_{1} x} \cdot \frac{\left(c_{1} x\right)^{2}}{2} \int_{-\frac{c_{1} x}{2}}^{\frac{c_{1} x}{2}} h_{t}^{2}(x, t) d t+\int_{\frac{\varepsilon}{c_{1}}}^{1-\frac{\varepsilon}{c_{1}}} \frac{d x \varepsilon}{2} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} h_{t}^{2}(x, t) d t\right. \\
& \left.+\int_{1-\frac{\varepsilon}{c_{1}}}^{\frac{c_{1}-c_{1} x}{2}} \frac{\left(c_{1} x\right)^{2}}{2} \frac{d x}{c_{1}(1-x)} \int_{\frac{c_{1} x-c_{1}}{2}}^{c_{t}} h_{t}^{2}(x, t) d t\right\} \\
\leq & \frac{\varepsilon^{2}}{2} \frac{4^{n}}{3^{n}} \frac{1}{c_{0}} \int_{\Sigma_{\varepsilon}^{n}} w^{n}(\xi, \eta)\left|\nabla \nabla_{(\xi, \eta)} v_{n}\right|^{2} d \xi d \eta .
\end{aligned}
$$

By (3.3), since $\varepsilon=\varepsilon(n) \rightarrow 0$, the last term converges to zero as $n \rightarrow+\infty$. The proof of (3.17) is now complete. From (3.16), (3.17), since $v_{n}$ converges to $u$ weakly in $H_{0}^{1}(\Omega)$, we obtain that

$$
\left.\hat{\hat{v}}_{n}\right|_{K} \quad \text { converges to }\left.\quad u\right|_{K} \quad \text { in } \quad B_{s-1+d / 2}^{2,2}(K) \quad(s<1),
$$

hence

$$
\begin{equation*}
\left.\hat{\hat{v}}\right|_{K}=\left.u\right|_{K} . \tag{3.18}
\end{equation*}
$$

By the embedding of $D(E)$ into $C^{\delta}(K)$ (see Proposition 2.1 of [18]) and by the Ascoli-Arzelà theorem, we obtain that $H \tilde{v}_{n}$ converges to $u^{*}$ in the uniform norm, therefore in $L^{2}\left(K ; \mathcal{H}^{d}\right)$.

Let us show that

$$
\begin{equation*}
\lim _{h \rightarrow \infty}\left\|H \tilde{v}_{n}-\left.\hat{\hat{v}}_{n}\right|_{K}\right\|_{L^{2}\left(K, \mathcal{H}^{d}\right)}=0 \tag{3.19}
\end{equation*}
$$

Note that both sequences $H \tilde{v}_{n}$ and $\left.\hat{\hat{v}}_{n}\right|_{K}$ are uniformly bounded with the $L^{2}$ norm. As $H \tilde{v}_{n}$ and $\left.\hat{\hat{v}}_{n}\right|_{K}$ are continuous functions on $K$, we can evaluate the integral giving the $L^{2}$-norm in (3.19) as the limit of the corresponding Darboux sums. Therefore for every $\eta>0$ there exists $m=m(\eta)$, independent on $n$, such
that

$$
\begin{equation*}
\left\|H \tilde{v}_{n}-\left.\hat{\hat{v}}_{n}\right|_{K}\right\|_{L^{2}\left(K, \mathcal{H}^{d}\right)}^{2} \leq \eta+\sum_{j=1}^{4^{m}}\left|H \tilde{v}_{n}\left(R_{j}\right)-\hat{\hat{v}}_{n}\left(R_{j}\right)\right|^{2} \mathcal{H}^{d}\left(\tilde{K}_{j}\right) \tag{3.20}
\end{equation*}
$$

where the family $\tilde{K}_{j}$ is a decomposition of $K$ and $R_{j} \in \tilde{K}_{j}$. In particular, we can choose $R_{j} \in V^{n}, \tilde{K}_{j}=K \cap \bar{B}\left(R_{j}, 3^{-n}\right)$ and $\mu\left(\tilde{K}_{j}\right) \cong 3^{-m d}$. By (3.9) and (3.11), the sum in (3.20) vanishes for every $n \geq m$ and this proves (3.19). Therefore (3.14) follows from (3.18). By taking (3.14) into account we see that (3.13) gives

$$
\begin{equation*}
c_{0} E[u] \leq \underline{\lim } c_{0} E^{(n)}\left[\tilde{v}_{n}\right] \leq \frac{c^{*}}{c_{0}} . \tag{3.21}
\end{equation*}
$$

