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FINITE DIMENSIONAL GRADING OF THE VIRASORO ALGEBRA

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Abstract. The Virasoro algebra is a central extension of the Witt algebra, the complexified Lie algebra of the sense preserving diffeomorphism group of the circle Diff S^1. It appears in Quantum Field Theories as an infinite dimensional algebra generated by the coefficients of the Laurent expansion of the analytic component of the momentum-energy tensor, Virasoro generators. The background for the construction of the theory of unitary representations of Diff S^1 is found in the study of Kirillov's manifold Diff S^1/S^1. It possesses a natural Kählerian embedding into the universal Teichmüller space with the projection into the moduli space realized as an infinite-dimensional body of the coefficients of univalent quasiconformally extendable functions. The differential of this embedding leads to an analytic representation of the Virasoro algebra based on Kirillov's operators. In this paper we overview several interesting connections between the Virasoro algebra, Teichmüller theory, Löwner representation of univalent functions, and propose a finite-dimensional grading of the Virasoro algebra such that the grades form a hierarchy of finite dimensional algebras which, in their turn, are the first integrals of Liouville partially integrable systems for coefficients of univalent functions.

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1. Introduction

The Virasoro–Bott group *vir* appears in physics literature as the space of reparametrization of a closed string. It may be represented as the central extension of the infinite-dimensional Lie–Fréchet group of sense preserving diffeomorphisms of the unit circle. The corresponding Virasoro algebra *Vir* is realized as the central extension of the algebra of vector fields on S^1. The coadjoint orbits of the Virasoro–Bott group are related to the unitary representation of *vir* as an analogue to the representation of finite-dimensional compact semi-simple Lie groups given by the Borel–Weil–Bott theorem, see [36]. Two orbits are of particular importance because they carry the structure of infinite-dimensional homogeneous Kählerian manifolds. They are Diff S^1/S^1 and Diff $S^1/SL_2(\mathbb{R})$, both are homogeneous complex analytic Fréchet–Kähler manifolds.

We deal with the analytic representation of Diff S^1/S^1. Let U be the unit disk $U = \{z : |z| < 1\}$. Let **S** stand for the standard class of holomorphic

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univalent functions $f : U \to \mathbb{C}$ normalized by

$$f(z) = z\left(1 + \sum_{n=1}^{\infty} c_n z^n\right), \quad z \in U.$$

By $\tilde{\mathbf{S}}$ we denote the class of functions from $\mathbf{S}$ smooth (C^∞) on the boundary S^1 of U. Given a map $f \in \tilde{\mathbf{S}}$ we construct an adjoint univalent meromorphic map

$$g(z) = d_1 z + d_0 + \frac{d_{-1}}{z} + \cdots$$

defined in the exterior $U^* = \{z : |z| > 1\}$ of U, and such that $\hat{\mathbb{C}} \setminus \overline{f(U)} = g(U^*)$. This gives the identification of Diff S^1/S^1 with the space of smooth contours Γ that enclose univalent domains Ω of conformal radius 1 with respect to the origin and such that $\infty \notin \Omega$, $0 \in \Omega$, see [1, 19]. Being quasicircles, the smooth contours allow us to embed Diff S^1/S^1 into the universal Teichmüller space making use of the above conformal welding, and then, to project it to the set $\mathcal{M} \subset \mathbb{C}^{\mathbb{N}}$ which is the limiting set for the coefficient bodies $\mathcal{M} = \lim_{n\to\infty} \mathcal{M}_n$, where

$$\mathcal{M}_n = \{(c_1, \dots, c_n) : \ f \in \tilde{\mathbf{S}}\}. \tag{1}$$

Then the Virasoro generators can be realized by the first order differential operators

$$L_j = \partial_j + \sum_{k=1}^{\infty} (k+1) c_k \partial_{j+k}, \quad j \in \mathbb{N},$$

in terms of the affine coordinates of $\mathcal{M}$, acting over the set of holomorphic functions, where $\partial_k = \partial/\partial c_k$. A representation [1] of the Virasoro algebra into the Lie algebra of the differential operators on $\tilde{\mathbf{S}}$ can be given by means of the Neretin [27] homogeneous polynomials $P_k(c_1, \dots, c_k)$ given by the recurrence relation (2) below, where

$$\frac{12}{z^2} \sum_{n=0}^{\infty} P_n(c_1, \dots, c_n) z^n = S_f(z),$$

and

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

is the Schwarzian derivative of a univalent function $f \in \tilde{\mathbf{S}}$. Indeed, the polynomials P_n satisfy the following recurrence relations

$$L_k(P_j) = (j+k) P_{j-k} + \frac{1}{12} k(k^2 - 1)\delta_{j,k}, \quad P_0 \equiv P_1 \equiv 0, \ \ P_j(0) = 0. \tag{2}$$

Note that $L_k(P_j) \equiv 0$ for $k > j$, so we do not need to define the polynomials P_j for negative indices.

From the physics viewpoint the Virasoro generators naturally appear as coefficients of the Laurent mode expansion of a momentum-energy tensor in the 2-D Conformal Field Theory. The Schwarzian derivative comes into play as a defect term in the chart-change formula for this tensor (its analytic component).

