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BOUND SETS AND TWO-POINT BOUNDARY VALUE PROBLEMS FOR SECOND ORDER DIFFERENTIAL EQUATIONS

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To Professor I. Kiguradze with sincere congratulations on his 70th birthday anniversary

Abstract. Using Mawhin's continuation principle we obtain a general result on the existence of solutions to a boundary value problem for second order nonlinear vector ODEs. Applications are given to the existence of solutions which are contained in suitable bound sets with possibly non-smooth boundary.

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1. Introduction

This paper deals with the existence of solutions to a generalized two-point boundary value problem (of Floquet type) for a vector second order nonlinear equation. More precisely, we study the problem

$$\begin{cases} x'' = f(t, x, x'), \quad t \in [0, 1], \\ x(1) = Ax(0), \\ x'(1) = Bx'(0), \end{cases} \tag{P}$$

where $f : [0, 1] \times \mathbb{R}^{2m} \to \mathbb{R}^m$ is a continuous function and A and B are $m \times m$ square matrices, with A non-singular.

The proofs make use of a suitable version of Mawhin's continuation principle (see [21]), which we describe in Section 2 (see Theorem 2). We observe that our general result applies also when the vector field f is of Carathéodory type and the matrix A may be singular. However, in the applications, we will keep the above more restrictive assumptions in order to make the treatment more transparent and avoid some technicalities.

Our Theorem 2 requires the fulfillment of a transversality condition on the boundary of a suitable open and bounded subset of $\mathbb{R}^m$. To this end, we will use the concept of bound set defined as the intersection of sublevel sets of certain scalar functions called bounding functions.

The theory of the bound set was introduced by Gaines and Mawhin in [10] to get the existence of periodic solutions for nonlinear differential systems and then extended to Floquet and other boundary value problems by Mawhin in

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[20] and [23]. In the quoted papers the bounding functions are taken of class C^1 (in the case of first order ODEs). Nonsmooth bounding functions were employed in [28, 27, 26] to obtain existence results for first order equations with periodic, Floquet and other boundary conditions. Time-dependent bound sets (also known as curvature bound sets) were introduced by Gaines and Mawhin in [10], too. Variants of this concept were used by Fabry and Habets in [7] for the Picard boundary problem defined by means of a unique bounding function, and in Frigon [8, 9] for the periodic and Sturm–Liouville boundary conditions.

For a periodic boundary value problem associated to second order differential equations, e.g. when $A = B = I$ in (P), a lot of existence results were obtained by similar techniques. We refer, for instance, to Bebernes [3], Bebernes and Schmitt [4], Knoblock [17] and Hartman's book [11], where the bound set is a ball and the family of bounding functions reduces to a single one given by the Euclidean norm. Erbe–Palamides [5] and Erbe–Schmitt [6] applied analogous approaches to the investigation of problem (P) when both A and B are non-singular and satisfy an additional assumption.

In [19], Mawhin introduced the concept of bounding functions for second order periodic problems. In that paper the bounding functions are of class C^2, with a positive definite Hessian matrix, a condition which is related to the convexity of the bound set (see Section 3). Some partial results, where the positive definiteness of the Hessian matrix is not required, were obtained in [29], at the expense of some further restrictions on the functions or on the vector field. Following this direction, in Section 3, we try to outline a possible theory for problem (P) using not necessarily convex bound sets. We study, in particular, the Lipschitzian case in Theorem 3, while Theorem 4 reads as a slight modification of the result in [29].

Some applications of the continuation theorem depend also on a possibility of finding an a priori bound on the first derivative of a possible solutions of problem (P). For this purpose, we shall assume the usual Nagumo growth conditions on the vector field f. In this direction, our main result is Theorem 5 in Section 4, where the continuation principle is reformulated assuming the existence of bound sets and the Nagumo condition.

General linear and nonlinear boundary value problems for scalar and vector differential equations (see, for instance, [13], [16], [14], [15]) have been one of the main research topics of Professor Kiguradze, with a special emphasis on the study of non-autonomous equations [12]. It is our pleasure and honor to dedicate this paper to him with our best wishes for his future works.

2. The Continuation Theorem

Let $\mathbb{R}^m$ be the m-dimensional real Euclidean space with norm $|\cdot|$. For sake of simplicity and when no confusion may occur, we shall identify each vector $\operatorname{col}(x) = (x_1, \dots, x_m)^T \in \mathbb{R}^m$ with its transpose $x = (x_1, \dots, x_m)$. We denote by (u, v) the inner product of two vectors $u, v \in \mathbb{R}^m$.

Let $f : [0, 1] \times \mathbb{R}^{2m} \to \mathbb{R}^m$ satisfy the Carathéodory conditions, i.e.,

1) $f(t,\cdot,\cdot)$ is continuous for a.e. $t \in [0,1]$;
2) $f(\cdot,x,y)$ is measurable for every $(x,y) \in \mathbb{R}^{2m}$;
3) for every $r > 0$ there exists $g_r \in L^1([0,1],\mathbb{R}^+)$ such that $|f(t,x,y)| \leq g_r(t)$ for every $|x| \leq r, |y| \leq r$ and for a.e. $t \in [0,1]$.

Given two of $m \times m$ real matrices A and B, we consider the second order problem

$$\begin{cases} x'' = f(t,x,x'), \quad t \in [0,1], \\ x(1) = Ax(0), \\ x'(1) = Bx'(0). \end{cases} \tag{P}$$

Clearly, (P) is equivalent to:

$$\begin{cases} u' = \tilde{f}(t,u), \quad t \in [0,1], \;\; u \in \mathbb{R}^{2m}, \\ u(1) = Cu(0), \end{cases}$$

where

$$u = (x,y), \quad \tilde{f}(t,u) = (y, f(t,x,y))$$

and

$$C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

is a $2m \times 2m$ matrix.