Together with (3.6) and (3.7), this inequality leads to

$$
\begin{equation*}
c_{0} E[u] \leq \lim _{n \rightarrow+\infty} \int_{\Sigma_{\varepsilon}^{n}} a^{n}\left|\nabla v_{n}\right|^{2} d x d y . \tag{3.22}
\end{equation*}
$$

Finally, from (3.22) and (3.15), we get

$$
\begin{aligned}
F[u]= & \int_{\Omega}|\nabla u|^{2} d x d y+c_{0} E[u] \leq \underset{n \rightarrow+\infty}{\lim _{\Omega \backslash \Sigma_{\varepsilon}^{n}} \int_{n}\left|\nabla v_{n}\right|^{2} d x d y} \\
& +\underline{\lim }_{n \rightarrow+\infty} \int_{\Sigma_{\varepsilon}^{n}} a^{n}\left|\nabla v_{n}\right|^{2} d x d y \leq \underline{\lim }_{n \rightarrow+\infty} F^{n}\left[v_{n}\right]
\end{aligned}
$$

and this concludes the proof of Proposition 3.1.

Step 2. We now remove the assumption that $v_{n} \in C_{0}^{1}(\Omega)$. Thus we actually prove the following

Proposition 3.2. In the preceding assumptions, for every sequence $v_{n}$ that converges strongly to a function $u$ in $L^{2}(\Omega)$ as $n \rightarrow \infty$, we have

$$
\begin{equation*}
F[u] \leq \underline{\lim } F^{n}\left[v_{n}\right] \tag{3.23}
\end{equation*}
$$

as $n \rightarrow \infty$.
Proof. We can assume that condition (3.3) holds and we set

$$
c^{*}=\lim _{n \rightarrow+\infty} F^{n}\left[v_{n}\right] .
$$

By the density of $C_{0}^{1}(\Omega)$ in the space $H_{0}^{1}\left(\Omega ; w^{n}\right)$, we find that for every $n$ there exists a function $v_{n}^{*} \in C_{0}^{1}(\Omega)$, such that

$$
\begin{gathered}
v_{n}^{*} \rightarrow v \quad \text { weakly in } \quad H_{0}^{1}(\Omega), \\
\lim F^{n}\left[v_{n}^{*}\right] \leq c^{*}, \\
\lim F^{n}\left[v_{n}^{*}-v_{n}\right]=0 .
\end{gathered}
$$

By proceeding as in Step 1 with respect to the functions $v_{n}^{*}$, we obtain

$$
\begin{aligned}
F[u] & \leq \lim F^{n}\left[v_{n}^{*}\right] \leq \lim \left\{F^{n}\left[v_{n}^{*}-v_{n}\right]+F^{n}\left[v_{n}\right]\right\} \\
& \leq \lim F^{n}\left[v_{n}^{*}-v_{n}\right]+\lim F^{n}\left[v_{n}\right] \leq \lim F^{n}\left(\left[v_{n}\right] .\right.
\end{aligned}
$$

Remark 3.1. As a by-product of the Proposition 3.1 and Proposition 3.2, we also obtain the following inequality

$$
\begin{equation*}
c_{0} E[u] \leq \underline{\lim } \int_{\Sigma_{\varepsilon}^{n}} a^{n}\left|\nabla v_{n}\right|^{2} d x d y \tag{3.24}
\end{equation*}
$$

for every sequence $v_{n}$ weakly convergent to $u$ in $H_{0}^{1}(\Omega)$.

