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DUCK TRAJECTORIES OF THREE-DIMENSIONAL SINGULARLY PERTURBED SYSTEMS

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Still I do not believe that my dear Vano is 70 ...

Abstract. For the singularly perturbed system of three equations with one fast variable and two slow ones the problem of the emergence of duck trajectories is considered in the case with two different slow motion trajectories intersecting in a general manner.

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The problem of the emergence of duck trajectories in the case with two different slow motion trajectories intersecting in a general manner was considered in [1] for a scalar singularly perturbed differential equation of first order, which is equivalent to a system of two equations with a fast variable and a slow one (for the notions and terms used here see [2], [3]). An analogous investigation is carried out below for the singularly perturbed system of three equations with one fast variable and two slow ones

$$\dot{x} = f(x, y), \quad \varepsilon\dot{y} = g(x, y), \quad x \in \mathbb{R}^2, \quad y \in \mathbb{R}, \quad 0 < \varepsilon \ll 1, \tag{1}$$

where the functions f and g are infinitely differentiable in the domain

$$\Omega = \Omega_x \times \Omega_y; \quad x \in \Omega_x,\ y \in \Omega_y.$$

Let us formulate four suppositions under which this system will be considered.

1. The equation $g(x, y) = 0$ has exactly two solutions $y = \varphi(x)$ and $y = \psi(x)$, where $\varphi, \psi \in C^\infty(\Omega_x)$, while the corresponding slow motion surfaces

$$\Gamma_1 = \{(x, y) : x \in \Omega_x,\ y = \varphi(x)\}, \quad \Gamma_2 = \{(x, y) : x \in \Omega_x,\ y = \psi(x)\} \tag{2}$$

intersect on some smooth curve l in a general manner, i.e., the inequality $\nabla\varphi(x) - \nabla\psi(x) \neq 0$ is fulfilled at each point of this curve; moreover, at the points of the curve l we have $g''_{yy} \neq 0$.

We introduce into consideration the parts

$$\Gamma_1^- = \{(x, y) \in \Gamma_1 : g'_y(x, y) < 0\}, \quad \Gamma_1^+ = \{(x, y) \in \Gamma_1 : g'_y(x, y) > 0\} \tag{3}$$

of the surface Γ_1, which are called its stable part and unstable one, respectively.

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2. The inequality $g'_y(x, y) \neq 0$ holds for $(x, y) \in \Gamma_1 \setminus l$ and each of sets (3) is nonempty.

Denote by l_0 the projection of the curve l onto the plane $y = 0$, and by Γ_0^- and Γ_0^+ the projections of the respective parts (3) of the surface Γ_1 onto the plane $y = 0$. Construct the system

$$\dot{x} = f(x, \varphi(x)), \tag{4}$$

whose trajectories are assumed also to lie on the plane $y = 0$.

3. Each trajectory of system (4) intersects the curve l_0 in a general manner in the direction from Γ_0^- to Γ_0^+.

Fix arbitrarily a point $x_0 \in l_0$ and denote by $x = x(t, x_0)$ a solution of system (4) with the initial condition $x|_{t=0} = x_0$. Let us introduce the degenerate system

$$\dot{x} = f(x, y), \quad g(x, y) = 0 \tag{5}$$

which corresponds to (1), and the family of its trajectories

$$(x(t), y(t)) : \ x = x(t, x_0), \quad y = \varphi(x(t, x_0)), \quad x_0 \in l_0, \tag{6}$$

lying on the surface Γ_1. All these trajectories are ducks, because when motion occurs along each of them, a phase point of system (5) moves from the stable segment of the surface Γ_1 to the unstable one.

Thus when conditions 1–3 are fulfilled, for $\varepsilon = 0$ system (1) has a one-parameter continuum of duck-trajectories (6). In this connection there naturally arises the following

Question: under what additional conditions is the segment $(x(t), y(t))$, $-t_1 \leq t \leq t_2$ of trajectory (6) of the degenerate system (5) with the preassigned values $x_0 \in l_0$, $t_j > 0$, $j = 1, 2$, the limit of some trajectory of the initial system (1) as $\varepsilon \to 0$?