Consider now the Banach spaces $X := C([0,1],\mathbb{R}^{2m})$ with the "sup norm" $|\cdot|_\infty$ and $Z := L^1([0,1],\mathbb{R}^{2m})$ with the L^1-norm $|\cdot|_1$ and the operators

$$\begin{array}{rcl} L : X \supset \operatorname{dom} L = \{u \in AC([0,1],\mathbb{R}^{2m}) : u(1) = Cu(0)\} & \to & Z \\ u & \mapsto & u' \end{array}$$

and

$$\begin{array}{rcl} N : X & \to & Z \\ u & \mapsto & \tilde{f}(\cdot, u(\cdot)). \end{array}$$

Thus problem (P) can be written as

$$Lu = Nu, \quad u \in \operatorname{dom} L. \tag{2.1}$$

Of course, L is a linear operator and the assumptions on f imply the continuity of N. Moreover, N takes the bounded sets in X to the bounded sets in Z which are formed by functions which are uniformly bounded by a positive L^1-function. We also have that

$$\ker L = \{u \equiv c : c \in \ker(I - C)\} = \ker(I - A) \times \ker(I - B)$$

and

$$\operatorname{Im} L = \left\{ w \in Z : \int_0^1 w(s)ds \in \operatorname{Im}(I - C) \right\}$$

which is clearly a closed set in Z. Hence we can conclude that codim $\operatorname{Im} L = \dim \ker L = \dim \ker(I - C) < +\infty$, i.e., L is a Fredholm map of index zero.

Calling P_1 and P_2 the (continuous) projections of $\mathbb{R}^{2m}$ onto $\ker(I-C)$ and $\mathrm{Im}(I-C)$, respectively, we can define the continuous projections as follows:

$$\begin{aligned} P : X &\rightarrow \ker L \\ u &\mapsto Pu \equiv P_1 u(0) \end{aligned}$$

and

$$\begin{aligned} Q : Z &\rightarrow Z \\ w &\mapsto Q(w) \equiv (I - P_2) \int_0^1 w(s)ds. \end{aligned}$$

Therefore the right inverse K of L is defined by

$$\begin{aligned} K = K_{P,Q} : \mathrm{Im}L &\rightarrow X \cap \ker P \\ w &\mapsto x_w(t) := c_w + \int_0^t w(s)ds, \end{aligned}$$

where c_w is a unique solution of

$$\begin{cases} (C-I)c + \int_0^1 w(s)ds = 0, \\ P_1 c = 0. \end{cases}$$

In this manner, we enter into the setting of [21] and have that equation (2.1) turns out to be equivalent to

$$u = Pu + JQNu + K(I-Q)Nu, \quad u \in X,$$

where, in general, $J : \mathrm{Im}Q \rightarrow \ker L$ is any linear isomorphism, but, in our case, it will be convenient to take for J the identity on $\ker(I-C)$. An application of the Ascoli–Arzelá theorem and the Lebesgue dominated convergence theorem shows that N is L-compact [21] on the bounded subsets of X.

In this framework, in order to prove an existence result for (P) one can apply the Mawhin's continuation theorem [21] as follows.

Theorem 1. *Let X and Z be normed spaces, $\Omega \subset X$ an open and bounded set and $L : \mathrm{dom}L \subset X \rightarrow Z$ a Fredholm map of index zero with associated projectors $P : X \rightarrow \ker L$ and $Q : Z \rightarrow \mathrm{coker}L$. Suppose that there is a L-compact operator $N^* : \bar{\Omega} \times [0,1] \rightarrow Z$. Assume*

1) $Lu \neq \lambda N^*(u,\lambda), \quad \forall \lambda \in]0,1[\,,\ \forall u \in \partial\Omega,$
2) $QN^*(z,0) \neq 0, \quad \forall z \in \ker L \cap \partial\Omega,$
3) $d[QN^*(\cdot,0)|_{\ker L}, \ker L \cap \Omega, 0] \neq 0$

(*where d is the Brouwer degree*). *Then, $Lu = N^*(u,1)$ has at least one solution in $\mathrm{dom}L \cap \bar{\Omega}$.*

In our approach to problem (P), we choose

$$N^*(u,\lambda)(t) := (y(t), f(t,x(t),\lambda y(t))), \quad u = (x,y),$$

so that

$$N^*(u,1)(t) = \tilde{f}(t,u(t)) = Nu(t).$$

Hence $u \in \mathrm{dom}L$ is a solution to the equation

$$Lu = \lambda N^*(u,\lambda),$$

for some $\lambda \in]0,1[$ if and only if $u(t) = (x(t),y(t)) \in \mathrm{dom}\, L$ with

$$\begin{cases} x' = \lambda y, \\ y' = \lambda f(t,x,\lambda y). \end{cases}$$

This, in turn, holds if and only if $u(t) = (x(t),x'(t))$ with

$$\begin{cases} x'' = \lambda^2 f(t,x,x'), \quad t \in [0,1], \\ x(1) = Ax(0), \\ x'(1) = Bx'(0). \end{cases}$$

By the above choice of N^*, we have that for $z = (a,b) \in \ker L$,

$$QN^*(z,0) = (I - P_2)\left(b, \int_0^1 f(s,a,0)\,ds\right)$$

$$= \left((I-P_A)b, (I-P_B)\int_0^1 f(s,a,0)\,ds\right),$$

P_A and P_B being the projections of $\mathbb{R}^m$ onto $\mathrm{Im}(I-A)$ and $\mathrm{Im}(I-B)$, respectively.