## 4. Some Properties of Variational Solutions

The weights $w=w_{\varepsilon}^{n}$, defined in (1.1), (1.2) belong to the Muckenhoupt class $\mathcal{A}_{2}$ for any fixed $n$ and $\varepsilon$. Degenerate elliptic equations with weights in this class were thoroughly investigated by Fabes, Kenig and Serapioni in [5] as an extension of the classical theory of De Giorgi, Nash and Moser. They considered weak solutions of equations of the form

$$
\begin{equation*}
L u=\operatorname{div}(w \nabla u)=f \tag{4.1}
\end{equation*}
$$

and studied both the local and the global properties of these solutions. The main objective of their theory is to establish the Harnack principle for local weak solutions, which is known to imply that these solutions are Hölder continuous. Instrumental to this study are certain weighted Sobolev and Poincaré inequalities. Even if the scope of their theory is well beyond the special class of weights occurring in this paper, their results provide us with some useful analytic tools to apply to the special weighted operators considered in this paper.

As in Section 1, we fix $n \in \mathbb{N}$ and $0<\varepsilon \leq \varepsilon_{0} \leq c_{1} / 2$ and we consider the spaces $H^{1}\left(\Omega ; w_{\varepsilon}^{n}\right), H_{0}^{1}\left(\Omega ; w_{\varepsilon}^{n}\right)$ (see (1.3)) and the quadratic functionals $F_{\varepsilon}^{n}[u]$ defined in (1.4). By $F_{\varepsilon}^{n}(u, v)$ we now denote the bilinear forms with domain $H_{0}^{1}\left(\Omega ; w_{\varepsilon}^{n}\right)$, obtained from the functionals $F_{\varepsilon}^{n}$ by polarization. In the following, as in Theorem 1.1 and its proof, for every $n$ we choose $\varepsilon=\varepsilon(n)$, with $\varepsilon(n) \rightarrow 0$ as $n \rightarrow \infty$, and in the notation we suppress the subscript $\varepsilon=\varepsilon(n)$ by writing simply $H_{0}^{1}\left(\Omega ; w^{n}\right), F^{n}[u], F^{n}(u, v)$ and similar expressions.

By the Poincaré inequality in Theorem 1.3 of [5] we get
Proposition 4.1. For every $n$, the space $H_{0}^{1}\left(\Omega, w^{n}\right)$ is a Hilbert space under the norm

$$
\begin{equation*}
\|u\|_{H_{0}^{1}\left(\Omega ; w^{n}\right)}=\left(F^{n}[u]\right)^{\frac{1}{2}} . \tag{4.2}
\end{equation*}
$$

Moreover, the bilinear form $F^{n}(u, v)$, with domain $H_{0}^{1}\left(\Omega ; w^{n}\right)$, is a regular, strongly local Dirichlet form in $L^{2}(\Omega)$.

By the Lax-Milgram theorem we obtain

Corollary 4.1. Given $f \in L^{2}(\Omega)$, for every $n$, there exists a unique solution $u_{n} \in H_{0}^{1}\left(\Omega ; w^{n}\right)$ of the problem

$$
\begin{equation*}
F^{n}\left(u_{n}, v\right)=\int_{\Omega} f v d x d y \quad \text { for every } \quad v \in H_{0}^{1}\left(\Omega ; w^{n}\right) \tag{4.3}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
\left\|u_{n}\right\|_{H_{0}^{1}\left(\Omega ; w^{n}\right)} \leq c\|f\|_{L^{2}(\Omega)} \tag{4.4}
\end{equation*}
$$

with a constant $c$ independent of $n$.
The solution $u_{n}$ of (4.3) satisfies the variational principle

$$
\begin{equation*}
F^{n}\left[u_{n}\right]-2 \int_{\Omega} f u_{n} d x d y=\min _{v \in H_{0}^{1}\left(\Omega ; w^{n}\right)}\left\{F^{n}\left([v]-2 \int_{\Omega} f v d x d y\right\}\right. \tag{4.5}
\end{equation*}
$$

As already mentioned, the solutions $u_{n}$ are Hölder continuous in $\Omega$ by the results in [5]. Together with Theorem 8.15 in [7] and by analogous results for weighted operators in [2] and [4], these results provide us with the following regularity property of our weak solutions $u_{n}$.