To solve the problem posed we fix arbitrarily a point $x_0 \in l_0$ and a time interval $-t_1 \leq t \leq t_2$, $t_j > 0$, $j = 1, 2$, and denote by $x = x_0(t)$ the respective segment of the trajectory $x = x(t, x_0$ of system (4). (The numbers t_j, $j = 1, 2$, are chosen finite, i.e., not depending on ε, but not too large so that this trajectory segment would wholly lie in an admissible neighborhood of the curve l_0.) Furthermore, on the curve l_0 we give a smooth parametrization $l_0 = \{x = \gamma(s),\ a \leq s \leq b\}$ and introduce into consideration the function

$$\Phi(s) = (\nabla\varphi(x), f(x, \varphi(x))\big|_{x=\gamma(s)}. \tag{7}$$

Let to a chosen value of the parameter $x_0 \in l_0$ there correspond the value of the parameter $s_0 \in (a, b)$, i.e., $x_0 = \gamma(s_0)$.

4. $\Phi(s_0) = 0$, $\Phi'(s_0) \neq 0$.

We have to perform some additional constructions. Let $x^- = x_0(-t_1)$, $x^+ = x_0(t_2)$. In the three-dimensional space (x_1, x_2, y) we consider two curves l^- and l^+ whose projections l_0^- and l_0^+onto the plane $y = 0$ intersect in a general manner the trajectory $x - x_0(t)$ at the points x^- and x^+, respectively. Finally, we define y^- and y^+ from the conditions $(x^-, y^-) \in l^-$ and $(x^+, y^+) \in l^+$.

Let $y^-(\tau)$ and $y^+(\tau)$ be solutions in term of "fast" time $\tau = t/\varepsilon$ of the problems

$$dy/d\tau = g(x^-, y), \quad y(0) = y^-; \quad dy/d\tau = g(x^+, y), \quad y(0) = y^+. \tag{8}$$

Assume that

$$y^-(\tau) \to \varphi(x^-) \quad \text{as} \quad \tau \to +\infty; \quad y^+(\tau) \to \varphi(x^+) \quad \text{as} \quad \tau \to -\infty. \tag{9}$$

It should be emphasized that one can always succeed in fulfilling equalities (9) by choosing arbitrarily curves l^- and l^+. Indeed, by virtue of conditions 1 and 2, equations (8) are in the states of equilibrium $y = \varphi(x^-)$ and $y = \varphi(x^+)$, respectively, the first of which is exponentially stable as τ increases, while the second one as $\tau \to -\infty$. Therefore it is obvious that equalities (9) will be valid if y^- and y^+ are taken sufficiently close to $\varphi(x^-)$ and $\varphi(x^+)$, respectively.

Now let us define the curve L consisting of two segments parallel to the y-axis and connecting the points (x^-, y^-) and $(x^-, \varphi(x^-))$, (x^+, y^+) and $(x^+, \varphi(x^+))$, and also of the curve segment $(x_0(t), \varphi(x_0(t)))$ lying on Γ_1 for $-t_1 \le t \le t_2$.

Theorem. *If conditions* 1–4 *are fulfilled, then there exists a unique trajectory* $L(\varepsilon)$ *of system* (1) *such that its ends lie on* l^- *and* l^+*, respectively, while the curve* L *serves as its limit as* $\varepsilon \to 0$*.*

The trajectory $L(\varepsilon)$ of system (1) gives the answer to the question posed, since as $\varepsilon \to 0$ it infinitely approaches the segment of the curve L which lies between the points $(x^-, \varphi(x^-))$ and $(x_0, \varphi(x_0))$ on the stable part of Γ_1^-, as well as the segment of the curve L which lies on the unstable part of Γ_1^+. The trajectory possessing such properties is called the duck-trajectory of system (1).

Proof. We prove the theorem in several stages.