The Virasoro algebra *Vir* is an infinite-dimensional vector space, moreover, the dimension of this space is uncountably infinite. It is natural to reduce *Vir* to a finite-dimensional grading by truncating and to study the resulting finite-dimensional grades with respect to standard bases. Indeed, the recurrence relation (2) does not require a complete series of operators ∂_j at each step. Moreover, the limiting coefficient body $\mathcal{M}$ is of very complicated form although we know thanks to de Branges [8] that its projection to $\mathbb{C}^n$ is bounded for any fixed $n \in \mathbb{N}$. Note that $\mathcal{M} = \lim_{n\to\infty} \mathcal{M}_n \neq \cup_{n=1}^{\infty}\mathcal{M}_n$. Let us construct the truncated generators

$$\mathcal{L}_j = \partial_j + \sum_{k=1}^{n-j}(k+1)c_k\partial_{j+k}, \tag{3}$$

for $j = 1, \dots, n$, $\mathcal{L}_j = 0$ for $j > n$, defined on $\mathcal{M}_n$. For each fixed n we get a finite-dimensional algebra $\mathcal{A}_n = \operatorname{span}(\mathcal{L}_1, \dots, \mathcal{L}_n)$ with the same commutator relation as for $A = \operatorname{span}(L_1, \dots, L_n, \dots)$, and satisfying the same relations (2) for Neretin polynomials. All these algebras are disjoint and form a kind of hierarchy with respect to n. We use here the attribution 'hierarchy' because it turns out that the vector fields $\mathcal{L}_j$ naturally appear in the hierarchies of the Hamiltonian systems for the coefficient bodies for univalent functions (see [22, 30]). These systems are partially integrable in the sense of Liouville and the vector fields $\mathcal{L}_j$ serve as first integrals replacing ∂_j by formal variables corresponding to the velocity components of the Hamiltonian systems.

We also discuss the Kählerian embedding of the Virasoro–Bott group/ Virasoro algebra into the universal Teichmüller space/the space of harmonic Beltrami differentials and connections between corresponding embedding of the finite-dimensional grades and Teichmüller spaces of Fuchsian groups.

2. $\mathbb{C}^{\mathbb{N}}$ and $\mathbb{C}^{\infty}$

The infinite vector space $\mathbb{C}^\infty$ consists of infinite sequences of elements from $\mathbb{C}$ such that only finitely many elements are non-vanishing. Algebraic operations for such sequences can be introduced in the same way as for $\mathbb{C}^n$. The dimension of $\mathbb{C}^\infty$ is *countably infinite*. So $\mathbb{C}^\infty$ is a coproduct of countably many copies of $\mathbb{C}$. The space $\mathbb{C}^{\mathbb{N}}$ consists of sequences with possibly infinitely many non-vanishing elements. The dimension of this space is *uncountably infinite* and $\mathbb{C}^{\mathbb{N}}$ is the product of countably many copies of $\mathbb{C}$. The space $\mathbb{C}^{\mathbb{N}}$ is dual to $\mathbb{C}^\infty$, but these spaces are not isomorphic (see, e.g., [15, Section 1.3]). The Virasoro algebra topologically is uncountably infinite-dimensional.

3. Virasoro Algebra and Kirillov's Operators

We denote by Diff S^1 the infinite-dimensional Lie–Fréchet group of sense preserving diffeomorphisms of the unit circle. The associated Lie algebra is the Lie algebra $\operatorname{Vect} S^1$ of right-invariant vector fields $\phi(\theta)\frac{d}{d\theta}$ with the vanishing mean value on S^1, and with the usual Lie–Poisson bracket

$$[\phi_1, \phi_2] = \phi_1\phi_2' - \phi_1'\phi_2, \quad \phi_1, \phi_2 \in \operatorname{Vect} S^1,$$

where the derivative is taken with respect to the angular variable. Fixing the trigonometric basis in Vect S^1, the commutator relations admit the form

$$\begin{aligned}
[\cos n\theta, \cos m\theta] &= \frac{n-m}{2}\sin(n+m)\theta + \frac{n+m}{2}\sin(n-m)\theta,\\
[\sin n\theta, \sin m\theta] &= \frac{m-n}{2}\sin(n+m)\theta + \frac{n+m}{2}\sin(n-m)\theta,\\
[\sin n\theta, \cos m\theta] &= \frac{m-n}{2}\cos(n+m)\theta - \frac{n+m}{2}\cos(n-m)\theta.
\end{aligned}$$

Let $\mathrm{Vect}_{\mathbb{C}}\, S^1$ be the complexified Lie algebra of Vect S^1. The topological complex basis $v_n = -ie^{in\theta}\frac{d}{d\theta}$ satisfies the commutation rule of the Witt algebra

$$[v_n, v_m] = (m-n)v_{n+m}.$$

The Lie algebra Vect S^1 contains the span$(v_1, \dots, v_n, \dots)$ as a dense Lie subalgebra in the Fréchet topology.

