Georgian Mathematical Journal
Volume 14 (2007), Number 1, 195–202

SOLUTION OF A GOURSAT PROBLEM IN OPTIMAL DOMAINS

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Dedicated to the memory of Gaetano Fichera
on the occasion of his 85th birth anniversary

Abstract. In case a non-linear differential equation is reduced to a fixed-point problem, one has to apply a fixed-point theorem to a suitably chosen subset of the underlying function space. Generally speaking, in view of the non-linearity of the differential equation the restrictions for the data of the problem will be the more extensive the greater the chosen subset of the function space. The problem is to find an optimal subset leading to a domain of existence for the desired solution being as large as possible. The present paper will discuss this question for a Goursat problem.

2000 Mathematics Subject Classification: 35L70, 47J05, 47H10, 47N20.
Key words and phrases: Goursat problem, Schauder fixed-point theorem, optimal domain of existence.

1. Optimization of Fixed-Point Methods

Consider an operator $U = u_0 + \mathcal{A}\mathcal{F}u$ defined in a Banach space, where $\mathcal{A}$ is linear and bounded, while $\mathcal{F}$ is non-linear and only locally bounded. The latter assumption implies that fixed-point theorems can be applied only to bounded subsets of the Banach space, not to the whole space. The larger this subset the larger the bound for $\|\mathcal{F}u\|$. This, however, leads to a stronger restriction of $\|\mathcal{A}\|$. To optimize the fixed-point method means to find such subsets for which the restriction of $\|\mathcal{A}\|$ is as small as possible.

The simplest subsets for the application of fixed-point theorems are balls (centred at u_0). Optimal balls for the application of the contraction-mapping principle are constructed in the paper [1] which is contained in the collection [2] of papers. The paper [3] determines an optimal ball for the application of the Schauder fixed-point theorem. More general sets are polycylinders (because they depend on several parameters). S. Graubner's Thesis [4], his paper [5] in [2] and his paper [6] in [7] deal with the application of the contraction-mapping principle to optimal polycylinders.

In case the operator $\mathcal{A}$ means integration over a one-dimensional interval, the norm $\|\mathcal{A}\|$ is proportional to the length of this interval. Then optimization of the fixed-point method means to find the largest possible existence interval. So far this special case has been considered only under the assumption that the assumed estimates of $\mathcal{F}u$ do not depend on the length of the existence interval.

ISSN 1072-947X / \$8.00 /

The present paper optimizes the domain of existence for a Goursat problem in the x, y-plane in the case where the estimates of $\mathcal{F}$ are correlated with those of $\mathcal{A}$.

The optimization of fixed-point methods is a new branch in mathematical analysis. Its main goal is to find as weak as possible conditions for solving an operator equation by a fixed-point method. In the present paper this will be done for a Goursat problem.

2. Statement of the Problem

Consider the Goursat problem

$$\frac{\partial^2 u}{\partial x \partial y} = F(x, y, u), \tag{1}$$

$$u(x, 0) = \varphi(x), \tag{2}$$

$$u(0, y) = \psi(y), \tag{3}$$

where the right-hand side $F(x, y, u)$ is given for $|x| \geq 0$, $|y| \geq 0$ and for arbitrary values of u. Suppose the initial functions φ and ψ satisfy the compatibility condition

$$\varphi(0) = \psi(0).$$

Solutions of the Goursat problem are fixed points of the operator

$$U(x, y) = \varphi(x) + \psi(y) - \psi(0) + \int_0^x \int_0^y F\Big(\xi, \eta, u(\xi, \eta)\Big) d\xi d\eta \tag{4}$$

and vice versa.