1) For $t > 0$ we consider the bundle of trajectories of system (1) whose initial conditions (x_*, y_*) lie, for $t = 0$, on the curve l^- in some sufficiently small neighborhood U of the point (x^-, y^-). Let us describe the behavior of each of these trajectories when t increases.

According to [4], first (on a segment of the form $0 \le t \le N\varepsilon \ln(1/\varepsilon)$, where $N = const > 0$) there occurs fast motion in an asymptotically small neighborhood of the straight line $x = x_*$. This motion is described by the Cauchy problem ($\tau = t/\varepsilon$)

$$dy/d\tau = g(x_*, y), \quad y(0) = y_*. \tag{10}$$

It should be emphasized that by virtue of the first equality from (9) and the proximity of (x_*, y_*) to (x^-, y^-), a solution $y(\tau)$ of problem (10) is close to a solution of the first of the Cauchy problems (8) and thus it is defined for all $\tau \ge 0$ and satisfies the limit equality

$$\lim_{\tau \to +\infty} y(\tau) = \varphi(x_*).$$

Therefore each of the considered trajectories "falls" on the stable manifold Γ_1^- and then the motion continues in its neighborhood of order ε approximately by the law $(x(t), y(t))$, $t \geq 0$, where $y(t) = \varphi(x(t))$ and $x(t)$ is a solution of the Cauchy problem

$$\dot{x} = f(x, \varphi(x)), \quad x|_{t=0} = x_*. \tag{11}$$

Moreover, by virtue of condition 3 and the proximity of x_* to x^-, after a certain time the solution of problem (11) intersects in a general manner the curve l_0 at some point close to x_0.

Thus the considered bundle of trajectories of system (1) forms the smooth (with respect to the set variables) integral manifold of slow motions ("ribbon") $y = S_1(x, \varepsilon)$, where $\varepsilon > 0$, $S_1(x, 0) = \varphi(x)$, and, which is the main thing, this tape continues to the intersection with any neighborhood od the point $(x_0, \varphi(x_0))$ that does not depend on ε or, speaking more exactly, to the intersection with the part of this neighborhood which is projected onto Γ_1^-.

Analogously, considering a small neighborhood $V \subset l^+$ of the point (x^+, y^+) and drawing – for $t < 0$ – the trajectories of system (1) from the points of the set V, we obtain the ribbon $y = S_2(x, \varepsilon)$, where $\varepsilon > 0$, $S_2(x, 0) = \varphi(x)$, which can also be continued to the intersection with the above-mentioned neighborhood of the point $(x_0, \varphi(x_0))$, namely, with its part projected onto Γ_1^+.

Thus the proof is reduced to the local problem of finding – in some small neighborhood of the above-mentioned point – a trajectory of system (1), whose ends lie on the ribbons $y = S_1(x, \varepsilon)$ and $y = S_2(x, \varepsilon)$.

2) The next stage of the proof consists in reducing the initial system (1) in a neighborhood of the point $(x_0, \varphi(x_0))$ to the normal form. Its construction in turn is divided into several steps.

2.1. In the first step, using conditions 1 and 2, we obtain the equality

$$g(x, y) = \omega(x, y)(y - \varphi(x))(y - \psi(x)),$$

where the multiplier $\omega(x, y) \in C^\infty(\Omega)$ is different from zero at all points of the surface Γ_1. Therefore without loss of generality the inequality $\omega(x, y) > 0$ can be assumed to be fulfilled (the case $\omega(x, y) < 0$ is reduced to this one by replacing y by $-y$). After successively performing in system (1) the replacements

$$\tau = \int_0^t \omega(x(s, \varepsilon), y(s, \varepsilon))ds, \quad \tau \to t, \quad f(x, y)/\omega(x, y) \to f(x, y), \quad z = y - \varphi(x),$$

we reduce it to the form

$$\begin{aligned} \dot{x} &= f(x, z + \varphi(x)), \\ \varepsilon \dot{z} &= (\varphi(x) - \psi(x))z + z^2 - \varepsilon(\nabla\varphi(x), f(x, z + \varphi(x))). \end{aligned} \tag{12}$$

