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THE GROWTH CONDITION GUARANTEEING SMALL SOLUTIONS FOR A LINEAR OSCILLATOR WITH AN INCREASING ELASTICITY COEFFICIENT

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Dedicated to Professor Ivan T. Kiguradze on his 70th birthday

Abstract. The second order linear differential equation

$$\ddot{x} + q^2(t)x = 0$$

is considered, where $q : [0, \infty) \to (0, \infty)$ is continuous, piecewise continuously differentiable, non-decreasing, and $\lim_{t\to\infty} q(t) = \infty$. A solution x_0 is called *small* if $\lim_{t\to\infty} x_0(t) = 0$. It is known that the equation always has at least one nontrivial small solution, but, in general, it can have also solutions not small. The Armellini–Tonelli–Sansone Theorem says that if the function q grows "regularly" in some sense, then all solutions are small. A generalization of this growth condition is given in terms only of the integral of q and $\dot{q}/q$. It is proved that the result is a real generalization of the earlier theorems.

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1. Introduction

Consider the linear second order differential equation

$$\ddot{x} + q^2(t)x = 0, \tag{1.1}$$

where $q : [0, \infty) \to (0, \infty)$ is continuous, piecewise continuously differentiable, nondecreasing, and satisfies

$$\lim_{t\to\infty} q(t) = \infty. \tag{1.2}$$

Equation (1.1) describes the oscillation generated by the restoring force $-q^2(t)x$ with elasticity coefficient $q^2(t)$ tending to infinity as $t \to \infty$. H. Milloux [11] proved that equation (1.1) always has a nontrivial *small solution*, i.e., a solution $x_0 : [0, \infty) \to \mathbb{R}$ possessing the property

$$\lim_{t\to\infty} x_0(t) = 0.$$

Milloux showed by an example that, in general, (1.2) does not imply that *all* solutions are small. The celebrated Armellini–Tonelli–Sansone Theorem [1, 15, 14] (abbreviated as A–T–S Theorem) gave a growth condition for the function q guaranteeing that all solutions of (1.1) are small. A nondecreasing function $f : [0, \infty) \to (0, \infty)$ is said to grow *intermittently* if for every $\varepsilon > 0$ there is a

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sequence $A = \{(a_n, b_n)\}_{n=1}^{\infty}$ of disjoint intervals such that $a_n \to \infty$ as $n \to \infty$, and

$$\limsup_{n\to\infty} \sum_{k=1}^{n} \frac{b_k - a_k}{b_n} \le \varepsilon, \qquad \sum_{n=1}^{\infty} (f(a_{n+1}) - f(b_n)) < \infty \tag{1.3}$$

are satisfied. If this does not occur, then f is said to grow *regularly.* The A–T–S Theorem says:

If the function $\ln q$ *grows to infinity regularly, then all solutions of (1.1) are small.*

After Milloux's theorem and A–T–S Theorem many results have appeared on the zero tending solutions of (1.1) (see the surveys in [2, 9]). I. T. Kiguradze and T. A. Chanturia [8] described the class of equations (1.1) for which the space of small solutions is of dimension one (see [9, Theorem 4.10]). The growth condition of A–T–S Theorem was generalized and sharpened in many papers (see, e.g., [3, 4, 5, 10, 12, 13]). Most of these conditions, similarly to (1.3), use different sequences of intervals that make them rather sophisticated. In this paper, applying our earlier method for the investigation of the stability of equilibrium of the damped oscillator, we give a new A–T–S-type growth condition using only the integral of functions q and $\dot{q}/q$. We compare our results to the classical A–T–S Theorem and to a recent condition of P. Pucci and J. Serrin [13].