3. Application of the Schauder Fixed-Point Theorem

To solve the Goursat problem (1)–(3), we look for fixed points $u(x, y)$ of the operator (4) belonging to $\mathcal{C}(M_{\varrho_1,\varrho_2})$, where M_{ϱ_1,ϱ_2} is the closed rectangle

$$M_{\varrho_1,\varrho_2} = \Big\{(x, y) : 0 \leq x \leq \varrho_1,\ 0 \leq y \leq \varrho_2\Big\}.$$

To estimate the operator, we assume

$$|F(x, y, u)| \leq \alpha(\varrho_1, \varrho_2) K(R)$$

provided $(x, y) \in M_{\varrho_1,\varrho_2}$ and $|u| \leq R$. Clearly, the functions $\alpha(\varrho_1, \varrho_2)$ and $K(R)$ are monotonically increasing in their variables. Suppose, additionally, that the initial functions φ and ψ are bounded:

$$|\varphi| \leq C_1 \quad \text{and} \quad |\psi(y) - \psi(0)| \leq C_2.$$

Then the operator maps the ball

$$\mathcal{B}_R = \Big\{u \in \mathcal{C}(M_{\varrho_1,\varrho_2}) : \|u\| \leq R\Big\}$$

into itself in case the relation

$$C + \alpha(\varrho_1, \varrho_2)\varrho_1\varrho_2 K(R) \leq R \tag{5}$$

is satisfied, where $C = C_1 + C_2$.

Provided that this condition (5) is satisfied, the images U of all $u \in \mathcal{B}_R$ belong to $\mathcal{B}_R$ too, that is, the images are uniformly bounded. Note, further, that $U(x,y) - U(x_*, y_*)$ can be written in the form

$$\Big(\varphi(x) - \varphi(x_*)\Big) + \Big(\psi(x) - \psi(x_*)\Big) + \int\limits_0^x \Big(\int\limits_{y_*}^{y} F d\eta\Big) d\xi + \int\limits_0^{y_*} \Big(\int\limits_{x_*}^{x} F d\xi\Big) d\eta.$$

The third and the fourth term can be estimated by

$$\alpha(\varrho_1,\varrho_2)K(R)|y - y_*|\varrho_1 \quad \text{and} \quad \alpha(\varrho_1,\varrho_2)K(R)|x - x_*|\varrho_2, \quad \text{resp.},$$

where (x_*, y_*) is an arbitrary point of M_{ϱ_1,ϱ_2}. Thus the images of all $u \in \mathcal{B}_R$ are everywhere equi-continuous. Consequently, the image of $\mathcal{B}_R$ turns out to be relatively compact in view of the Arzelà–Ascoli Theorem. Moreover, the uniform continuity of $F(x,y,u)$ implies the continuity of the operator (4): If $\delta > 0$ is sufficiently small and $\|\tilde{u} - \hat{u}\| < \delta$, then one has $|F(x,y,\tilde{u}) - F(x,y,\hat{u})| < \varepsilon$ and thus $\|\tilde{U} - \hat{U}\| \le \varepsilon \cdot \varrho_1\varrho_2$ for the corresponding images.

To sum up, the Schauder fixed-point theorem (see [8]) shows the existence of at least one solution of the Goursat Problem in M_{ϱ_1,ϱ_2} in case condition (5) is satisfied.

4. An Optimal Ball in the Banach Space

Since (5) can be rewritten as

$$\alpha(\varrho_1,\varrho_2)\varrho_1\varrho_2 \le \frac{R - C}{K(R)}\ , \tag{6}$$

the proposed monotonicity of $\alpha(\varrho_1,\varrho_2)$ implies that the area of M_{ϱ_1,ϱ_2} can be maximal only if the right-hand side of (6) is maximal. The right-hand side of (6) is called the *associated limit function* of the optimization problem. The existence of an optimal domain of existence depends on the behaviour of the associated limit function.

Provided that $K(R)$ is continuously differentiable, the associated limit function can be maximal only if R satisfies the relation

$$\frac{1}{R - C} = \frac{d}{dR}\ln K(R). \tag{7}$$

Note that (7) can be written as

$$K(R) = (R - C)K'(R).$$

Thus the auxiliary function

$$\psi(R) = K(R) - (R - C)K'(R)$$

is zero at each maximum point of the associated limit function. Since

$$\psi'(R) = -(R - C)K''(R),$$

ψ is monotonically decreasing if $K'' > 0$, and so ψ can have at most one zero. Therefore we have

Lemma 1. *The associated limit function cannot have more than one maximum point if* $K'' > 0$.