The Virasoro algebra is a unique (up to an isomorphism) non-trivial central extension $\mathrm{Vect}_{\mathbb{C}}\, S^1 \oplus \mathbb{C}$ of $\mathrm{Vect}_{\mathbb{C}}\, S^1$ by $\mathbb{C}$ (or Vect S^1 by $\mathbb{R}$ in the real case) given by the *Gelfand–Fuchs 2-cocycle* [13]:

$$\omega(v_n, v_m) = \frac{1}{12}n(n^2-1)\delta_{n,-m},$$

and the commutation rule becomes

$$[v_n, v_m]_{Vir} = (m-n)v_{n+m} + \frac{c}{12}\omega(v_n, v_m), \tag{4}$$

(we put $c = 1$ in (2)). The above cocycle is cohomologically equivalent to the cocycle

$$\omega_0(v_n, v_m) = n^3\delta_{n,-m},$$

or, in the functional form, to

$$\omega_0(\phi_1, \phi_2) = \frac{-i}{2\pi}\int_0^{2\pi} \phi_1'\phi_2'' d\theta = \frac{-i}{4\pi}\int_0^{2\pi} (\phi_1'\phi_2'' - \phi_1''\phi_2')d\theta.$$

The Gelfand–Fuchs cocycle admits the following functional form

$$\omega(\phi_1, \phi_2) = -\frac{1}{2\pi i}\int_0^{2\pi} (\phi_1' + \phi_1''')\phi_2 d\theta. \tag{5}$$

Thus, we consider the Virasoro algebra Vir to be a Lie algebra over the space $\mathrm{Vect}_{\mathbb{C}}\, S^1 \oplus \mathbb{C}$ defined by the commutator

$$[(\phi_1, a), (\phi_2, b)]_{Vir} = ([\phi_1, \phi_2]_{\mathrm{Vect}\, S^1}, \frac{c}{12}\omega(\phi_1, \phi_2)),$$

where a and b are elements of the center $\mathbb{C}$, and $c \in \mathbb{C}$ is the central charge. Integration by parts leads to the Jacobi identity

$$\omega(\phi_1, [\phi_2, \phi_3]) + \omega(\phi_2, [\phi_3, \phi_1]) + \omega(\phi_3, [\phi_1, \phi_2]) = 0.$$

Following the construction for the algebra, let us consider the group Diff S^1. In the real case the *Virasoro–Bott group vir* is the unique (up to an isomorphism) non-trivial central extension of Diff S^1 given by the Thurston–Bott cocycle [7]

$$\Omega(f,g) = \frac{1}{2\pi}\int_0^{2\pi} \log((f\circ g)')d\log(g').$$

The Virasoro–Bott group is given by the following product on Diff $S^1 \times \mathbb{R}$

$$(f,\alpha)(g,\beta) = (f\circ g, \alpha+\beta+\frac{c}{12}\Omega(f,g)).$$

The Kirillov infinitesimal action [17] of Vect S^1 on $\tilde{\mathbf{S}}$ is given by the Goluzin–Schiffer variational formulas which lift the actions from the Lie algebra Vect S^1 onto $\tilde{\mathbf{S}}$. Let $f \in \tilde{\mathbf{S}}$ and let $\nu(e^{i\theta}) \in \operatorname{Vect} S^1$ be a C^∞ real-valued function in $\theta \in (0, 2\pi]$. The infinitesimal action $\theta \mapsto \theta + \varepsilon\nu(e^{i\theta})$ yields a variation of the univalent function $f^*(z) = f + \varepsilon\, \delta_\nu f(z) + o(\epsilon)$, where

$$\delta_\nu f(z) = \frac{f^2(z)}{2\pi i}\int_{S^1}\left(\frac{wf'(w)}{f(w)}\right)^2 \frac{\nu(w)dw}{w(f(w)-f(z))}. \tag{6}$$

Kirillov and Yuriev [19], [20] (see also [1]) established that the variations $\delta_\nu f(\zeta)$ are closed with respect to the commutator (4) and the induced Lie algebra is the same as Vect S^1. Moreover, Kirillov's result [16] states that there is an exponential map Vect $S^1 \to$ Diff S^1 such that the subgroup S^1 coincides with the stabilizer of the map $f(z) \equiv z$ from $\tilde{\mathbf{S}}$.

Taking the complexification $\operatorname{Vect}_{\mathbb{C}} S^1$ of Vect S^1 and the basis $\nu = -iz^k$ in the integrand of (6) we calculate the residue in (6) and obtain

$$L_k(f)(z) = \delta_\nu f(z) = z^{k+1}f'(z), \quad k = 1, 2, \dots .$$

In terms of the affine coordinates in $\mathcal{M}$ we get Kirillov's operators as

$$L_j = \partial_j + \sum_{k=1}^{\infty}(k+1)c_k\partial_{j+k}.$$

Kirillov's operators act over Neretin's polynomials as was shown in the Introduction. In general, we have real vector fields from Vect S^1. The computation of L_k must be carried out with respect to the basis $1, e^{\pm ki\theta}$, which leads to L_k with $k \le 0$. However, we deal with holomorphic functions, and L_k with $k > 0$ are to be treated as holomorphic vector fields (see the discussion in [17, p. 738], [1, p. 632–634]).

4. Kählerian Embedding into the Universal Teichmüller Space

The universal Teichmüller space T is a holomorphically homogeneous complex Banach manifold $\mathrm{QS}(S^1)/\operatorname{Möb}(S^1)$, the quotient of the space of quasisymmetric maps (QS) over the space of Möbius maps (Möb) over the unit circle S^1. All finite Teichmüller spaces of arbitrary Fuchsian groups are holomorphically embedded into T.