2. The Main Result

Theorem 2.1. *Suppose that the function* $\dot{q}/q^2$ *is bounded on* $[0, \infty)$*, and for every sequence* $\{t_1, t_2, \ldots\}$ $(\lim_{n\to\infty} t_n = \infty)$ *the inequalities*

$$\pi \le \liminf_{n\to\infty} \int_{t_n}^{t_{n+1}} q, \qquad \int_{t_n}^{t_{n+1}} q \le 2\pi \ (n = 1, 2, \ldots) \tag{2.1}$$

imply

$$\sum_{n=1}^{\infty} \int_{t_n}^{t_{n+1}} \frac{\dot{q}(t)}{q(t)} \left[\min \left\{ \int_{t_n}^{t} q; \int_{t}^{t_{n+1}} q \right\} \right]^2 dt = \infty. \tag{2.2}$$

Then every solution of (1.1) *is small.*

In the proof we will use the basic theorem of [6] on the asymptotic stability of the zero solution of the damped oscillator

$$\ddot{x} + h(t)\dot{x} + k^2 x = 0, \tag{2.3}$$

where $h : [0, \infty) \to [0, \infty)$ is a bounded piecewise continuous function (damping coefficient), and k is a positive constant. This theorem needs the following concept. Let a constant $\alpha > 0$ be given. A discrete set $A = \{\tau_n\}$, $\tau_n < \tau_{n+1}$, $n = 1, 2, \ldots$, on the real line is called *asymptotically* α*-discrete* if

$$\liminf_{n\to\infty} (\tau_{n+1} - \tau_n) \ge \alpha.$$

Theorem 2.2. [6, Theorem 4.1] *If*

$$\int_0^\infty h(t)[\mathrm{dist}(t,A)]^2\,dt = \infty \tag{2.4}$$

for every asymptotically (π/k)*-discrete set* A*, then the zero solution of* (2.3) *is asymptotically stable, i.e., for every solution* x *of* (2.3)

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \dot{x}(t) = 0$$

is satisfied.

Proof of Theorem 2.1. A suitable transformation turns equation (1.1) into an equation of form (2.3). Indeed, introduce the new independent variable

$$\tau = \varphi(t) := \int_0^t q(s)\,ds.$$

Then equation (1.1) is transformed into the form

$$y'' + \frac{\dot{q}(\varphi^{-1}(\tau))}{q^2(\varphi^{-1}(\tau))}\,y' + y = 0, \tag{2.5}$$

where $(\cdot)^{\cdot} := d(\cdot)/dt$, $(\cdot)' := d(\cdot)d\tau$, and φ^{-1} denotes the inverse function of φ. Let us apply Theorem 2.2 to (2.5). Since

$$\int_{\tau_n}^{\tau_{n+1}} d\tau = \int_{\varphi^{-1}(\tau_n)}^{\varphi^{-1}(\tau_{n+1})} q(t)\,dt,$$

a sequence $\{\tau_n\}$ is α-discrete if and only if

$$\liminf_{n\to\infty} \int_{t_n}^{t_{n+1}} q(t)\,dt \geq \alpha \qquad (t_n := \varphi^{-1}(\tau_n)).$$

On the other hand, condition (2.4) has the form

$$\int_0^\infty \frac{\dot{q}\left(\varphi^{-1}(\tau)\right)}{q^2\left(\varphi^{-1}(\tau)\right)}\left[\mathrm{dist}\left(\tau,\{\tau_n\}\right)\right]^2 d\tau$$
$$= \sum_{n=1}^\infty \int_{t_n}^{t_{n+1}} \frac{\dot{q}(t)}{q(t)}\left[\min\left\{\int_{t_n}^{t} q;\ \int_{t}^{t_{n+1}} q\right\}\right]^2 dt = \infty. \tag{2.6}$$

It remains only to explain the appearance of the second condition in (2.1). In Theorem 2.2, divergence (2.4) is required for every asymptotically π/k-discrete set A. However, checking (2.4) for equation (2.5) we may restrict ourselves to the sets $A = \{\tau_n\}$ satisfying also

$$\tau_{n+1} - \tau_n = \int_{t_n}^{t_{n+1}} q(t)\,dt \le 2\pi \quad (n = 1, 2, \dots). \tag{2.7}$$