On the other hand, since $1/R - C$ is monotonically decreasing, the equation (7) has at least one solution $R_* > C$ in case condition

$$\lim_{R\to\infty} \frac{d}{dR} \ln K(R) > 0$$

is satisfied. At such an R_* the second derivative of the associated limit function equals

$$-\frac{R-C}{K^2} K'',$$

and so the associated limit function must be maximal if $K'' > 0$. Taking into account Lemma 1, the following statement has been proved:

Lemma 2. *In case conditions*

$$K'' > 0 \quad \text{and} \quad \lim_{R\to\infty} \frac{d}{dR} \ln K(R) > 0$$

are satisfied, there exists a uniquely determined $R_* > C$ *for which the associated limit function is maximal.*

Next we are going to compare different types of bounds $K(R)$:

a) $K(R)$ is said to be of *weak growth* if

$$K(R) = k_1 + k_2 R^\sigma \quad \text{with} \quad 0 < \sigma < 1$$

is satisfied ($k_1 \geq 0$, $k_2 > 0$). Then the associated limit function is monotonically increasing and tends to ∞ as $R \to \infty$.

b) $K(R)$ has *linear growth* if

$$K(R) = k_1 + k_2 R,$$

where $k_1 \geq 0$ and $k_2 > 0$. Here the associated limit function is again monotonically increasing, but it is bounded by $1/k_2$.

c) $K(R)$ has *polynomial growth* if

$$K(R) = k_1 + k_2 R^\sigma, \;\; k_1 \geq 0, \;\; k_2 > 0 \;\text{ and }\; \sigma > 1.$$

In this case equation (7) can be rewritten in the form

$$(\sigma - 1) k_2 R^\sigma - \sigma k_2 C R^{\sigma-1} = k_1. \tag{8}$$

Note that the associated limit functions tends to zero as $R \to C+0$ and also as $R \to \infty$. Therefore the associated limit function must have at least one maximum in (C, ∞). On the other hand, the left-hand side of (8) is monotonically increasing in R, and so (8) cannot have more than one solution. This implies that there exists a uniquely determined $R_* > C$ such that the associated limit function is maximal at R_*.

d) Finally, $K(R)$ has *exponential growth* if

$$K(R) = k_1 \exp(k_2 \sigma), \;\; k_1 > 0, \;\; k_2 > 0 \;\text{ and }\; \sigma > 0.$$

Here equation (7) reads as

$$R = C + \frac{1}{k_2\sigma} R^{1-\sigma}. \tag{9}$$

If $\sigma > 1$, then the right-hand side of (9) is decreasing in R and tends to C as $R \to \infty$. Since its left-hand side is increasing, equation (9) must have a uniquely determined solution $R > C$. If $\sigma = 1$, then we get the unique solution

$$R = C + \frac{1}{k_2\sigma}.$$

It remains to discuss the case $\sigma < 1$. Here we write (7) in the form

$$R - C = \frac{1}{k_2\sigma} R^{1-\sigma}. \tag{10}$$

Both sides of (10) tend to ∞ as $R \to \infty$, the right-hand side, however, more weakly than the left-hand one. Thus there must be at least one intersection point $R > C$ of the corresponding curves. While the derivative of the left-hand side of (10) equals 1 everywhere, the right-hand side has the derivative

$$\frac{1}{k_2} \cdot \frac{1-\sigma}{\sigma} \cdot \frac{1}{R^\sigma}. \tag{11}$$

In view of (9) this equals

$$(1-\sigma)\left(1 - \frac{C}{R}\right) < 1$$

at the intersection point of the left-hand and the right-hand side of (10). Further, the derivative of (11) is given by

$$-\frac{1}{k_2} \cdot (1-\sigma) \cdot \frac{1}{R^{1+\sigma}}\ ,$$

and thus the derivative of the right-hand side of (10) is always less than 1, at least at the points R which are larger than the above introduced intersection point. Since the left-hand side has derivative 1, another intersection point cannot exist. Consequently, in the case $0 < \sigma < 1$ too the existence of a unique maximum of the associated limit function has been proved.