The analytic representation of T is based on quasiconformally extendable holomorphic mappings. Given a function $\mu(z)$, $z \in \hat{\mathbb{C}}$, from the class $L_1^\infty(U^*)$ in U^* (of essentially bounded functions $\|\mu\|_\infty < 1$), let us solve the Beltrami equation $w_{\bar{z}} = \mu(z) w_z$ extending μ by 0 into U. Normalizing the solution $w = f(z)$ by $f(0) = 0$, $f(\infty) = \infty$, $f'(0) = 1$, we obtain a unique quasiconformal map, conformal in the unit disk U. It is said that two Beltrami coefficients μ_1 and μ_2 represent the same point of the Teichmüller space $x \in T$ if the normalized solutions to the Beltrami equation f^{μ_1} and f^{μ_2} map the unit disk U onto one and the same domain in $\hat{\mathbb{C}}$. So the universal Teichmüller space T can be thought of as the family $\mathbf{S}^{qc}$ of all normalized conformal maps of U admitting quasiconformal extension. Any compact subspace of T consists of conformal maps f of U that admit quasiconformal extension to U^* with $\|\mu_f\| \leq k < 1$ for some k.

Let $x, y \in T$ and $f, g \in S^{qc}$ be such that $\mu_f \in x$ and $\mu_g \in y$. Then, *the Teichmüller distance* $\tau(x, y)$ on T is defined as

$$\tau(x, y) = \inf_{\mu_f \in x,\ \mu_g \in y} \frac{1}{2} \log \frac{1 + \|\mu_{g \circ f^{-1}}\|_\infty}{1 - \|\mu_{g \circ f^{-1}}\|_\infty}.$$

Unlike Teichmüller spaces of Fuchsian groups, geodesics in the universal Teichmüller space are not unique. Indeed, for a given $x \in T$ we consider the extremal Beltrami coefficient μ^* such that $\|\mu^*\|_\infty = \inf_{\nu \in x} \|\nu\|_\infty$. The extremal μ^* does not need to be unique. A geodesic on T can be described in terms of the extremal coefficient μ^* as a continuous homomorphism $x_t : [0, 1] \mapsto T$ such that $\tau(0, x_t) = t\tau(0, x_1)$. Hence the geodesic does not need to be unique either.

We consider the Banach space $H^{2,0}(U)$ of all quadratic differentials $\varphi(z) dz^2$, with holomorphic φ in U, equipped with the norm

$$\|\varphi\|_{H^{2,0}} = \sup_{z \in U} |\varphi(z)|(1 - |z|^2)^2,$$

which is finite. For a function f from $\mathbf{S}$ the Schwarzian derivative

$$S_f(z) = \frac{\partial}{\partial z} \left(\frac{f''(z)}{f'(z)} \right) - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2$$

is defined and Nehari's [25] estimate $\|S_f\|_{H^{2,0}} \leq 6$ holds. Given $x \in T$, $\mu \in x$, we construct the mapping $f^\mu \in \mathbf{S}^{qc}$ and have the holomorphic Bers embedding $T \mapsto H^{2,0}(U)$ by the Schwarzian derivative. This embedding models the universal Teichmüller space as an infinite-dimensional complex manifold on the Banach space $H^{2,0}(U)$. All finite Teichmüller spaces of Fuchsian groups are holomorphically embedded into T. They possess an additional Hilbert structures given by the Kählerian Weil–Petersson metric. Hans Petersson first introduced his scalar product in 1939 [28] and then André Weil used it for the moduli spaces [35] (see also [12, 23, 37]).

The Banach space $H^{2,0}(U)$ is an infinite-dimensional vector space that can be thought of as a co-tangent space to T at the initial point. Let us give an explicit realization of tangent and cotangent spaces. The map $f^\mu \in \mathbf{S}^{qc}$ has a Fréchet derivative at a point μ in a direction ν. Let us construct the variation in $\mathbf{S}^{qc}$

$$f^{\tau\nu}(z) = z + \tau V(z) + o(\tau), \quad z \in U.$$

Taking the Schwarzian derivative in U we get

$$S_{f^{\tau\nu}} = \tau V'''(z) + o(\tau), \quad z \in U,$$

locally uniformly in U. Taking into account the normalization of the class $\mathbf{S}^{qc}$ we have (see, e.g., [14, 21])

$$V(z) = -\frac{z^2}{\pi} \iint\limits_{U^*} \frac{\nu(w) d\sigma_w}{w^2(w-z)},$$

$$V'''(z) = -\frac{6}{\pi} \iint\limits_{U^*} \frac{\nu(w) d\sigma_w}{(w-z)^4} = -\frac{6}{\pi} \iint\limits_{U} \frac{\bar{w}^2}{w^2} \frac{\nu(1/\bar{w}) d\sigma_w}{(1-\bar{w}z)^4}. \tag{7}$$

The integral formula implies $V'''(A(z))A'(z)^2 = V'''(z)$ (subject to the relation for the Beltrami coefficient $\mu(A(z))\overline{A'(z)} = \mu(z)A'(z)$) for any Möbius transform A.

Let us extend $\nu(z)$, $z \in U^*$ into U by putting $\nu(1/\bar{z}) = \overline{\nu(z)} z^2/\bar{z}^2$, $z \in U$. Taking $\Lambda_\nu(z) = S_{f^{\tau\nu}}(z)$ and $\dot{\Lambda}_\nu(z) = V'''(z)$ we have (see, e.g., [12, Section 6.5, Theorem 5])

$$\Lambda_\nu(z) - \tau \dot{\Lambda}_\nu(z) = \frac{o(\tau)}{(1-|z|^2)^2}, \quad z \in U,$$

or $\dot{\Lambda}_\nu$ is the derivative of Λ_ν at the initial point of the universal Teichmüller space with respect to the norm of the Banach space $H^{2,0}(U)$. The reproducing formula for the Bergman integral gives

$$\varphi(z) = \frac{3}{\pi} \iint\limits_{U} \frac{\varphi(w)(1-|w|^2)^2 d\sigma_w}{(1-\bar{w}z)^4}, \quad \varphi \in H^{2,0}(U). \tag{8}$$