Indeed, if $\tau_{n+1} - \tau_n > 2\pi$ for some n in the sequence $A = \{\tau_n\}$, then we keep on adding points $\tau_{n_j} := \tau_n + j\pi$, $j = 1, 2, \dots$, to the set $A = \{\tau_n\}$ until the distance $\tau_{n+1} - \tau_{n_j}$ becomes less than or equal to 2π. Doing this for all such n's, we get a modification A' of A, which already satisfies both conditions in (2.1). Consequently, (2.2) has to hold for A'. However, the infinite sum (2.2) with the original A is greater than the same infinite sum with the modification A', so (2.2) automatically holds for A, too. This means that we can require the second condition in (2.1) without loss of the generality. □

Let us mention that the boundedness of $\dot q/q^2$ is not a strict assumption. Indeed, for any given $c > 0$, the measure of the set of t's where the inequality $\dot q(t)/q^2(t) \ge c > 0$ holds is finite; namely, it is at most $1/(q(0)c)$. Actually, this assumption could be removed by the use of Theorem 5.1 instead of Theorem 4.1 of [6] (see [7]), but this would make Theorem 2.2 technically difficult.

3. Applications

In this section we compare Theorem 2.1 to earlier results in the case with $\dot q/q^2$ bounded. For convenience, we formulate a corollary of our main result.

Corollary 3.1. *Suppose that the function $\dot q/q^2$ is bounded on $[0, \infty)$, and there is an ε $(0 < \varepsilon < 1)$ such that for every sequence $\{(a_n, b_n)\}_{n=1}^\infty$ $(\lim_{n\to\infty} a_n = \infty)$ of disjoint intervals with properties*

$$\int_{a_n}^{b_n} q = \varepsilon, \quad \pi - 2\varepsilon \le \int_{b_n}^{a_{n+1}} q \le 2\pi - 2\varepsilon, \qquad n = 1, 2, \dots, \tag{3.1}$$

the divergence

$$\sum_{n=1}^{\infty} [\ln q(a_{n+1}) - \ln q(b_n)] = \infty \tag{3.2}$$

holds.

Then every solution of (1.1) *is small.*

Proof. We prove that the condition of Theorem 2.1 is satisfied.

Let $\{t_n\}$ be a sequence satisfying (2.1). For every $n \in \mathbb{N}$ there are a_n, b_n such that

$$a_n < t_n < b_n < a_{n+1}, \quad \int_{a_n}^{t_n} q = \int_{t_n}^{b_n} q = \frac{\varepsilon}{2}, \quad n = 1, 2, \dots .$$

For n large enough we may assume $\int_{t_n}^{t_{n+1}} q \geq \pi - \varepsilon$, so we have

$$\int_{a_n}^{b_n} q = \varepsilon,$$

$$2\pi - 2\varepsilon \geq \int_{b_n}^{a_{n+1}} q = \int_{t_n}^{t_{n+1}} q - \int_{t_n}^{b_n} q - \int_{a_{n+1}}^{t_{n+1}} q \geq \pi - \varepsilon - \varepsilon = \pi - 2\varepsilon,$$

and

$$\int_{t_n}^{t_{n+1}} \frac{\dot{q}(t)}{q(t)} \left[\min \left\{ \int_{t_n}^{t} q; \int_{t}^{t_{n+1}} q \right\} \right]^2 dt$$

$$\geq \frac{\varepsilon^2}{4} \int_{b_n}^{a_{n+1}} \frac{\dot{q}(t)}{q(t)}\, dt = \frac{\varepsilon^2}{4} \left[\ln q(a_{n+1}) - \ln q(b_n) \right].$$

By the condition of the corollary, (3.2) implies (2.2), i.e., the condition of Theorem 2.1 is satisfied. □

Now we show that the A–T–S Theorem is a consequence of Corollary 3.1.