By the way, the above statement concerning the exponential growth with $\sigma \geq 1$ can also be obtained from Lemma 2 because $K'' > 0$ and

$$\frac{d}{dR} \ln K(R) = +k_2\sigma R^{\sigma-1} \to \begin{cases} +\infty & \text{as } R \to +\infty \text{ if } \sigma > 1, \\ k_2 > 0 & \text{as } R \to +\infty \text{ if } \sigma = 1. \end{cases}$$

To sum up, the following theorem has been proved:

Theorem. *For equations with exponential or polynomial growth there exists a uniquely determined optimal radius R_* for which the associated limit function is maximal. If the growth is weak, the associated limit function is unbounded and takes each positive value. If the growth is linear, $K(R) = k_1 + k_2R$, then the associated limit function takes each value between 0 and $1/k_2$.*

5. The Largest Existence Domain

Let L_* be the largest value of the associated limit function. If the associated limit function does not possess a maximal value (as this is the case for weak and linear growth of the right-hand side of the differential equation (1)), then L_* may be any value of the associated limit function. To maximize the area $\varrho_1\varrho_2$ of M_{ϱ_1,ϱ_2} under the side condition

$$\alpha(\varrho_1,\varrho_2)\varrho_1\varrho_2 = L_*, \tag{12}$$

consider the Lagrange function $\varrho_1\varrho_2 + \lambda\Big(\alpha(\varrho_1,\varrho_2)\varrho_1\varrho_2 - L_*\Big)$. Equating its derivatives to zero, the following lemma has been proved:

Lemma 3. *If the area of M_{ϱ_1,ϱ_2} is maximal, then ϱ_1 and ϱ_2 satisfy the relation*

$$\frac{\partial\alpha}{\partial\varrho_1}\varrho_1 - \frac{\partial\alpha}{\partial\varrho_2}\varrho_2 = 0.$$

6. Examples

For $\alpha(\varrho_1,\varrho_2) = \varrho_1 + \varrho_2$ Lemma 3 leads to the relation

$$\varrho_1 = \varrho_2,$$

that is, the optimal rectangle is a square, and relation (12) yields

$$\varrho_1 = \left(\frac{L_*}{2}\right)^{\frac{1}{3}}.$$

If $\alpha(\varrho_1,\varrho_2) = \varrho_1$, then Lemma 3 implies $\varrho_1 = 0$. Hence the smaller ϱ_1, the larger the area of M_{ϱ_1,ϱ_2} is. Indeed, here equation (12) gives

$$\varrho_1\varrho_2 = \frac{L_*}{\varrho_1}.$$

Finally, one obtains

$$2\varrho_1^2 = \varrho_2$$

if $\alpha(\varrho_1,\varrho_2) = \varrho_1^2 + \varrho_2$. By virtue of (12) one has

$$\varrho_1 = \left(\frac{L_*}{6}\right)^{\frac{1}{5}}.$$

Supposing $L_* = 6$, an optimal area turns out to equal 2. In case the optimization is carried out only among squares, the relation (12) leads to

$$\varrho_1^4 + \varrho_1^3 = L_*.$$

Again, for $L_* = 6$ an approximate value is value $\varrho_1 = 1.3641$ and thus the area of an optimal square is equal to 1.8608. Of course, this value is smaller than the value 2 for the area of an optimal rectangle.

For $\alpha(\varrho_1,\varrho_2) = \varrho_1\varrho_2$ the equation (12) becomes $\varrho_1^2\varrho_2^2 = L_*$, whereas Lemma 3 does not lead to any relation between ϱ_1 and ϱ_2. Consequently, all rectangles M_{ϱ_1,ϱ_2} with $\varrho_1\varrho_2 = \sqrt{L_*}$ are optimal. In other words, in this case the optimum is not unique.

If the associated limit function does not have a maximal value, one has to take a value near its supremum in order to get an existence domain whose area is large.

7. Concluding Remark

S. Graubner's papers [4, 5, 6] and the papers [1, 3] are concentrated on the conditions under which the optimization problem is uniquely solvable. The above examples show that the solution of the optimization problem under consideration here is not always uniquely determined. Therefore the optimization problem of this paper is not included neither in the criteria of S. Graubner's papers [4, 5, 6] nor in those of [1, 3]. So it would be useful to generalize these earlier theories in order to include therein the optimization of the Goursat problem as well.

Acknowledgement

The author is grateful to the referee who read the manuscript very carefully. All of his useful comments have been taken into consideration.

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(Received 21.04.2006)

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