ON A PRIORI ESTIMATES OF SOLUTIONS
OF SYSTEMS OF HIGHER ORDER NONLINEAR
FUNCTIONAL-DIFFERENTIAL INEQUALITIES

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*Dedicated to the blessed memory
of Academician I. Vekua*

Abstract. Systems of higher order nonlinear functional-differential inequalities arising in the theory of boundary value problems as well as in the stability theory are considered on a finite and an infinite interval. On a finite interval, a priori estimates are obtained for solutions of these systems satisfying boundary conditions of periodic type, and on an infinite interval, a priori estimates are established for the derivatives of bounded solutions.

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INTRODUCTION

The proof of theorems on the solvability of boundary value problems for nonlinear differential equations and systems led to finding a priori estimates of solutions of nonlinear differential inequalities and systems of such inequalities, satisfying the corresponding boundary conditions (see [1, 3–5, 7–12, 14], and the references therein). Such estimates are also applied to the stability theory (see, e.g., [2, 13]).

In this paper we investigate systems of higher order nonlinear functional-differential inequalities which arise in the theory of boundary value problems as well as in the stability theory.

For such systems, on a finite interval, new a priori estimates are established for solutions satisfying boundary conditions of periodic type (see Theorem 1.1 and Corollaries 1.1 and 1.2 below). These estimates clear the way for the investigation of boundary value problems with nonlinear boundary conditions of periodic type for systems of higher order nonlinear functional-differential equations.

On an infinite interval, for the above-mentioned systems of functional-differential inequalities, we prove a Esclangon–Landau type theorem (see Theorem 2.1), concerning a priori estimates of the derivatives of bounded solutions.

Throughout the paper, the following notation will be used:

$$\mathbb{R} = (-\infty, +\infty), \quad \mathbb{R}_+ = [0, +\infty);$$

$$[x]_+ = \frac{x+|x|}{2};$$

$\mathbb{R}^n$ is the n -dimensional real Euclidean space;

$x = (x_i)_{i=1}^n$ and $X = (x_{ik})_{i,k=1}^n$ are, respectively, the n -dimensional column vector and the $n \times n$ matrix with elements x_i and $x_{ik} \in \mathbb{R}$ ($i, k = 1, \dots, n$) and with the norms

$$\|x\| = \sum_{i=1}^n |x_i|, \quad \|X\| = \sum_{i,k=1}^n |x_{ik}|;$$

X^{-1} is the inverse of a non-singular matrix X ;

$r(X)$ is the spectral radius of X ;

I is the compact interval;

$C(I)$ is the space of continuous functions $u : I \rightarrow \mathbb{R}$ with the norm

$$\|u\|_{C(I)} = \max\{|u(t)| : t \in I\};$$

$\mu(u) = \min\{|u(t)| : t \in I\};$

$C^n(I)$ is the space of n -times continuously differentiable functions $u : I \rightarrow \mathbb{R}$ with the norm

$$\|u\|_{C^n(I)} = \sum_{k=0}^n \|u\|_{C^k(I)};$$

$\tilde{C}^n(I)$ is the space of functions $u : I \rightarrow \mathbb{R}$, absolutely continuous together with their derivatives up to order n inclusive.

$\tilde{C}_{loc}^n(\mathbb{R}_+)$ is the space of functions $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ whose restrictions to any compact interval $I \subset \mathbb{R}_+$ belong to $\tilde{C}^n(I)$.

1. SYSTEMS WITH BOUNDARY CONDITIONS OF PERIODIC TYPE

Let $-\infty < a < b < +\infty$, $I = [a, b]$, m and n_i ($i = 1, \dots, m$) be natural numbers. On the interval I , we consider the system of functional-differential inequalities

$$(-1)^{n_i-1} u_i^{(2n_i)}(t) \operatorname{sgn}(u_i(t)) \leq \sum_{j=1}^m \sum_{k=1}^{n_j} p_{ijk}(t) \|u_j^{(k-1)}\|_{C(I)} + q(t) \quad (1.1)$$

$$(i = 1, \dots, m)$$

with the boundary conditions of periodic type

$$u_i^{(k-1)}(b) = u_i^{(k-1)}(a), \quad (-1)^{n_i-k} [u_i^{(2n_i-k)}(b) - u_i^{(2n_i-k)}(a)] u_i^{(k-1)}(a) \geq \phi_{ik}(|u_i^{(k-1)}(a)|) \quad (k = 1, \dots, n_i; i = 1, \dots, m). \quad (1.2)$$

Here $p_{ijk} : I \rightarrow \mathbb{R}_+$ ($k = 1, \dots, n_j$; $i, j = 1, \dots, m$) and $q : I \rightarrow \mathbb{R}_+$ are Lebesgue integrable functions, and $\phi_{ik} : \mathbb{R}_+ \rightarrow \mathbb{R}$ ($k = 1, \dots, n_i$; $i = 1, \dots, m$) are non-decreasing functions.

A particular case of (1.2) is the periodic boundary conditions

$$u_i^{(k-1)}(b) = u_i^{(k-1)}(a) \quad (k = 1, \dots, 2n_i; i = 1, \dots, m) \quad (1.3)$$

(in this case $\phi_{ik}(x) \equiv 0$). Thus it is natural to call (1.2) boundary conditions of periodic type.