At this point, we have to make a choice for the set Ω. We consider an open bounded set

$$G \subset \mathbb{R}^m,$$

take a constant

$$K > 0$$

and define

$$\Omega = \{u = (x,y) \in X :\ x(t) \in G \wedge |y(t)| < K, \quad \forall\, t \in [0,1]\}.$$

In order to have condition 2) satisfied and observing that the first component of $QN^*(\cdot,0)$ is linear, we must require that

$$(I-P_A)b = 0 \ \text{ with } \ b \in \ker(I-B) \Longrightarrow b = 0.$$

This takes place if and only if

$$\ker(I-B) \cap \mathrm{Im}(I-A) = \{0\}. \tag{2.2}$$

Now, we consider the second component in $QN^*(\cdot,0)$ and get that 2) holds if and only if, besides (2.2), we also have that

$$(I-P_B)\bar{f}(a) \neq 0, \quad \forall\, a \in \ker(I-A) \cap \partial G, \tag{2.3}$$

where we have set

$$\bar{f}(a) := \int_0^1 f(s,a,0)\,ds.$$

If we assume now (2.2), assumption 3) reads as

$$d[g, (G \cap \ker(I-A)) \times (B(K) \cap \ker(I-B)), 0] \neq 0,$$

for the map $g : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ defined by

$$g : (a,b) \mapsto \big((I-P_A)b, (I-P_B)\bar{f}(a)\big).$$

At this point, we can observe that (2.2) ensures that the map

$$\ker(I-B) \ni b \mapsto (I-P_A)b$$

is a linear isomorphism and therefore, after an application of the degree formula to a product space, we can conclude that

$$|d[QN^*(\cdot,0)|_{\ker L}, \ker L \cap \Omega, 0]| = |d[(I-P_B)\bar{f}, G \cap \ker(I-A), 0]|.$$

Notice that, in a special case in which the set $\ker(I-A)$ is invariant for the map $\bar{f}$, (2.3) holds if

$$\ker(I-A) \cap \operatorname{Im}(I-B) = \{0\}, \quad \text{and } \bar{f}(a) \neq 0,\ \forall\, a \in \ker(I-A) \cap \partial G.$$

Moreover, in this situation

$$|d[(I-P_B)\bar{f}, G \cap \ker(I-A), 0]| = |d[\bar{f}, G \cap \ker(I-A), 0]|.$$

Now, we are in the position to state the continuation theorem for problem (P) as follows.

Theorem 2. *Let $f : [0,1] \times \mathbb{R}^{2m} \to \mathbb{R}^m$ be a Carathéodory function and let A and B be $m \times m$ real matrices. Suppose that $G \subset \mathbb{R}^m$ is an open bounded set such that*

(BS) *there is no solution $x(\cdot)$ for some $\lambda \in]0,1[$ to*

$$\begin{cases} x'' = \lambda f(t,x,x'), & t \in [0,1], \\ x(1) = Ax(0), \\ x'(1) = Bx'(0) \end{cases} \tag{2.4}$$

such that $x(t) \in \bar{G}$, for all $t \in [0,1]$ and $x(\tilde{t}) \in \partial G$ for some $\tilde{t} \in [0,1]$;

(NC) *there is $K > 0$ such that*

$$|x'|_\infty < K$$

for each solution $x(\cdot)$ to (2.4), for $\lambda \in]0,1[$, such that $x(t) \in \bar{G}$, for all $t \in [0,1]$.

Assume further that

$$\ker(I-B) \cap \operatorname{Im}(I-A) = \{0\}$$

and

$$d[(I-P_B)\bar{f}, G \cap \ker(I-A), 0] \neq 0,$$

where P_B is the projection of $\mathbb{R}^m$ onto $\mathrm{Im}(I-B)$ *and*

$$\bar{f}(a) := \int_0^1 f(s,a,0)\,ds.$$

Then, problem (P) has at least one solution x with $x(t) \in \bar{G}$, for all $t \in [0,1]$.

3. Bound Sets for Second Order Equations

In the sequel, we shall consider a special case of the second order problem (P) with continuous right hand side. More precisely, let $f : [0,1] \times \mathbb{R}^{2m} \to \mathbb{R}^m$ be a continuous function and let A and B be $m \times m$ real matrices, with A nonsingular, and consider the boundary value problem

$$\begin{cases} x'' = f(t,x,x'), \quad t \in [0,1], \\ x(1) = Ax(0), \\ x'(1) = Bx'(0). \end{cases} \tag{P}$$

In this section we are interested in obtaining sufficient conditions for a set $G \subset \mathbb{R}^m$ (open and bounded) in order that the transversality condition (BS) be satisfied. For this, we introduce the following definition.

Definition 1. An open bounded subset $G \subset \mathbb{R}^m$ is said to be a *bound set for the boundary value problem* (P) if there exists no solution x of (P) such that $x(t) \in \bar{G}$ for every $t \in [0,1]$ and $x(t_0) \in \partial G$ for some $t_0 \in [0,1]$.

According to this definition, the transversality condition (BS) is satisfied if and only if G is a bound set for (2.4) for every $\lambda \in]0,1[$.

A useful tool to locate bound sets consists in defining them as the intersection of sublevel sets of scalar functions. More precisely, we assume that for each $u \in \partial G$ there exists a continuous function $V_u : \mathbb{R}^m \to \mathbb{R}$ such that

$$V_u|_{\bar{G}} \le 0 \tag{$H1$}$$

and

$$V_u(u) = 0. \tag{$H2$}$$

The function V_u is called a *bounding function for G at u.*

Bounding functions and bound sets were introduced by Gaines and Mawhin [10] and Mawhin [20], generalizing the concept of Krasnosel'skiĭ's guiding functions [18].

We also need to introduce the following definition.