Changing the variables $w \to 1/\bar{w}$ in the latter integral we come to the so-called harmonic (Bers') Beltrami differential

$$\nu(z) = \Lambda^*_\varphi(z) = -\frac{1}{2}\overline{\varphi(z)}(1-|z|^2)^2, \quad z \in U.$$

Let us denote by $A(U)$ the Banach space of analytic functions with the finite L^1-norm. Then $A(U) \hookrightarrow H^{2,0}(U)$ is a continuous inclusion ([23], Section 1.4.2). On $L^\infty(U) \times A(U)$ one can define the coupling

$$\langle \nu, \varphi \rangle := \iint\limits_{U} \nu(z)\varphi(z)\, d\sigma_z,$$

where $d\sigma_z$ means the area element in U. Denote by N the space of *locally trivial Beltrami coefficients* which is a subspace of $L^\infty(U)$ that forms the kernel of the operator $\langle \cdot, \varphi \rangle$ for all $\varphi \in A(U)$. Then one can identify the tangent space to T at the initial point with the space $H := L^\infty(U)/N$. It is natural to relate it to a subspace of $L^\infty(U)$. The superposition $\dot{\Lambda}_\nu \circ \Lambda^*_\varphi$ acts identically on $A(U)$ due to

(7), (8). The space N is also the kernel of the operator $\dot{\Lambda}_\nu$. Thus, the operator Λ^* splits the following exact sequence

$$0 \longrightarrow N \hookrightarrow L^\infty(U) \xrightarrow{\dot{\Lambda}_\nu} A(U) \longrightarrow 0.$$

Then, $H = \Lambda^*(A(U)) \cong L^\infty(U)/N$. The coupling $\langle \mu, \varphi \rangle$ defines $A(U)$ as a cotangent space.

Let $A^2(U)$ denote the separable Hilbert space of analytic functions φ with the finite norm

$$\|\varphi\|^2_{A^2(U)} = \iint\limits_U |\varphi(z)|^2(1-|z|^2)^2 d\sigma_z.$$

Then $A(U) \hookrightarrow A^2(U)$ and Petersson's Hermitian product [32, 37] is defined on $A^2(U)$ as

$$(\varphi_1, \varphi_2) = \iint\limits_U \varphi_1(z)\overline{\varphi_2(z)}(1-|z|^2)^2 d\sigma_z.$$

The Kählerian Weil–Petersson metric $\{\nu_1, \nu_2\} = \langle \nu_1, \dot{\Lambda}_{\nu_2} \rangle$ can be defined on the tangent space to T which gives a Kählerian manifold structure to T (see a recent monograph [32] by Takhtajan and Teo). In [32] it was proved that the Weil–Petersson metric is right-invariant and continuous. Moreover, the connected component T_0 of the origin in T is a topological group.

As we have already mentioned, Kirillov's manifold $\mathrm{Diff}\, S^1/S^1$ is naturally embedded into T by the conformal welding. Moreover, a complex structure on $\mathrm{Diff}\, S^1/S^1$ is defined as follows. We identify $\mathrm{Vect}_0\, S^1 = \mathrm{Vect}\ S^1/S^1$ with the functions with the vanishing mean value over S^1. This gives

$$\phi(\theta) = \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta.$$

Let us define a complex structure by the operator

$$J(\phi)(\theta) = \sum_{n=1}^{\infty} -a_n \sin n\theta + b_n \cos n\theta.$$

Then $J^2 = -id$. On $\mathrm{Vect}_0\, S^1 \otimes \mathbb{C}$, the operator J diagonalizes and we have

$$\phi \to v := \frac{1}{2}(\phi - iJ(\phi)) = \sum_{n=1}^{\infty} (a_n - ib_n)e^{in\theta},$$

and the latter extends into the unit disk as a holomorphic function. Then $\mathrm{Diff}\, S^1/S^1$ is embedded into T as a complex submanifold.

Simple calculations give

$$\omega(\phi, J(\phi)) = \frac{1}{2i}\sum_{n=1}^{\infty}(n^3 - n)(a_n^2 + b_n^2).$$

A homogeneous Kählerian metric on Diff S^1/S^1 is of the form [17]

$$\|v\|^2_{a,b} = \sum_{n=1}^{\infty}(an^3+bn)|c_n|^2, \quad c_n = a_n - ib_n.$$

The Gelfand–Fuchs cocycle represents a Kählerian form and the homogeneous Kählerian metric is compatible with the Gelfand–Fuchs cocycle if $a = -b = 1/12$.

Let us realize the Lie algebra Vect S^1 within the space of harmonic Beltrami differentials H. In [33] it was proved that a vector $v(e^{i\theta}) \in \operatorname{Vect} S^1$ generates the harmonic Beltrami differential as

$$\nu(z) = \frac{3}{2\pi}\int_0^{2\pi}\left(\frac{1-|z|^2}{(1-e^{i\theta}\bar{z})^2}\right)^2 e^{2i\theta}v(e^{i\theta})d\theta. \tag{9}$$

This formula was derived making use of the Douady–Earle quasiconformal extension of diffeomorphisms of S^1 instead of the extension by Beurling and Ahlfors used in [32], because the former extension is compatible with Möbius transforms, which is important when working with Teichmüller spaces. There exist constants M_1 and M_2 independent of z such that $|\nu(z)| \leq M_1(1-|z|^2)/|z|^2$ and $|\nu(z)| \leq M_2/(1-|z|^2)$, see [33]. Formula (9) gives an explicit Kählerian embedding of Diff S^1/S^1 into T as a holomorphic submanifold.