To describe the next replacement of the variables we fix arbitrarily a point $x \in \Omega_x$ and draw from it the trajectories of system (4), i.e., we consider the solution $w(t, x)$ of this system with the initial condition $w(0, x) = x$. According to condition 3, there exists a unique moment of time $t_0 = t_0(x)$ (positive for $x \in \Gamma_0^-$, and negative for $x \in \Gamma_0^+$), for which $w(t_0, x) \in l_0$. Denote by $s = s(x)$ the value of the parameter s which corresponds to the point $w(t_0(x), x)$ and proceed, in system (4), to the new coordinates:

$$s(x) - s_0 = v, \quad t_0(x) = \xi, \tag{13}$$

where s_0 is the value of the parameter s corresponding to the chosen point $s_0 \in l_0$. As a result, the considered system is reduced to the form

$$\dot{\xi} = 1, \quad \dot{v} = 0, \tag{14}$$

while the solution $x = x_0(t)$ we are interested in is given in terms of the new variables by the equalities

$$\xi = t, \quad v = 0. \tag{15}$$

2.2. In the next step of the construction of the normal form we perform in system (12) the replacement (13). As a result, taking into account equalities (14), system (12) acquires the form

$$\begin{gathered} \dot{\xi} = 1 + z\gamma_1(\xi, v, z), \quad \dot{v} = z\gamma_2(\xi, v, z), \\ \varepsilon \dot{z} = \varkappa(\xi, v)z + z^2 + \varepsilon\Delta(\xi, v, z), \end{gathered} \tag{16}$$

where γ_1, and γ_2 are some sufficiently smooth functions, while the functions $\varkappa(\xi, v)$ and $\Delta(\xi, v, z)$ are obtained from

$$\varphi(x) - \psi(x) \quad \text{and} \quad -(\nabla\varphi(x), f(x, z + \varphi(x))),$$

respectively, by the above-mentioned replacement of the coordinates. Note that since system (16) is considered in some sufficiently small neighborhood of the origin, by virtue of the first equality (16) the variable ξ can be assumed to be the new time.Then we obtain the system

$$\varepsilon dz/d\xi = \varkappa(\xi, v)z + H(\xi, v, z) + \varepsilon\Delta_1(\xi, v, z), \quad dv/d\xi = z\Delta_2(\xi, v, z), \tag{17}$$

where the smooth function H is such that

$$H(\xi, v, 0) \equiv H'_z(\xi, v, 0) \equiv 0, \tag{18}$$

while Δ_1 and Δ_2 are given by the equalities

$$\Delta_1 = \Delta(\xi, v, z)/(1 + z\gamma_1(\xi, v, z)), \quad \Delta_2 = \gamma_2(\xi, v, z)/(1 + z\gamma_1(\xi, v, z)). \tag{19}$$

2.3. Prior to performing the third step of normalization let us recall that in reducing the problem to the local one we considered the smooth surfaces (ribbons) $y = S_j(x, \varepsilon)$, $j = 1, 2$, formed by the trajectories of system (1). In the considered neighborhood of the point $(x_0, \varphi(x_0))$ these surfaces are at a distance of order ε from the surface Γ_1, i.e., in terms of the new coordinates, from the surface $z = 0$. Hence it is advisable first to perform the replacement $z/\varepsilon \to z$ in system (17) in order to reduce, taking into account properties (18), the function H to the form

$$\varepsilon dz/d\xi = \varkappa(\xi, v)z + \Delta_3(\xi, v) + \varepsilon\Delta_4(\xi, v, z, \varepsilon), \quad dv/d\xi = \varepsilon\Delta_5(\xi, v, z, \varepsilon). \tag{20}$$

Here Δ_j, $j = 3, 4, 5$, are some sufficiently smooth (with respect to the set of variables) functions, and by virtue of (19)

$$\Delta_3(\xi, v) = \Delta(\xi, v, 0). \tag{21}$$

Next, we observe that by virtue of conditions 1, 2

$$\varkappa(0, v) = (\varphi(x) - \psi(x))\big|_{x \in l_0} \equiv 0,$$

$$\varkappa'_\xi(0,v) = (\nabla\varphi(x) - \nabla\psi(x), f(x,\varphi(x)))\big|_{x\in l_0} > 0.$$