Definition 2. An open bounded subset $G \subset \mathbb{R}^m$ is said to have the boundary invariant with respect to the subgroup of $GL^m(\mathbb{R})$ generated by a nonsingular $m \times m$ real matrix A if

$$Au \in \partial G \Leftrightarrow u \in \partial G. \tag{IC}$$

Now we are in the position to prove sufficient conditions for the existence of a bound set for a second order Floquet problem. We start with the case of locally Lipschitzian bounding functions. We recall that a function $V:\mathbb{R}^m\to\mathbb{R}$ is said to be *locally Lipschitzian at a point* x_0 if there exist a positive constant L (called Lipschitz constant of V at x_0) and a neighbourhood U of x_0 such that

$$|V(x_1)-V(x_2)|\le L|x_1-x_2|\,,\ \forall\, x_1,x_2\in U. \tag{L}$$

Theorem 3. *Let G be an open bounded subset of $\mathbb{R}^m$ whose boundary is invariant with respect to the subgroup generated by A. Let $\{V_u\}_{u\in\partial G}$ be a family of bounding functions for G, with V_u locally Lipschitzian at u, for each $u\in\partial G$, (i.e. such that $(H1),(H2)$ and (L) are satisfied). Assume that for every $u\in\partial G$ the following statements are valid:*

$(H3)$ *$\forall\, t\in]0,1[,\ \forall\, v\in\mathbb{R}^m$: $\limsup_{h\to0^+}\frac{V_u(u+hv)}{h}\le 0\le\liminf_{h\to0^-}\frac{V_u(u+hv)}{h}$, it follows that*

$$\limsup_{h\to0}\frac{V_u(u+hv+\frac{h^2}{2}f(t,u,v))}{h^2}>0;$$

$(H4)$ *$\forall\, v\in\mathbb{R}^m$: $\limsup_{h\to0^+}\frac{V_u(u+hv)}{h}\le 0\le\liminf_{h\to0^-}\frac{V_{Au}(Au+hBv)}{h}$, it follows that*

$$\max\left\{\limsup_{h\to0^+}\frac{V_u(u+hv+\frac{h^2}{2}f(0,u,v))}{h^2},\right.$$
$$\left.\limsup_{h\to0^-}\frac{V_{Au}(Au+hBv+\frac{h^2}{2}f(1,Au,Bv))}{h^2}\right\}>0.$$

Then G is a bound set for (P).

Proof. Suppose that G is not a bound set for (P). Let x be a solution of (P) and $t_0\in[0,1]$ be such that $x(t)\in\bar G$ for each $t\in[0,1]$ and $x(t_0)\in\partial G$. Since $x\in C^2([0,1],\mathbb{R}^m)$ is a solution of $x''=f(t,x,x')$, there exists a function $\delta(h)$, infinitesimal when $h\to0$, such that, for each h,

$$x(t_0+h)=x(t_0)+h[x'(t_0)+\delta(h)].$$

If $t_0<1$, it follows by $(H1),(H2)$ and (L) that, for each $h>0$,

$$\begin{aligned}0&\ge\frac{V_{x(t_0)}(x(t_0+h))}{h}\\&=\frac{V_{x(t_0)}(x(t_0)+h[x'(t_0)+\delta(h)])}{h}\\&\ge\frac{V_{x(t_0)}(x(t_0)+hx'(t_0))}{h}-L_{x(t_0)}|\delta(h)|,\end{aligned}$$

which yields

$$\limsup_{h\to0^+}\frac{V_{x(t_0)}(x(t_0)+hx'(t_0))}{h}\le0, \tag{3.5}$$

since $\delta(h)\to0$ when $h\to0$.

In a similar way one can prove that, if $t_0 > 0$, then

$$\liminf_{h\to 0^-} \frac{V_{x(t_0)}(x(t_0)+hx'(t_0))}{h} \geq 0. \tag{3.6}$$

Moreover, again by Taylor's formula, it follows that there exists another function $\epsilon(h)$, infinitesimal when $h \to 0$, such that, for each h,

$$\begin{aligned} x(t_0+h) &= x(t_0) + hx'(t_0) + \frac{h^2}{2}\left[x''(t_0) + \epsilon(h)\right] \\ &= x(t_0) + hx'(t_0) + \frac{h^2}{2}\left[f(t_0, x(t_0), x'(t_0)) + \epsilon(h)\right]. \end{aligned}$$

Thus

$$\begin{aligned} 0 &\geq \frac{V_{x(t_0)}(x(t_0+h))}{h^2} \\ &= \frac{V_{x(t_0)}(x(t_0) + hx'(t_0) + \frac{h^2}{2}[f(t_0, x(t_0), x'(t_0)) + \epsilon(h)])}{h^2} \\ &\geq \frac{V_{x(t_0)}(x(t_0) + hx'(t_0) + \frac{h^2}{2}f(t_0, x(t_0), x'(t_0)))}{h^2} - L_{x(t_0)}|\epsilon(h)|, \end{aligned}$$

which implies that

$$\limsup_{h\to 0} \frac{V_{x(t_0)}(x(t_0) + hx'(t_0) + \frac{h^2}{2}f(t_0, x(t_0), x'(t_0)))}{h^2} \leq 0, \tag{3.7}$$

since $\epsilon(h) \to 0$ when $h \to 0$.

Hence, if $t_0 \in]0,1[$, then (3.5), (3.6) and (3.7) contradict $(H3)$.