Proposition 3.2. *If* $q(t) \nearrow \infty$ $(t \to \infty)$ *regularly in the A–T–S sense* (*see Introduction*)*, then q satisfies condition* (3.1)–(3.2) *of Corollary* 3.1*.*

Proof. Suppose that q grows regularly, i.e., there exists an $\varepsilon_0 > 0$ such that for every sequence $\{(a_n, b_n)\}$ of disjoint intervals the relation

$$\limsup_{n\to\infty} \sum_{i=1}^{n} \frac{b_i - a_i}{b_n} \leq \varepsilon_0 \tag{3.3}$$

implies (3.2). Without loss of generality, we may suppose $\varepsilon_0 < 1$. We prove that condition (3.1)–(3.2) of Corollary 3.1 is satisfied with the same ε_0. Indeed, it is enough to prove that every sequence $\{(a_n, b_n)\}$ with properties (3.1) satisfies (3.3).

Let $\{(a_n, b_n)\}$ satisfy (3.1). Then

$$\begin{aligned} q(a_n)(b_n - a_n) &\leq \int_{a_n}^{b_n} q = \varepsilon_0, \\ q(b_{n-1})(a_n - b_{n-1}) &\leq \int_{b_{n-1}}^{a_n} q \leq q(a_n)(a_n - b_{n-1}), \end{aligned} \tag{3.4}$$

whence

$$b_n - a_n \leq \frac{\varepsilon_0}{q(a_n)}, \qquad \frac{\pi - 2\varepsilon_0}{q(a_n)} \leq a_n - b_{n-1} \leq \frac{2\pi - 2\varepsilon_0}{q(b_{n-1})}, \tag{3.5}$$

and, finally,

$$b_n - a_n \le \frac{\varepsilon_0}{\pi - 2\varepsilon_0}(a_n - b_{n-1}) \le \varepsilon_0(a_n - b_{n-1}), \quad n = 2, 3, \dots . \tag{3.6}$$

Now we can estimate the "density" of $\{(a_n, b_n)\}$ in the following way:

$$\sum_{i=1}^{n} \frac{b_i - a_i}{b_n} \le \frac{b_1 - a_1}{b_n} + \frac{\varepsilon_0}{b_n} \sum_{i=2}^{n} (a_i - b_{i-1}) \le \frac{b_1 - a_1}{b_n} + \varepsilon_0,$$

which means that (3.3) is satisfied. □

After the classical A–T–S Theorem let us turn to a recent result. As a corollary of their very general (nonlinear!) main theorem, P. Pucci and J. Serrin formulated the following nice result [13, Corollary A1]:

Suppose that there exists a nonnegative continuous function ψ of bounded variation on $[0, \infty)$, such that

$$\dot{q}(t) \ge \psi(t) q(t) \quad (t \ge 0), \qquad \int_0^\infty \psi = \infty. \tag{3.7}$$

Then every solution of (1.1) *is small.*

We show that this assertion is also a consequence of Corollary 3.1.

Proposition 3.3. *Condition* (3.7) *implies condition* (3.1)–(3.2).

Proof. Suppose that there is a function ψ satisfying the conditions in (3.7). Let ε $(0 < \varepsilon < 1)$ be arbitrary and consider a sequence $\{(a_n, b_n)\}$ with properties (3.1). We have to show that (3.2) holds. Since $\dot{q}/q \ge \psi$, for this it is sufficient to prove $\sum_{k=1}^\infty \int_{b_k}^{a_{k+1}} \psi = \infty$.

Introduce the notation $I_k := \int_{a_k}^{b_k} \psi$, $J_k := \int_{b_k}^{a_{k+1}} \psi$ $(k = 1, 2, \dots)$. Obviously, $\sum_{k=1}^\infty (I_k + J_k) = \infty$. Let us use the method of indirect proof: suppose

$$\sum_{k=1}^{\infty} J_k < \infty. \tag{3.8}$$

Then

$$\sum_{k=1}^{\infty} I_k = \infty. \tag{3.9}$$

We will show that these two relations contradict the fact that ψ is of bounded variation on $[0, \infty)$.