Therefore we can perform a smooth replacement such that the following equality is fulfilled:

$$d\tilde{\xi}/d\xi = \varkappa(\xi,v)/\tilde{\xi}$$

(the variable v is considered here as a parameter). Indeed, solving this differential equation, we obtain a smooth solution

$$\tilde{\xi} = \xi\sqrt{(2/\xi^2)\int_0^\xi \varkappa(s,v)ds}. \tag{22}$$

Thus, in the third (last) step of normalization we perform in system (20) replacement (22) and then again denote by ξ the variable $\tilde{\xi}$. As a result, we obtain the desired normal form of system (1) which is written as

$$\varepsilon dz/d\xi = \xi z + \Omega(\xi,v) + \varepsilon\Theta_1(\xi,v,z,\varepsilon), \quad dv/d\xi = \varepsilon\Theta_2(\xi,v,z,\varepsilon). \tag{23}$$

Here all functions depend smoothly enough on its variables, and by virtue of (19), (21) the following equality is valid:

$$\Omega(0,v) = -\Phi(v+s_0)\Big/\left(\sqrt{\varkappa'_\xi(0,v)}\,\omega(x,\varphi(x))|_{x=\gamma(s_0+v)}\right),$$

where $\Phi(s)$ is function (7). From this and condition 4 it follows that

$$\Omega(0,0) = 0, \quad \Omega'_v(0,0) \neq 0. \tag{24}$$

3) In the final stage of the proof of the theorem we fix a sufficiently small number $q > 0$ and, for $-q \le \xi \le q$, consider for system (23) the boundary value problem with the boundary conditions

$$z\big|_{\xi=-q} = \Sigma_1(\xi,v,\varepsilon)\big|_{\xi=-q}, \quad z\big|_{\xi=q} = \Sigma_2(\xi,v,\varepsilon)\big|_{\xi=q}, \tag{25}$$

where the smooth (with respect to the set of variables) functions Σ_j, $\Sigma_j|_{\varepsilon=0} = -\Omega(\xi,v)/\xi$, $j = 1,2$, defined in some sufficiently small neighborhoods of the points $\xi = -q$, $v = 0$ and $\xi = q$, $v = 0$, respectively, are actually the above-introduced integral manifolds (ribbons) $y = S_j(x,\varepsilon)$, $j = 1,2$, in terms of the new variables.

By virtue of equalities (15) we are interested in the solution of the boundary value problem (23) (25) (if such a solution exists) with zero approximation

$$v = 0, \quad z = z_0(\xi) \equiv -\Omega(\xi,0)/\xi, \tag{26}$$

where $z_0(\xi) \in C^\infty[-q,q]$ (see the first relation (24)).

To complete the proof of the theorem, we use the following lemma.

Lemma. *For all sufficiently small $\varepsilon > 0$, the boundary value problem* (23), (25) *has a unique solution $v = v(\xi,\varepsilon)$, $z = z(\xi,\varepsilon)$ with zero approximation* (26), *for which the asymptotic representations*

$$v = \sum_{k=1}^\infty \varepsilon^k v_k(\xi), \quad z = z_0(\xi) + \sum_{k=1}^\infty \varepsilon^k z_k(\xi) \tag{27}$$

hold uniformly with respect to $-q \le \xi \le q$.

Proof. First we have to present the algorithm of defining the coefficients of series (27). For this, these series are substituted into system (23) and equate successively the coefficients at the equal degrees ε.