If $t_0 = 0$ or $t_0 = 1$, then the invariance condition (IC) implies that both $x(0)$ and $x(1)$ belong to ∂G. Thus, by the boundary conditions, (3.6) and (3.7), with $t_0 = 1$, can be rewritten as

$$\liminf_{h\to 0^-} \frac{V_{Ax(0)}(Ax(0) + hBx'(0))}{h} \geq 0$$

and

$$\limsup_{h\to 0^-} \frac{V_{Ax(0)}(Ax(0) + hBx'(0) + \frac{h^2}{2}f(1, Ax(0), Bx'(0)))}{h^2} \leq 0.$$

The above inequalities, together with analogous formulas for (3.5) and (3.7), written for $t_0 = 0$, give a contradiction to $(H4)$. □

Similar assumptions involving contingent derivatives instead of Dini derivatives can be obtained to prove the existence of a bound set defined as intersection of sublevel sets of a family of continuous bounding functions. Since the aim of this paper is to describe the method, with a limited amount of technical details, we do not consider this case. We recall [28] and [27] for analogous theorems in the framework of periodic and Floquet boundary value problems for first order differential equations.

When f is of Carathéodory type, to obtain the existence of a bound set, it is necessary to require that, for each u in ∂G, the hypothesis on the bounding function V_u be assumed in a neighborhood of u. We leave this investigation for a future paper. We recall [10] and [25] for a comparison between the hypotheses concerning the continuous and the Carathéodory right hand side for the periodic problem associated to first order differential equations and to [1] and [2] in the case of the Floquet problem associated to differential inclusions. We also recall the recent interesting work by Mawhin and Thompson [24] for differential systems satisfying the Carathéodory assumptions

When the bounding functions $\{V_u\}_{u\in\partial G}$ are of class C^2 at u, then conditions $(H3)$ and $(H4)$ can be rewritten in terms of the gradient and the Hessian matrix of V_u and the following corollary of Theorem 3 is true.

Corollary 1. *Let G be an open bounded subset of $\mathbb{R}^m$ whose boundary is invariant with respect to the subgroup generated by A. Let $\{V_u\}_{u\in\partial G}$ be a family of C^2-bounding functions for G, i.e. such that $(H1)$ and $(H2)$ are satisfied. Assume that for every $u \in \partial G$ the following statements are valid:*

$(H5)$ $\forall\, t \in]0,1[,\ \forall\, v \in \mathbb{R}^m : (\nabla V_u(u), v) = 0,$ *it follows that*

$$(V''_u(u)v, v) + (\nabla V_u(u), f(t,u,v)) > 0;$$

$(H6)$ $\forall\, v \in \mathbb{R}^m : (\nabla V_u(u), v) \le 0 \le (\nabla V_{Au}(Au), Bv),$ *it follows that*

$$\max\Big\{ \big(V''_u(u)v, v\big) + \big(\nabla V_u(u), f(0,u,v)\big), \\ \big(V''_{Au}(Au)Bv, Bv\big) + \big(\nabla V_{Au}(Au), f(1, Au, Bv)\big)\Big\} > 0.$$

Then G is a bound set for problem (P).

In [19] and [22], the sufficient conditions on a function $V \in C^2([0,1], \mathbb{R}^m)$ were introduced to prove that the set $G = \{x \in \mathbb{R}^m : V(x) < 0\}$ is a bound set for the periodic boundary value problem, i.e., when $A = B = I$. Then the invariance condition (IC) is clearly satisfied and conditions $(H5)$ and $(H6)$ of Corollary 1 are equivalent to the following: for every $(u,v) \in \partial G \times \mathbb{R}^m$, with $(\nabla V_u(u), v) = 0$, and for all $t \in [0,1]$,

$$(V''_u(u)v, v) + (\nabla V_u(u), f(t,u,v)) > 0.$$

Hence, in the particular case where G is the set with a C^2-function V taking its negative values and we also have $V_u = V$ for every u, Corollary 1 is equivalent to Proposition 4.1 in [19].

In [5], a special family of C^2-bounding functions is used to prove that the ball centered at the origin and having radius R is a bound set for a Floquet boundary value problem, e.g. when both A and B are nonsingular, under the further hypothesis that A is orthogonal and that it has a certain property of symmetry with respect to B. According to this approach, the invariance condition (IC) for a ball is equivalent to the orthogonality of A. The assumption considered in [5, Theorem 3.1] is the following: $\forall\, t \in [0,1], (u,v) \in \mathbb{R}^{2m}$, with $|u| = R$ and $(u,v) = 0$, it follows that

$$|v|^2 + (u, f(t,u,v)) > 0.$$

In that case, the function $V_u(x) = \frac{1}{2}(|x|^2 - |u|^2)$ is a bounding function for the set $G = \{x \in \mathbb{R}^m : |x| < R\}$ at $u \in \partial G$. Moreover, $\nabla V_u(u) = u$ and $V_u'' = I$. Therefore Theorem 3.1 of [5] is a special case of Corollary 1 in the above setting.

We now consider the case in which the set G is convex. Then, G is equipped with a family of outward normals $\{\eta(u)\}_{u \in \partial G}$, i.e., for each $u \in \partial G$ there exists a versor $\eta(u)$ such that $(x - u, \eta(u)) < 0$ for every $x \in G$. Thus the position $V_u(x) = (x - u, \eta(u))$ defines the collection of C^2-bounding functions for G. Moreover, $\nabla V_u = \eta(u)$ and $V_u'' = 0$. Therefore we have the following corollary.

Corollary 2. *Let G be an open bounded convex subset of $\mathbb{R}^m$ whose boundary is invariant with respect to the subgroup generated by A. Assume that for every $u \in \partial G$ the following statements are true:*

1) $\forall\, t \in]0,1[, \forall\, v \in \mathbb{R}^m : (\eta(u), v) = 0,$ *it follows that*
$$(\eta(u), f(t,u,v)) > 0;$$
2) $\forall\, v \in \mathbb{R}^m : (\eta(u), v) \le 0 \le (\eta(Au), Bv),$ *it follows that*
$$\max\{(\eta(u), f(0,u,v)), (\eta(Au), f(1, Au, Bv))\} > 0.$$

Then G is a bound set for problem (P).

$$\begin{cases} x'' = f(t,x,x'), \quad t \in [0,1], \\ x(1) = Ax(0), \\ x'(1) = Bx'(0). \end{cases}$$

The following result states the sufficient conditions for the existence of a not necessarily convex bound set defined by means of a family of C^2-bounding functions.