5. Finite Grading and Coefficients of Univalent Functions

Let us recall that Teichmüller spaces $T(\Gamma)$ of Fuchsian groups Γ are contained canonically and holomorphically in T. Nag and Verjovsky [24] discovered that Teichmüller spaces (finite in particular) of Fuchsian groups are embedded transversely to Diff S^1/S^1 thought of as a leaf of a holomorphic foliation of T. This happens first of all because of the fractal structure of quasicircles invariant under a Fuchsian group. This fact implies, in particular, that the Weil–Petersson metric defined on T is not "universal" for the corresponding Weil–Petersson metrics defined on $T(\Gamma)$ although the construction is similar. On the other hand, given the dimension m of $T(\Gamma)$, we can see more common features by projecting both T and $T(\Gamma)$ to $\mathbb{C}^{\mathbb{N}}$ and $\mathbb{C}^m$, respectively.

Here we would like to mention a lemma by Krushkal [18, Ch. V]. Let $T(\Gamma)$ be a finite Teichmüller space of a Fuchsian group Γ and let m be its dimension. There is a holomorphic embedding j of $T(\Gamma)$ into $\mathcal{M}_m$ as a bounded analytic surface in $\mathbb{C}^{m+1}$, where $\mathcal{M}_m$ is the coefficient body for univalent functions from $\mathbf{S}^{qc}$. The first m coefficients $(c_1,\dots,c_m)$ in the Taylor expansion

$$f(z) = z\left(1+\sum_{n=1}^{\infty}c_n z^n\right), \quad z \in U,$$

of functions f from $\mathbf{S}^{qc}$ play the role of local parameters of $T(\Gamma)$. This embedding is represented in the diagram

$$
\begin{array}{ccc}
T(\Gamma) & \xrightarrow{j} & \mathcal{M}_m \subset \mathbb{C}^{m+1} \\
\uparrow & & \uparrow \\
T_0T(\Gamma) & \xrightarrow{j^*} & T\mathcal{M}_m \subset T\mathbb{C}^{m+1},
\end{array}
$$

and, for the universal Teichmüller space, in

$$
\begin{array}{ccccccc}
\operatorname{Diff} S^1/S^1 & \overset{i}{\hookrightarrow} & T & \xrightarrow{j} & \mathcal{M} \subset \mathbb{C}^{\mathbb{N}} & \xrightarrow{\pi} & \mathbb{C}^{m+1} \\
\uparrow & & \uparrow & & & & \uparrow \\
\operatorname{Vect} S^1 & \overset{i^*}{\hookrightarrow} & T_0T & \xrightarrow{j^*} & T\mathcal{M} \subset T\mathbb{C}^{\mathbb{N}} & \xrightarrow{\pi^*} & T\mathbb{C}^{m+1}.
\end{array}
$$

We are going to construct the projections π and π^* aiming at a finite-dimensional grading (with respect to m) of the Virasoro algebra. At the same time we want to preserve the Virasoro algebra structure of the new finite basis. We choose $(\partial_1, \dots, \partial_m)$ as a standard affine basis in $T\mathbb{C}^m$, where $\partial_k := \partial/\partial c_k$. The Virasoro basis is the embedding of a complex Fourier basis under the mapping $i^* \circ j^*$. The result of this embedding is the operators

$$L_j = \partial_j + \sum_{k=1}^{\infty}(k+1)c_k\partial_{j+k}, \quad j \in \mathbb{N}.$$

Acting over the functions from $\tilde{\mathbf{S}}$ as $L_k(f)(z) = z^{k+1}f'(z)$, $k \in \mathbb{N}$. Let $g = a_0 + a_1 z + \dots$ be an analytic function. Fix m. We denote by $[g]_m$ the truncation of the function g: $[g(z)]_m = a_0 + a_1 z + \dots + a_m z^m$, and construct the operators $\mathcal{L}_k \equiv \mathcal{L}_k^m$ by the formula $\mathcal{L}_k = [z^{k+1}f'(z)]_{m+1}$. This leads to the expression (3) in terms of the affine coordinates or

$$
\begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{L}_3 \\ \dots \\ \mathcal{L}_m \end{pmatrix}
=
\begin{pmatrix}
1 & 2c_1 & \dots & (m-1)c_{m-2} & mc_{m-1} \\
0 & 1 & \dots & (m-2)c_{m-3} & (m-1)c_{m-2} \\
0 & 0 & \dots & (m-3)c_{m-4} & (m-2)c_{m-3} \\
\dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & 0 & 1
\end{pmatrix}
\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \\ \dots \\ \partial_m \end{pmatrix}. \quad (10)
$$

The projection π^* is defined by the above truncation.