In the first step of the algorithm we come to the system

$$dv_1/d\xi = \Theta_2(\xi, 0, z_0(\xi), 0), \quad \xi z_1 = z_0'(\xi) - \Omega_v'(\xi, 0)v_1 - \Theta_1(\xi, 0, z_0(\xi), 0), \tag{28}$$

which is simple to analyze. Indeed, from the first equation (28) the function $v_1(\xi)$ is defined to within a constant that we choose so that the function $z_1(\xi)$ be smooth in the neighborhood of the point $\xi = 0$, Namely, we assume that

$$v_1(0) = (z_0'(0) - \Theta_1(0, 0, z_0(0), 0))/\Omega_v'(0, 0),$$

which can be done by virtue of the second relation (24).

In the algorithm step with a number $k \geq 2$ we obtain a system analogous to (28)

$$dv_k/d\xi = R_k(\xi), \quad \xi z_k = G_k(\xi) - \Omega_v'(\xi, 0)v_k,$$

where the functions $R_k, G_k \in C^\infty[-q, q]$ depend only on $v_m(\xi)$ and $z_m(\xi)$ with numbers $m \leq k-1$. Solving then the equation for v_k and choosing the initial condition $v_k(0) = G_k(0)/\Omega_v'(0, 0)$, we obtain the smooth function $z_k(\xi)$ in the neighborhood of the point $\xi = 0$. We have thus described the algorithm.

To justify the algorithm we need to define the following two special families of solutions of system (23), each of which satisfies one of the boundary conditions (25). Let us fix an arbitrary natural number $k_0 \geq 2$ and substitute into (23) the expressions

$$v = v_{k_0}^j(\xi, \varepsilon) + h_j, \quad z = z_{k_0}(\xi, \varepsilon) + g_j, \quad j = 1, 2, \tag{29}$$

where

$$v_{k_0}^j(\xi, \varepsilon) = \sum_{k=1}^{k_0-1} \varepsilon^k v_k(\xi) + \varepsilon^{k_0}\delta_j, \quad z_{k_0}(\xi, \varepsilon) = z_0(\xi) + \sum_{k=1}^{k_0-1} \varepsilon^k z_k(\xi)$$

and δ_j, $j = 1, 2$, are for the time being arbitrary constants (to be defined later). As a result, for $h_j = h_j(\xi, \varepsilon, \delta_j)$, $g_j = g_j(\xi, \varepsilon, \delta_j)$, $j = 1, 2$, we obtain some systems of differential equations which we supplement with the initial conditions

$$h_j\big|_{\xi=(-1)^j q} = 0, \quad g_j\big|_{\xi=(-1)^j q} = \big(\Sigma_j(\xi, v_{k_0}^j(\xi, \varepsilon), \varepsilon) - z_{k_0}(\xi, \varepsilon)\big)\big|_{\xi=(-1)^j q}. \tag{30}$$

When $j = 1$, to define the functions $h_1(\xi, \varepsilon, \delta_1)$, $g_1(\xi, \varepsilon, \delta_1)$ the above reasoning brings us to the system

$$\varepsilon dg_1/d\xi = a(\xi, \varepsilon, \delta_1)g_1 + b(\xi, \varepsilon, \delta_1)h_1 + C(g_1, h_1, \xi, \varepsilon, \delta_1) + \varepsilon^{k_0}u_*(\xi, \varepsilon, \delta_1),$$
$$dh_1/d\xi = \varepsilon D(g_1, h_1, \xi, \varepsilon, \delta_1) + \varepsilon^{k_0}u_{**}(\xi, \varepsilon, \delta_1),$$

where all coefficients smoothly depend on their variables and also

$$\begin{aligned} a(\xi, \varepsilon, \delta_1) = \xi + O(\varepsilon), \quad b(\xi, \varepsilon, \delta_1) = \Omega_v'(\xi, 0) + O(\varepsilon), \\ C(0, 0, \xi, \varepsilon, \delta_1) \equiv C_{g_1}'(0, 0, \xi, \varepsilon, \delta_1) \equiv C_{h_1}'(0, 0, \xi, \varepsilon, \delta_1) \\ \equiv D(0, 0, \xi, \varepsilon, \delta_1) \equiv 0. \end{aligned} \tag{31}$$