Theorem 4. *Let G be an open bounded subset of $\mathbb{R}^m$ whose boundary is invariant with respect to the subgroup generated by A. Let $\{V_u\}_{u \in \partial G}$ be a family of C^2-bounding functions for G, i.e. such that $(H1)$ and $(H2)$ are satisfied. Assume that*

(H^*) *for every $u \in \partial G$ there exists $k_u > 0$ such that there is no solution $x(\cdot)$ of (P) with $x(t) \in \bar{G}$, $-k_u < V_u(x(t)) \le 0$, $\forall\, t \in [0,1]$ and $x(t_0) = u$ for some $t_0 \in [0,1]$.*

Suppose also that there is $a_u \in [0, \frac{\pi^2}{4}[$ such that

$(H7)$ $\forall\, t \in [0,1], \forall\, x \in \bar{G} : V_u(x) > -k_u, \forall\, v \in \mathbb{R}^m,$ *it follows that*
$$(V_u''(x)v, v) + (\nabla V_u(x), f(t,x,v)) \ge -a_u[V_u(x) + k_u].$$

Finally, assume that

$(H8)$ $\forall\, v \in \mathbb{R}^m : (\nabla V_u(u), v) \le 0 \le (\nabla V_{Au}(Au), Bv),$ *it follows that*
$$(\nabla V_u(u), v)k_u \ge (\nabla V_{Au}(Au), Bv)k_{Au}.$$

Then G is a bound set for problem (P).

Proof. Suppose the thesis is false. Then there exists a solution $x(\cdot)$ of (P) and $t_0 \in [0,1]$ such that $x(t) \in \bar{G}$ for each $t \in [0,1]$ and $x(t_0) \in \partial G$.

If $t_0 \in]0,1[$, by $(H1)$ and $(H2)$ it follows that t_0 is a local maximum point for the function $v(t) := V_{x(t_0)}(x(t))$, therefore $v'(t_0) = 0$. Moreover, as a consequence of (H^*) there exists $t_1 < t_0$, such that for all $t \in [t_1, t_0]$

$$-k_{x(t_0)} = v(t_1) \leq v(t) \leq 0 = v(t_0)$$

or there exists $t_2 > t_0$ such that for all $t \in [t_0, t_2]$

$$-k_{x(t_0)} = v(t_2) \leq v(t) \leq 0 = v(t_0).$$

We proceed by considering the first situation, the latter one being treated in the same manner. By differentiation, we get

$$v'(t) = \big(\nabla V_{x(t_0)}(x(t)), x'(t)\big) \tag{3.8}$$

and

$$\begin{aligned} v''(t) &= \big(V''_{x(t_0)}(x(t))x'(t), x'(t)\big) + \big(\nabla V_{x(t_0)}(x(t)), x''(t)\big) \\ &= \big(V''_{x(t_0)}(x(t))x'(t), x'(t)\big) + \big(\nabla V_{x(t_0)}(x(t)), f(t, x(t), x'(t))\big) \\ &\geq -a_{x(t_0)}[v(t) + k_{x(t_0)}]. \end{aligned}$$

Let us set now $z(t) := v(t) + k_{x(t_0)}$. Then, by (3.8), $z(t) \geq 0$ for every $t \in [t_1, t_0]$, $z(t_1) = 0$, $z'(t_1) \geq 0 = z'(t_0)$ and

$$a_{x(t_0)} z(t) \geq -z''(t).$$

Thus, multiplying both sides of the above inequality by $z(t)$ and integrating by parts between t_1 and t_0, we get

$$a_{x(t_0)} \int_{t_1}^{t_0} z^2(t)dt \geq -[z'(t)z(t)]_{t_1}^{t_0} + \int_{t_1}^{t_0} z'(t)^2 dt = \int_{t_1}^{t_0} z'(t)^2 dt. \tag{3.9}$$

On the other hand, since $z(\cdot)$ satisfies the Sturm–Liouville type condition $z(t_1) = z'(t_0) = 0$, we also know that

$$\int_{t_0}^{t_1} z(t)^2 dt \leq \frac{1}{\Lambda^2} \int_{t_0}^{t_1} z'(t)^2\, dt, \quad \text{with } \Lambda = \frac{\pi}{2(t_0 - t_1)} > \frac{\pi}{2}\,.$$

Hence we find a contradiction to (3.9) since $z \not\equiv 0$.

If $t_0 \notin]0,1[$, then, by the boundary invariance condition, $x(0), x(1) \in \partial G$. Thus, by the same reasoning as above, setting $v(t) := V_{x(0)}(x(t))$ and $w(t) := V_{Ax(0)}(x(t))$, from

$$0 \geq v'(0) = \big(\nabla V_{x(0)}(x(0)), x'(0)\big) \ \text{ and } \ 0 \leq w'(1) = \big(\nabla V_{Ax(0)}(Ax(0)), Bx'(0)\big),$$

we get

$$\big(\nabla V_{x(0)}(x(0)), x'(0)\big) \leq 0 \leq \big(\nabla V_{Ax(0)}(Ax(0)), Bx'(0)\big). \tag{3.10}$$