Proposition 4.2. The solutions $u_{n}$ of problems (4.3) are continuous and uniformly bounded in $\bar{\Omega}$, that is

$$
\begin{equation*}
\max _{x \in \bar{\Omega}}\left|u_{n}(x)\right| \leq c \tag{4.6}
\end{equation*}
$$

with a constant c independent of $n$.
We now consider the quadratic functional $F[u]$ occurring in the statement of Theorem 1.1

$$
\begin{equation*}
F[u]=\int_{\Omega}|\nabla u|^{2} \mathrm{~d} x \mathrm{~d} y+c_{0} E[u] \tag{4.7}
\end{equation*}
$$

as defined on the domain

$$
\begin{equation*}
D_{0}[F]=\left\{u \in H_{0}^{1}(\Omega):\left.u\right|_{K} \in D_{0}[E]\right\} \tag{4.8}
\end{equation*}
$$

where it assumes finite values. Then, again by polarization, we associate with the functional $F[u]$ the bilinear form $F(u, v)$ with domain $D_{0}[F]$.

Proposition 4.3. The bilinear form $F(\cdot, \cdot)$ with domain $D_{0}[F]$ is a regular and strongly local Dirichlet form in $L^{2}(\Omega)$. The domain $D_{0}[F]$ is a Hilbert space with respect to the scalar product associated with the norm

$$
\begin{equation*}
\|u\|_{D_{0}[F]}=F[u]^{\frac{1}{2}} . \tag{4.9}
\end{equation*}
$$

In [11], Jonsson proves Proposition 4.3 when $K$ is the Sierpinski gasket. His proof can be adapted easily to the present case.

Corollary 4.2. For every $f \in L^{2}(\Omega)$, there exists a unique function $u \in$ $D_{0}[F]$ such that

$$
\begin{equation*}
F(u, v)=\int_{\Omega} f v d x d y \quad \text { for every } \quad v \in D_{0}([F]) \tag{4.10}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
\|u\|_{D_{0}[F]} \leq c\|f\|_{L^{2}(\Omega)} . \tag{4.11}
\end{equation*}
$$

Again, the solution $u$ in (4.10) obeys the variational principle

$$
\begin{equation*}
F[u]-2 \int f u d x d y=\min _{v \in D_{0}[F]}\left\{F[v]-2 \int_{\Omega} f v d x d y\right\} . \tag{4.12}
\end{equation*}
$$

Below, we may denote occasionally both the function $u$ and its trace $\left.u\right|_{K}$ on $K$ by the same symbol leaving interpretation according to the context.
The Koch curve $K$ divides the domain $\Omega$ into two adjacent subdomains, $\Omega^{1}$ and $\Omega^{2}$. The solution $u$ satisfies, formally, the conditions

$$
\left\{\begin{array}{llr}
-\Delta u^{k}=f & \text { in } \Omega^{k}, k=1,2, & \text { j) }  \tag{4.13}\\
c_{0} \Delta_{K} u=\left[\frac{\partial u}{\partial \nu}\right] & \text { in } K, u=0 \text { on } \partial K=\{A, B\}, & \text { jj) } \\
u=0 & \text { on } \partial \Omega,[u]=0 \operatorname{across} K . & \text { jjj) }
\end{array}\right.
$$

Here by $u^{k}$ we denote the restriction of $u$ to $\Omega^{k}$ and we use the notation $[\partial u / \partial \nu]=\partial u^{1} / \partial \nu_{1}+\partial u^{2} / \partial \nu_{2}$ for the jump of the normal derivative of $u$ across $K$, where $\partial u^{k} / \partial \nu_{k}$ denotes a normal derivative of $u^{k}$ with respect to the outward normal $\nu_{k}$ to $\Omega^{k}, k=1,2$.