We supplement the obtained system with the initial conditions (30) for $j = 1$ and proceed to the system of integral equations

$$g_1 = g_1(-q, \varepsilon, \delta_1) \exp\left\{\frac{1}{\varepsilon}\int_{-q}^{\xi} a(s, \varepsilon, \delta_1) ds\right\}$$

$$+\frac{1}{\varepsilon}\int_{-q}^{\xi} \exp\left\{\frac{1}{\varepsilon}\int_{\tau}^{\xi} a(s, \varepsilon, \delta_1) ds\right\}\Big[b(\tau, \varepsilon, \delta_1)\Big(\varepsilon\int_{-q}^{\tau} D(g_1, h_1, \sigma, \varepsilon, \delta_1) d\sigma$$

$$+\varepsilon^{k_0}\int_{-q}^{\tau} u_{**}(\sigma, \varepsilon, \delta_1) d\sigma\Big) + C(g_1, h_1, \tau, \varepsilon, \delta_1) + \varepsilon^{k_0} u_*(\tau, \varepsilon\delta_1)\Big] d\tau, \tag{32}$$

$$h_1 = \varepsilon\int_{-q}^{\xi} D(g_1, h_1, s, \varepsilon, \delta_1) ds + \varepsilon^{k_0}\int_{-q}^{\xi} u_{**}(s, \varepsilon, \delta_1) ds. \tag{33}$$

The investigation of system (32), (33) rests on the asymptotic equality

$$g_1(-q, \varepsilon, \delta_1) = O(\varepsilon^{k_0})$$

which is fulfilled by virtue of the above-described technique of defining the coefficients of series (27), also on the estimate

$$\max_{-q\le\xi\le 0} \frac{1}{\varepsilon}\int_{-q}^{\xi} \exp\left\{\frac{1}{\varepsilon}\int_{\tau}^{\xi} a(s, \varepsilon, \delta_1) ds\right\} d\tau \le \frac{M}{\sqrt{\varepsilon}}, \quad M = const > 0,$$

easy to verify.

Using these facts together with properties (31), we see that the operator generated by the right-hand sides of equations (32), (33) in the space

$$(g_1, h_1) \in C[-q, 0] \times C[-q, 0]$$

transforms into itself the set $B_1 \times B_2$, where B_j, $j = 1, 2$, are some balls in $C[-q, 0]$ with centers at zero and radii of order $\varepsilon^{k_0-1/2}$ and ε^{k_0}, respectively, and is a contracting one. Hence by (32), (33) we define uniquely the smooth (by the set of variables ξ, δ_1) functions $g_1(\xi, \varepsilon, \delta_1)$, $h_1(\xi, \varepsilon, \delta_1)$ as follows:

$$\max_{-q\le\xi\le 0}\left(|g_1| + \left|\frac{\partial g_1}{\partial \delta_1}\right|\right) \le M_1\varepsilon^{k_0-1/2}, \quad \max_{-q\le\xi\le 0}\left(|h_1| + \left|\frac{\partial h_1}{\partial \delta_1}\right|\right) \le M_2\varepsilon^{k_0}, \tag{34}$$

where $M_j = const > 0$, $j = 1, 2$. In an analogous manner we define – on a segment $0 \le \xi \le q$ – the functions $g_2(\xi, \varepsilon, \delta_2)$, $h_2(\xi, \varepsilon, \delta_2)$ figuring in (29).

The above constructions reduce the problem of finding the desired solution of the boundary value problem (23), (25) to defining two parameters δ_1, δ_2, which have in stock, by means of the conditions of "sewing" solutions (29) for $\xi = 0$, i.e., by means of the equalities

$$\varepsilon^{k_0}\delta_1 + h_1|_{\xi=0} = \varepsilon^{k_0}\delta_2 + h_2|_{\xi=0}, \quad g_1|_{\xi=0} = g_2|_{\xi=0}. \tag{35}$$