Moreover, there exists $t_1 > 0$ such that, for all $t \in [0, t_1]$, $-k_{x(0)} = v(t_1) \leq v(t) \leq 0$ and $v''(t) \geq -a_{x(0)}[v(t)+k_{x(0)}]$, and also there exists $t_2 < 1$ such that, for all $t \in [t_2, 1]$, $-k_{Ax(0)} = w(t_2) \leq w(t) \leq 0$ and $w''(t) \geq -a_{Ax(0)}[w(t)+k_{Ax(0)}]$. Therefore, introducing auxiliary functions $z(t) := v(t) + k_{x(0)}$ and $y(t) := w(t) + k_{Ax(0)}$ and, keeping in mind $z'(t_1) \le 0$ and $y'(t_2) \ge 0$, by (3.10) and $(H8)$ we obtain

$$
\begin{aligned}
a_{x(0)} \int_0^{t_1} z^2(t)dt + a_{Ax(0)} \int_{t_2}^1 y^2(t)dt \\
&\ge -[z'(t)z(t)]_0^{t_1} + \int_0^{t_1} z'(t)^2 dt - [y'(t)y(t)]_{t_2}^1 + \int_{t_2}^1 y'(t)^2 dt \\
&\ge \big(\nabla V_{x(0)}(x(0)), x'(0)\big) k_{x(0)} + \int_0^{t_1} z'(t)^2 dt \\
&\quad - \big(\nabla V_{Ax(0)}(Ax(0)), Bx'(0)\big) k_{Ax(0)} + \int_{t_2}^1 y'(t)^2 dt \\
&\ge \int_0^{t_1} z'(t)^2 dt + \int_{t_2}^1 y'(t)^2 dt \\
&\ge \frac{\pi^2}{4t_1^2} \int_0^{t_1} z(t)^2 dt + \frac{\pi^2}{4(1-t_2)^2} \int_{t_2}^1 y(t)^2 dt \\
&> a_{x(0)} \int_0^{t_1} z(t)^2 dt + a_{Ax(0)} \int_{t_2}^1 y^2(t)dt,
\end{aligned}
$$

and we get again a contradiction. □

In [29, Theorem 1 and Remark 2], the sufficient conditions on a family of C^2-bounding functions are obtained for the existence of a not necessarily convex bound set for the periodic boundary value problem associated to a second order forced equation. In that case, assumption $(H8)$ is trivially fulfilled. Therefore Theorem 4 restricted to the periodic boundary conditions provides an alternative version of the results in [29].

As a further remark we also note that if the bound set is convex, the assumptions of Corollary 1 concern only the behavior of V_u at u and require the positivity of $(V_u''(u)v, v) + (\nabla V_u(u), f(t, u, v))$. On the other hand, the assumptions of Theorem 4 concern the behavior of V_u in a neighborhood of u and require of $(V_u''(x)v, v) + (\nabla V_u(x), f(t, x, v))$ to be bounded from below by a strictly negative constant. Hence, the requirements of the theorem are weaker at u, but must be satisfied in a whole neighborhood of it. The disadvantage of our condition on the Hessian matrix (for the nonconvex case) is the fact that if $(V_u''(x)v, v)$ is negative on some vector v, then it must be unbounded from below on vectors of the form θv with $\theta \in \mathbb{R}$. Fortunately, in the applications to problem (P), the vector v is not an arbitrary point of $\mathbb{R}^m$, but, in fact, we have $v = x'(t_0)$ with $x(\cdot)$ being a solution of (P). Thus our condition may find some effective applications when we have some extra information about x', like, for instance, when we possess some a priori bound (say K) for the norm of the derivative. In such a case, we have to check $(H7)$ only for v's with $|v| \leq K$ (see [29] for a more detailed discussion). We also refer to [29] for some examples (in the periodic case) in which assumption (H^*) can be easily checked. Condition (H^*) prevents a possibility to have solutions in $\bar{G}$ which are permanently too close to a point $u \in \partial G$. Usually, it is satisfied by requiring that the inner product between the vector field f and some vector $\eta(u)$ be nonzero in a neighborhood of u (see [29, (vii)]).

4. Existence Result for Two-Point Boundary Value Problems

Modifying in a suitable manner the assumptions found in the preceding section on the set $G \subset \mathbb{R}^m$, we can guarantee that the transversality condition (BS) of Theorem 2 is satisfied. At this step, to get the existence of a solution for problem (P), it remains to guarantee the existence of an upper bound for the first derivative of each solution of every problem (2.4). To this end, as usual in this framework, we recall the concept of Nagumo equation.

Definition 3. A second order differential equation $x'' = f(t, x, x')$ is said to be a Nagumo equation with respect to a set $G \subset \mathbb{R}^m$ if there exists $K > 0$ such that $|x'|_\infty \leq K$ holds for each $\lambda \in]0, 1[$ and all solutions x of $x'' = \lambda f(t, x, x')$, with $x(t) \in \bar{G}$ for every $t \in [0, 1]$.

In [11, Lemma 5.2], the following lemma is proved, assuming that $x'' = f(t, x, x')$ is a Nagumo equation with respect to G when the growth of $f(t, x, y)$ is less than quadratic y for $x \in \bar{G}$.

Lemma 1. *If there exists a continuous function $\varphi : [0, +\infty[\to [0, +\infty[$, with $\int^{+\infty} \frac{s}{\varphi(s)} ds = \infty$, and $\alpha, \beta \geq 0$ such that $|f(t, x, y)| \leq \varphi(|y|)$ and $|f(t, x, y)| \leq 2\alpha[(x, f(t, x, y)) + |y|^2] + \beta$ in $[0, 1] \times \bar{G} \times \mathbb{R}^m$, then $x'' = f(t, x, x')$ is a Nagumo equation with respect to G.*

We recall that, according to [11, Lemma 5.1], the existence of $\alpha, \beta \geq 0$ such that $|f(t, x, y)| \leq 2\alpha[(x, f(t, x, y)) + |y|^2] + \beta$ is not necessary in the scalar case, i.e. when $m = 1$.