Using the notation

$$\mathbb{N}_+ := \{k \in \mathbb{N}, k \ge 2 : I_k - J_{k-1} \ge 0\}, \quad \mathbb{N}_- := \{k \in \mathbb{N}, k \ge 2 : I_k - J_{k-1} < 0\},$$

we have the estimate $\sum_{k \in \mathbb{N}_-} I_k \le \sum_{k \in \mathbb{N}_-} J_{k-1} < \infty$. Hence by (3.9)

$$\sum_{k \in \mathbb{N}_+} I_k = \infty. \tag{3.10}$$

Now we estimate the total variation $\vee_0^\infty \psi$ of ψ. For an arbitrary K $(2 < K \in \mathbb{N})$ we have

$$\vee_0^\infty \psi \geq \sum_{k=2}^{K} \left| \frac{I_k}{b_k - a_k} - \frac{J_{k-1}}{a_k - b_{k-1}} \right| \geq \sum_{k\in\mathbb{N}_+, k\leq K} \frac{1}{a_k - b_{k-1}} \left| \frac{a_k - b_{k-1}}{b_k - a_k} I_k - J_{k-1} \right|.$$

(3.5) and (3.6) imply the inequalities

$$\frac{1}{a_k - b_{k-1}} \geq \frac{q(b_{k-1})}{2(\pi - \varepsilon)}, \qquad \frac{a_k - b_{k-1}}{b_k - a_k} \geq \frac{\pi - 2\varepsilon}{\varepsilon} \geq 1 \quad (k \geq K_0) \tag{3.11}$$

with some $K_0 > 2$ large enough. From (3.8), (3.10) and the previous estimate, for $K > K_0$ we obtain

$$\begin{aligned} \vee_0^\infty \psi &\geq \sum_{k\in\mathbb{N}_+, K_0\leq k\leq K} \frac{q(b_{k-1})}{2(\pi-\varepsilon)} (I_k - J_{k-1}) \\ &\geq \frac{q(b_1)}{2\pi} \Big(\sum_{k\in\mathbb{N}_+, K_0\leq k\leq K} I_k - \sum_{k\in\mathbb{N}_+, K_0\leq k\leq K} J_{k-1} \Big) \to \infty \quad (K \to \infty), \end{aligned}$$

i.e., $\vee_0^\infty \psi = \infty$, which is a contradiction. □

In the next section we show by an example that neither the converse of Proposition 3.2 nor that of Proposition 3.3 is true; in other words, Corollary 3.1 is a real generalization both of the A–T–S Theorem and of the cited result of Pucci and Serrin.

4. Example

In this section we construct a function $q : [0, \infty) \to (0, \infty)$ which satisfies condition (3.1)–(3.2), but $\ln q$ grows to infinity intermittently, and q does not satisfy condition (3.7).

Let $T_n := (n-1)^2$, $n = 1, 2, \ldots$, and define successively

$$q(t) := \begin{cases} 1 & \text{if } t = 0, \\ q(T_n)e^{n^2(t-T_n)} & \text{if } T_n < t \leq T_n + 1/n^2, \\ q(T_n)e & \text{if } T_n + 1/n^2 < t \leq T_{n+1} \quad (n = 1, 2, \ldots). \end{cases}$$

Then $q(T_{n+1}) = eq(T_n)$; consequently, $q(T_{n+1}) = e^n \to \infty$ $(n \to \infty)$. It is easy to see that $\ln q$ grows to infinity intermittently. Indeed, consider the sequence of intervals $(a_n, b_n) = (T_n, T_n + 1/n^2)$, $n = 1, 2, \ldots$. Then

$$\limsup_{n\to\infty} \sum_{k=1}^{n} \frac{b_k - a_k}{b_n} = \limsup_{n\to\infty} \sum_{k=1}^{n} \frac{(T_k + \frac{1}{k^2}) - T_k}{T_n + \frac{1}{n^2}} \leq \limsup_{n\to\infty} \frac{n}{(n-1)^2} = 0.$$

At the same time,

$$\sum_{n=1}^{\infty} [\ln q(a_{n+1}) - \ln q(b_n)] = 0.$$

This means that for every $\varepsilon > 0$ (1.3) is satisfied by the same sequence of intervals $\{(a_n, b_n)\}$, i.e., $\ln q$ grows intermittently.