6. Connections with the Löwner Theory

6.1. **Coefficient bodies.** By the *coefficient problem for univalent functions* we mean the problem of finding precisely the regions $\mathcal{M}_n$ defined above (1). These sets have been investigated by a great number of authors, but the most remarkable source is the monograph [31] written by Schaeffer and Spencer in 1950. Among other contributions to the coefficient problem we distinguish the monograph by Babenko [4] that contains a good collection of qualitative results on coefficient bodies $\mathcal{M}_n$. The results concerning the structure and properties of $\mathcal{M}_n$ include (see [4], [31])

(i) $\mathcal{M}_n$ is homeomorphic to a $(2n-2)$-dimensional ball and its boundary $\partial\mathcal{M}_n$ is homeomorphic to a $(2n-3)$-dimensional sphere;

(ii) every point $x \in \partial\mathcal{M}_n$ corresponds to exactly one function $f \in \mathbf{S}$ which is called a *boundary function* for $\mathcal{M}_n$;
(iii) with the exception for a set of smaller dimension, at every point $x \in \partial\mathcal{M}_n$ there exists a normal vector satisfying the Lipschitz condition;
(iv) there exists a connected open set X_1 on $\partial\mathcal{M}_n$ such that the boundary $\partial\mathcal{M}_n$ is an analytic hypersurface at every point of X_1, the points of $\partial\mathcal{M}_n$ corresponding to the functions that give the extremum to a linear functional belong to the closure of X_1.

It is worth to note again that all boundary functions have a similar structure. They map the unit disk U onto the complex plane $\mathbb{C}$ minus piecewise analytic Jordan arcs forming a tree with a root at infinity and having at most n tips. The uniqueness of boundary functions implies that each point of $\partial\mathcal{M}_n$ defines the rest of coefficients uniquely.

6.2. **Hamiltonian dynamics and integrability.** Let us recall briefly the Hamiltonian and symplectic definitions and concepts that will be used in the sequel. There exists a vast amount of modern literature dedicated to different approaches to and definitions of *integrable systems* (see, e.g., [2], [3], [6], [38]).

The classical definition of a *completely integrable system* in the sense of Liouville applies to a Hamiltonian system. If we can find independent conserved integrals which are pairwise involutory, this system is completely integrable (see e.g., [2], [3], [6]). That is each first integral allows us to reduce the order of the system not just by one, but by two. We formulate this definition in a slightly adopted form as follows.

A dynamical system in $\mathbb{C}^{2n}$ is called *Hamiltonian* if it is of the form

$$\dot{x} = \nabla_s H(x), \tag{11}$$

where ∇_s denotes the *symplectic gradient* given by

$$\nabla_s = \left(\frac{\partial}{\partial \bar{x}_{n+1}}, \dots, \frac{\partial}{\partial \bar{x}_{2n}}, -\frac{\partial}{\partial x_1}, \dots, -\frac{\partial}{\partial x_n}\right).$$

The function H in (11) is called the *Hamiltonian function* of the system. It is convenient to redefine the coordinates $(x_{n+1}, \dots, x_{2n}) = (\psi_1, \dots, \psi_n)$, and rewrite the system (11) as

$$\dot{x}_k = \frac{\partial H}{\partial \overline{\psi}_k}, \quad \dot{\overline{\psi}}_k = -\frac{\partial H}{\partial x_k}, \quad k = 1, 2 \dots, n. \tag{12}$$

The system has n degrees of freedom. The two-form $\omega = \sum_{k=1}^{n} dx \wedge d\bar{\psi}$ admits the Lie–Poisson bracket $[\cdot,\cdot]$

$$[f,g] = \sum_{k=1}^{n}\left(\frac{\partial f}{\partial x_k}\frac{\partial g}{\partial \overline{\psi}_k} - \frac{\partial f}{\partial \overline{\psi}_k}\frac{\partial g}{\partial x_k}\right)$$

associated with ω. The symplectic pair $(\mathbb{C}^{2n}, \omega)$ defines the Poisson manifold $(\mathbb{C}^{2n}, [\cdot,\cdot])$.

The system (12) may be rewritten as

$$\dot{x}_k = [x_k, H], \quad \dot{\overline{\psi}}_k = [\overline{\psi}_k, H], \quad k = 1, 2, \dots, n, \tag{13}$$

and the *first integrals* L of the system are characterized by

$$[L, H] = 0. \tag{14}$$

In particular, $[H, H] = 0$, and the Hamiltonian function H is an integral of the system (11). If the system (13) has n functionally independent integrals $L_1, \dots, L_n$, which are pairwise involutory $[L_k, L_j] = 0$, $k, j = 1, \dots, n$, then it is called *completely integrable* in the sense of Liouville. The function H is included in the set of first integrals. The classical theorem of Liouville and Arnold [2] gives a complete description of the motion generated by the completely integrable system (13). It states that such a system admits action-angle coordinates about connected regular compact invariant manifold.

If the Hamiltonian system admits only $1 \leq k < n$ independent involutory integrals, then it is called *partially integrable*. The case $k = 1$ is known as the Poincaré–Lyapunov theorem which states that a periodic orbit of an autonomous Hamiltonian system can be included in a one-parameter family of such orbits under a non-degeneracy assumption. A bridge between these two extreme cases $k = 1$ and $k = n$ has been proposed by Nekhoroshev [26] and justified later in [5], [10], [11]. The result states the existence of k-parameter families of tori under suitable non-degeneracy conditions.