Proposition 4.4. Let $u$ be the solution of problem (4.10). Then $u \in C^{0}(\bar{\Omega})$ and $u^{k} \in H_{l o c}^{2}\left(\Omega^{k}\right), k=1,2$. Moreover, the normal derivative $\frac{\partial u^{k}}{\partial \nu_{k}}$ belongs to the dual $\left(B_{\beta, 0}^{2,2}(K)\right)^{\prime}$ of the space $B_{\beta, 0}^{2,2}(K), \beta=d / 2$.

For the definition of the Besov spaces $B_{\beta}^{2,2}(K), B_{\beta, 0}^{2,2}(K)$ and more details on this result we refer to Lancia, [15], [16], where it is also proved that $D_{0}[K]$ is a subspace of $B_{\beta, 0}^{2,2}(K)$ (Proposition 6.1 in [15]). As a consequence of Proposition 4.4, we then get that the transmission condition ((4.13) jj$)$ ) holds in the dual space $\left(D_{0}(E)\right)^{\prime}$ of $D_{0}(E)$. Moreover, by the regularity property of the functions in $D_{0}[E]$, mentioned in Section 1, we also have $\left.u\right|_{K} \in C^{\delta}(K)$, where $\delta=d / 2$.

We conclude this section by giving some convergence properties of the solutions $u_{n}$ as $n \rightarrow \infty$.

Theorem 4.1. Let $u_{n}, n \in \mathbb{N}$, and $u$ be the solutions of problems (4.3) and (4.10). Then, we have
(i) $u_{n} \rightarrow u$ strongly in $H_{0}^{1}(\Omega)$,
(ii) $\left.\left.u_{n}\right|_{K} \rightarrow u\right|_{K}$ strongly in $B_{\frac{d}{2}}^{2,2}(K)$
as $n \rightarrow \infty$.
Proof. We first observe that from Theorem 1.1 we get the convergence of the minimum values of the variational problems solved by $u_{n}$ and $u$, respectively, that is

$$
\begin{equation*}
F^{n}\left[u_{n}\right] \rightarrow F[u] . \tag{4.14}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
u_{n} \text { converges weakly to } u \text { in } H_{0}^{1}(\Omega) . \tag{4.15}
\end{equation*}
$$

Since these properties are standard consequences of the $M$-convergence (and coerciveness) of functionals, we omit the proof and refer to [22].

We now prove that $u_{n}$ converges strongly to $u$ as $n \rightarrow \infty$, as stated in (i). From (4.14) (4.15) and the weak convergence of $u_{n}$ to $u$ in $H_{0}^{1}(\Omega)$ we get

$$
\begin{equation*}
\overline{\lim }\left\|u_{n}-u\right\|_{H_{0}^{1}(\Omega)}^{2}=E[u]-\underline{\lim } \frac{1}{c_{0}} \int_{\Sigma_{\varepsilon}^{n}} a^{n}\left|\nabla u_{n}\right|^{2} d x d y . \tag{4.16}
\end{equation*}
$$

We now apply Remark 3.1 to the weakly converging sequence $u_{n}$. From (4.16) and (3.24) we easily get that $u_{n}$ is strongly convergent to $u$ in $H_{0}^{1}(\Omega)$. Finally, we observe that, by the continuity properties of the trace operators on $K$, the strong convergence ( $i$ ) of the functions $u_{n}$ implies the strong convergence of their traces on $K$, namely, $\left.\left.u_{n}\right|_{K} \rightarrow u\right|_{K}$ in $B_{d / 2}^{2,2}(K)$.

Remark 4.1. We point out that the convergence in $(i)$ is a stronger property than the usual one we expect in the variational homogenization theory, where perturbed solutions, in general, only converge weakly in $H^{1}(\Omega)$. The additional information that leads to the strong convergence in our present setting is the estimate obtained in the proof of Theorem 1.1, stated in Remark 3.1.

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(Received 9.11.2006)
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