We are now able to state two existence results for two-point boundary value problems associated to second order differential equations. We start with the theorem involving locally Lipschitzian bounding functions.

Theorem 5. *Let $f : [0, 1] \times \mathbb{R}^{2m} \to \mathbb{R}^m$ be a continuous function and let A and B be $m \times m$ real matrices, with A nonsingular. Let $G \subset \mathbb{R}^m$ be an open bounded set whose boundary is invariant with respect to the subgroup generated by A. Suppose that for each $u \in \partial G$ there exists a function $V_u : \mathbb{R}^m \to \mathbb{R}$ locally Lipschitzian at u, such that $(H1)$ and $(H2)$ are satisfied. Assume also that*

$(H3a)$ $\forall\, \lambda, t \in]0,1[, v \in \mathbb{R}^m$: $\limsup_{h\to 0^+} \frac{V_u(u+hv)}{h} \le 0 \le \liminf_{h\to 0^-} \frac{V_u(u+hv)}{h}$, *it follows that*

$$\limsup_{h\to 0} \frac{V_u(u+hv+\frac{h^2}{2}\lambda f(t,u,v))}{h^2} > 0;$$

$(H4a)$ $\forall\, \lambda \in]0,1[, v \in \mathbb{R}^m : \limsup_{h\to 0} \frac{V_u(u+hv)}{h} \le 0 \le \liminf_{h\to 0} \frac{V_{Au}(Au+hBv)}{h}$, *it follows that*

$$\max\left\{ \limsup_{h\to 0^+} \frac{V_u(u+hv+\frac{h^2}{2}\lambda f(0,u,v))}{h^2}, \limsup_{h\to 0^-} \frac{V_{Au}(Au+hBv+\frac{h^2}{2}\lambda f(1,Au,Bv))}{h^2} \right\} > 0.$$

Suppose further that there exists $\varphi : [0,+\infty[\to [0,+\infty[$, *with* $\int^{+\infty} \frac{s}{\varphi(s)} ds = \infty$, *and* $\alpha, \beta \ge 0$ *such that* $|f(t,x,y)| \le \varphi(|y|)$ *and*

$$|f(t,x,y)| \le 2\alpha[(x, f(t,x,y)) + |y|^2] + \beta$$

in $[0,1] \times \bar{G} \times \mathbb{R}^m$.
Assume finally that

$$\ker(I-B) \cap \operatorname{Im}(I-A) = \{0\}$$

and

$$d[(I-P_B)\bar{f}, G \cap \ker(I-A), 0] \ne 0,$$

where P_B *is the projection of* $\mathbb{R}^m$ *onto* $\operatorname{Im}(I-B)$ *and*

$$\bar{f}(a) := \int_0^1 f(s,a,0)\, ds.$$

Then, problem (P) *has at least one solution* x *with* $x(t) \in \bar{G}$, *for all* $t \in [0,1]$.

Proof. By Theorem 3 and Lemma 1 we get respectively that the transversality condition (BS) and the Nagumo growth condition (NC) of Theorem 2 are satisfied. Then the thesis follows by Theorem 2. □

Now we can present a theorem concerning C^2-bounding functions. For simplicity, we confine ourselves to the convex case and obtain the following result (see Corollary 1).

Theorem 6. *Let* $f : [0,1] \times \mathbb{R}^{2m} \to \mathbb{R}^m$ *be a continuous function and let* A *and* B *be* $m \times m$ *real matrices, with* A *nonsingular. Let* $G \subset \mathbb{R}^m$ *be an open bounded set whose boundary is invariant with respect to the subgroup generated by* A. *Suppose that, for each* $u \in \partial G$ *there exist* $V_u : \mathbb{R}^m \to \mathbb{R}$ *of class* C^2, *satisfying* $(H1), (H2)$ *and such that* $V_u''(u) \ge 0$ *(i.e.,* $V_u''(u)$ *is positive semi-definite). Assume also that*

$(H5a)$ $\forall\, t \in]0,1[, \forall\, v \in \mathbb{R}^m : (\nabla V_u(u), v) = 0$, *it follows that*

$$(\nabla V_u(u), f(t,u,v)) > 0;$$

(H6a) $\forall\, v \in \mathbb{R}^m : (\nabla V_u(u), v) \le 0 \le (\nabla V_{Au}(Au), Bv)$, *it follows that*

$$\max\Big\{ \big(\nabla V_u(u), f(0,u,v)\big), \big(\nabla V_{Au}(Au), f(1,Au,Bv)\big) \Big\} > 0.$$

Suppose further that there exists $\varphi : [0,+\infty[\to [0,+\infty[$, *with* $\int^{+\infty} \frac{s}{\varphi(s)} ds = \infty$, *and* $\alpha, \beta \ge 0$ *such that* $|f(t,x,y)| \le \varphi(|y|)$ *and*

$$|f(t,x,y)| \le 2\alpha[(x, f(t,x,y)) + |y|^2] + \beta$$

in $[0,1] \times \bar{G} \times \mathbb{R}^m$.
Assume finally that

$$\ker(I - B) \cap \operatorname{Im}(I - A) = \{0\}$$

and

$$d[(I - P_B)\bar{f}, G \cap \ker(I - A), 0] \ne 0,$$

where P_B *is the projection of* $\mathbb{R}^m$ *onto* $\operatorname{Im}(I - B)$ *and*

$$\bar{f}(a) := \int_0^1 f(s,a,0)\, ds.$$

Then problem (P) *has at least one solution* x *with* $x(t) \in \bar{G}$, *for all* $t \in [0,1]$.

Proof. The proof follows by Corollary 1 and Lemma 1. □

The corresponding theorem for the nonconvex case is more complicated to state, but it can be easily adapted from Theorem 4 and Lemma 1.

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