6.3. **Hamiltonian system for coefficients.** The Löwner–Kufarev parametric method (see, e.g., [9, 29]) is based on a representation of any function f from the class $\mathbf{S}$ by the limit

$$f(z) = \lim_{t \to \infty} e^t w(z, t), \tag{15}$$

where the function

$$w(z, t) = e^{-t} z \left(1 + \sum_{n=1}^{\infty} c_n(t) z^n\right)$$

is a solution to the Löwner–Kufarev equation

$$\frac{dw}{dt} = -w p(w, t), \tag{16}$$

with the initial condition $w(z, 0) \equiv z$. The function $p(z, t) = 1 + p_1(t) z + \cdots$ is holomorphic in U and has the positive real part for all $z \in U$ almost everywhere in $t \in [0, \infty)$. If $f \in \tilde{\mathbf{S}}$, then

$$\begin{aligned} \dot{c}_n &= c_n - \frac{e^t}{2\pi i} \int\limits_{S^1} w(z, t) p(w(z, t), t) \frac{dz}{z^{n+2}}, \\ &= -\frac{1}{2\pi i} \int\limits_{S^1} \sum_{k=1}^{n} e^{-kt} (e^t w)^{k+1} p_k \frac{dz}{z^{n+2}}, \quad n \geq 1. \end{aligned} \tag{17}$$

We consider the adjoint vector $\bar\psi(t) = (\bar\psi_1(t), \dots, \bar\psi_n(t))^T$ with complex-valued coordinates $\psi_1, \dots, \psi_n$, and the complex Hamiltonian function

$$H(a, \psi, u) = \sum_{k=1}^{n} \bar\psi_k \left(c_k - \frac{e^t}{2\pi i} \int_{S^1} w(z,t) p(w(z,t),t) \frac{dz}{z^{k+2}} \right).$$

To come to the Hamiltonian formulation for the coefficient system we require that $\bar\psi$ satisfy the adjoint to (17) system of differential equations

$$\dot{\bar\psi}_j = -\frac{\partial H}{\partial c_j}, \quad 0 \le t < \infty,$$

or

$$\dot{\bar\psi}_j = -\bar\psi_j + \frac{1}{2\pi i} \sum_{k=1}^{n} \bar\psi_k \int_{S^1} (p + wp') \frac{dz}{z^{k-j+1}}, \quad j = 1, \dots, n-1, \tag{18}$$

and

$$\dot{\bar\psi}_n = 0. \tag{19}$$

6.4. **First integrals and partial integrability.** Let us construct the following series

$$\sum_{k=1}^{n} \bar v_{n-k+1} z^{k-1} = e^t w'(z,t) \sum_{k=1}^{n} \bar\psi_{n-k+1} z^{k-1} + e^t w'(z,t) \sum_{k=n}^{\infty} b_k z^k. \tag{20}$$

Taking into account (18) and the formula for the derivative

$$\frac{\partial (e^t w')}{\partial t} = e^t w'(1 - p(w,t) - wp'(w,t)),$$

we come to the conclusion that $\dot{\bar v} = 0$ and $\bar v$ is constant. We denote by $(\mathcal{L}_1, \dots, \mathcal{L}_n)^T$ the vector of the first integrals of the Hamiltonian system (17–19). It is easily seen that they are the same as given by (10).

Indeed, equality (20) implies that $\mathcal{L}_k = \bar v_k$ are constants for all t and $k = 1, \dots, n$. Naturally,

$$[\mathcal{L}_j, H] = \sum_{k=1}^{n} \frac{\partial \mathcal{L}_j}{\partial c_k} \frac{\partial H}{\partial \overline{\psi_k}} - \frac{\partial \mathcal{L}_j}{\partial \overline{\psi_k}} \frac{\partial H}{\partial c_k} = \sum_{k=1}^{n} \frac{\partial \mathcal{L}_j}{\partial c_k} \dot c_k + \frac{\partial \mathcal{L}_j}{\partial \overline{\psi_k}} \dot{\bar\psi}_k = \dot{\mathcal{L}}_j = 0.$$

The commutator relations are

$$[\mathcal{L}_j, \mathcal{L}_k] = (j-k) L_{k+j}, \quad \text{when } k + j \le n, \tag{21}$$

or 0 otherwise. This implies that

- the first integrals $(\mathcal{L}_{[(n+1)/2]}, \dots, L_n)$ are pairwise involutory;
- the integrals $(\mathcal{L}_1, \dots, L_{[(n-1)/2]})$ are not pairwise involutory, but their Lie–Poisson brackets give all the rest of integrals.

Here $[\cdot]$ within the index field means the integer part. It is clear from the form of the matrix in the above representation of $\mathcal{L}_k$, $k = 1, \dots, n$, that all these integrals are algebraically (even linearly) independent. Therefore, the Hamiltonian system (17–19) is partially integrable in the Liouville sense.

Remark. All previous considerations have been done for the class $\tilde{\mathbf{S}}$. But the result on partial integrability is still valid for the whole class $\mathbf{S}$ going inside the unit disk by $f \to \frac{1}{r} f(rz)$, and letting $r \to 1$.

Remark. The complete integration of this Hamiltonian system requires additional information on the trajectories, in particular, on the controls $p_1, p_2, \dots$. One way to perform such integration is to solve of the extremal problem of finding the boundary hypersurfaces of $\mathcal{M}_n$ by optimal control methods, see [30].

Remark. In view of Hamiltonian mechanics, our Hamiltonian system describes 'trivial' motion with constant velocity because the Hamiltonian function is linear with respect to ψ. An attempt to get a non-trivial description of the Löwner–Kufarev motion was launched in [34] by introducing into consideration a special Lagrangian.

Remark. The coefficient bodies $\mathcal{M}_1$, $\mathcal{M}_2, \dots$ generate a hierarchy of Hamiltonian systems (17-19)

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