On the other hand,

$$\frac{\dot{q}(t)}{q(t)} = \begin{cases} n^2 & \text{if } T_n < t < T_n + 1/n^2, \\ 0 & \text{if } T_n + 1/n^2 < t < T_{n+1} \quad (n = 1, 2, \ldots). \end{cases}$$

If $\psi : [0, \infty) \to [0, \infty)$ is a function of bounded variation on $[0, \infty)$, then ψ is bounded on $[0, \infty)$. Let $\overline{\psi}$ denote an upper bound of ψ on $[0, \infty)$. If $\dot{q}(t) \geq \psi(t)q(t)$ $(t \geq 0)$, then $\psi(t) = 0$ for $t \in \bigcup_{n=1}^{\infty}(T_n + 1/n^2, T_{n+1})$; therefore,

$$\int_0^{\infty} \psi = \sum_{n=1}^{\infty} \int_{T_n}^{T_n + \frac{1}{n^2}} \psi \leq \overline{\psi} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

This means that there is no function ψ satisfying condition (3.7) of Pucci and Serrin.

Finally, we prove that condition (3.1)–(3.2) is satisfied for q. Let ε $(0 < \varepsilon < 1/e)$ be arbitrary, and choose a sequence $\{(a_n, b_n)\}$ having properties (3.1). Introduce the notation

$$\mathbb{N}_n := \left\{ k \in \mathbb{N} : T_n \leq a_k,\ b_{k+1} \leq T_n + \frac{1}{n^2} \right\}.$$

If $T_n < \alpha < \beta < T_{n+1}$, then

$$e^{n-1}(\beta - \alpha) \leq \int_{\alpha}^{\beta} q \leq e^n(\beta - \alpha).$$

From these inequalities and properties (3.1) we obtain the estimates

$$\frac{\varepsilon}{e^n} \leq b_k - a_k \leq \frac{\varepsilon}{e^{n-1}}; \quad \frac{\pi - \varepsilon}{e^n} \leq a_{k+1} - b_k \leq \frac{2\pi}{e^{n-1}} \qquad (k \in \mathbb{N}_n),$$

whence

$$a_{k+1} - b_k \geq \frac{\pi - \varepsilon}{e^n} = \frac{\pi - \varepsilon}{\varepsilon e} \frac{\varepsilon}{e^{n-1}} \geq \frac{\pi - \varepsilon}{\varepsilon \cdot e} (b_k - a_k) \geq 2(b_k - a_k) \quad (k \in \mathbb{N}_n).$$

These estimates show that for large n the interval $[T_n, T_n + 1/n^2]$ contains "many" elements of the sequence $\{(a_k, b_k)\}$, and

$$\text{measure}\Big\{ \bigcup_{k \in \mathbb{N}_n} (b_k, a_{k+1}) \Big\} \geq \frac{1}{2n^2}.$$

But $\dot{q}(t)/q(t) = n^2$ provided that $T_n < t < T_n + 1/n^2$; consequently,

$$\sum_{k \in \mathbb{N}_n} [\ln q(a_{k+1}) - \ln q(b_k)] \geq \sum_{k \in \mathbb{N}_n} \int_{b_k}^{a_{k+1}} \frac{\dot{q}(t)}{q(t)}\, dt \geq n^2 \frac{1}{2n^2} = \frac{1}{2},$$

which implies (3.2).

We can conclude that the function q defined by (4.1) satisfies condition (3.1)–(3.2). By Corollary 3.1, equation (1.1) with this q has only small solutions.

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