Let us first analyze the first equation of system (35). Note that $u_{**}(\xi, 0, \delta_1) = u^0_{**}(\xi)$, where $u^0_{**}(\xi)$, is some smooth function not depending on δ_1. Taking this fact into account and applying estimates (34) to equation (33), we see that

$$h_1 = \varepsilon^{k_0}\int_{-q}^{\xi} u^0_{**}(s) ds + O(\varepsilon^{k_0+1/2}), \quad \frac{\partial h_1}{\partial \delta_1} = O(\varepsilon^{k_0+1/2})$$

uniformly with respect to $\xi \in [-q, 0]$. Further, combining these equalities with analogous representations

$$h_2 = -\varepsilon^{k_0} \int_{\xi}^{q} u_{**}^0(s)ds + O(\varepsilon^{k_0+1/2}), \quad \frac{\partial h_2}{\partial \delta_2} = O(\varepsilon^{k_0+1/2}),$$

we conclude that δ_2 from the first equation (35) is expressed through δ_1:

$$\delta_2 = \delta_1 + \int_{-q}^{q} u_{**}^0(s)ds + O(\sqrt{\varepsilon}). \tag{36}$$

To investigate the second equation (35), we give a more detailed consideration to the structure of the function u_*:

$$u_* = \delta_1 \Omega_v'(\xi, 0) + u_*^0(\xi) + O(\varepsilon), \tag{37}$$

where $u_*^0(\xi)$ is some smooth function (its explicit expression is not needed). Let us make the substitutions $s/\sqrt{\varepsilon} \to s$, $\tau/\sqrt{\varepsilon} \to \tau$ in the integrals on the right-hand side of (32) and then use properties (31), (34), (37). As a result we come to the asymptotic equalities

$$g_1(0, \varepsilon, \delta_1) = \varepsilon^{k_0-1/2}(\varkappa_{1,1}\delta_1 + \varkappa_{1,2}) + O(\varepsilon^{k_0}),$$
$$\frac{\partial g_1}{\partial \delta_1}(0, \varepsilon, \delta_1) = \varepsilon^{k_0-1/2}\varkappa_{1,1} + O(\varepsilon^{k_0}),$$

where the value of the constant $\varkappa_{1,2}$ is insignificant, while

$$\varkappa_{1,1} = \Omega_v'(0,0) \int_{-\infty}^{0} \exp(-\tau^2/2)d\tau \neq 0$$

(see (24)). Combining them with analogous equalities for g_2

$$g_2(0, \varepsilon, \delta_2) = \varepsilon^{k_0-1/2}(\varkappa_{2,1}\delta_2 + \varkappa_{2,2}) + O(\varepsilon^{k_0}),$$
$$\frac{\partial g_2}{\partial \delta_2}(0, \varepsilon, \delta_2) = \varepsilon^{k_0-1/2}\varkappa_{2,1} + O(\varepsilon^{k_0}),$$

where

$$\varkappa_{2,1} = -\Omega_v'(0,0) \int_{0}^{\infty} \exp(-\tau^2/2)d\tau \neq 0,$$

we come to a conclusion that the second equation (35) transforms to

$$F(\delta_1, \delta_2, \varepsilon) = 0, \tag{38}$$

where

$$F = \Omega_v'(0,0)\sqrt{\pi/2}(\delta_1 + \delta_2) + \varkappa_{1,2} - \varkappa_{2,2} + O(\sqrt{\varepsilon}),$$
$$\frac{\partial F}{\partial \delta_j} = \Omega_v'(0,0)\sqrt{\pi/2} + O(\sqrt{\varepsilon}), \quad j = 1, 2.$$

Finally, substituting equality (36) into (38) and applying the theorem on an implicit function to the resulting equation, we define uniquely the bounded functions $\delta_1 = \delta_1(\varepsilon)$, $\delta_2 = \delta_2(\varepsilon)$. The lemma is proved. □

To complete the justification of the theorem, it remains to continue the trajectory of system (23), which we have defined due to the lemma, along the ribbons $y = S_1(x, \varepsilon)$ and $y = S_2(x, \varepsilon)$ to the intersection with the curves l^- and l^+, respectively. As a result, we obtain a unique trajectory of system (1) with the ends on l^- and l^+ and the required zero approximation of L. The theorem is proved. □

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