A vector function $(u_i)_{i=1}^m : I \rightarrow \mathbb{R}^m$ is said to be **a solution of system (1.1)** if

$$u_i \in \tilde{C}^{2n_i-1}(I) \quad (i = 1, \dots, n_i; \quad i = 1, \dots, m),$$

and almost everywhere on I the differential inequalities (1.1) are satisfied. A solution of system (1.1), that satisfies the boundary conditions (1.2) is said to be **a solution of problem (1.1), (1.2)**.

For any $\varepsilon > 0$ and $u \in C(I)$, set

$$\nu_i(u; \varepsilon) = [\mu^2(u) - \varepsilon(\phi_{i1}(\mu(u)) - \phi_{i1}(0))]_{+} + \sum_{k=1}^{n_i} |\phi_{ik}(0)| \quad (i = 1, \dots, m). \quad (1.4)$$

Moreover, put

$$h_{ij} = \frac{(b-a)^2}{4} \sum_{k=1}^{n_j} \left(\frac{b-a}{2\pi} \right)^{n_i+n_j-k-1} \int_a^b p_{ijk}(t) dt, \quad H = (h_{ij})_{i,j=1}^m. \quad (1.5)$$

Theorem 1.1. *If*

$$r(H) < 1, \quad (1.6)$$

then there exist positive constants ε and ρ , independent of the functions q and ϕ_{ik} ($k = 1, \dots, n_i; i = 1, \dots, m$), such that an arbitrary solution $(u_i)_{i=1}^m$ of problem (1.1), (1.2) admits the estimate

$$\sum_{i=1}^m \|u_i\|_{C^{n_i-1}(I)} < \rho \left(1 + \int_a^b q(t) dt + \sum_{i=1}^m \nu_i(u_i; \varepsilon) \right). \quad (1.7)$$

To prove this theorem, we need

Lemma 1.1. *If $n \geq 1$, $u \in C^n(I)$, and*

$$u^{(i-1)}(b) = u^{(i-1)}(a) \quad (i = 1, \dots, n), \quad (1.8)$$

then

$$\|u\|_{C(I)} \leq \mu(u) + \frac{(b-a)^{1/2}}{2} \left(\frac{b-a}{2\pi} \right)^{n-1} \left(\int_a^b |u^{(n)}(t)|^2 dt \right)^{1/2}. \quad (1.9)$$

Proof. We choose $t_0 \in [a, b)$ so that

$$|u(t_0)| = \mu(u).$$

Suppose

$$\omega = b - a, \quad v(t) = \begin{cases} u(t) - u(t_0) & \text{for } t_0 \leq t < b, \\ u(t - \omega) - u(t_0) & \text{for } b \leq t \leq t_0 + \omega. \end{cases}$$

Then $v \in \tilde{C}([t_0, t_0 + \omega])$,

$$v(t_0) = v(t_0 + \omega) = 0, \quad (1.10)$$

$$\int_{t_0}^{t_0+\omega} |v'(t)| dt = \int_a^b |u'(t)| dt, \quad (1.11)$$

and

$$\|u\|_{C(I)} \leq \mu(u) + \|v\|_{C(I)}. \quad (1.12)$$

Choose $t_1 \in [t_0, t_0 + \omega]$ so that

$$|v(t_1)| = \|v\|_{C(I)}.$$

Then in view of (1.10) we have

$$\begin{aligned} |v(t_1)| &= \left| \int_{t_0}^{t_1} v'(s) ds \right| \leq \int_{t_0}^{t_1} |v'(s)| ds, \\ |v(t_1)| &= \left| \int_{t_1}^{t_0+\omega} v'(s) ds \right| \leq \int_{t_1}^{t_0+\omega} |v'(s)| ds. \end{aligned}$$

Therefore

$$2|v(t_1)| \leq \int_{t_0}^{t_1} |v'(s)| ds + \int_{t_1}^{t_0+\omega} |v'(s)| ds = \int_{t_0}^{t_0+\omega} |v'(s)| ds.$$

Due to this inequality, from (1.11) and (1.12) we find

$$\|u\|_{C(I)} \leq \mu(u) + \frac{1}{2} \int_a^b |u'(s)| ds$$

and

$$\|u\|_{C(I)} \leq \mu(u) + \frac{(b-a)^{1/2}}{2} \left(\int_a^b |u'(s)|^2 ds \right)^{1/2}. \quad (1.13)$$

Thus the lemma is proved for the case $n = 1$.

Suppose now that $n > 1$. Then by the Wirtinger theorem (see [6], Theorem 258) we have

$$\left(\int_a^b |u'(s)|^2 ds \right)^{1/2} \leq \left(\frac{b-a}{2\pi} \right)^{n-1} \left(\int_a^b |u^{(n)}(t)|^2 dt \right)^{1/2}.$$

Therefore (1.13) results in (1.9). $\square$

Proof of Theorem 1.1. Due to (1.5) and (1.6), there exists $\delta \in (0, 1)$ so small that

$$r(H_\delta) < 1, \quad (1.14)$$

where

$$H_\delta = ((1 + \delta)h_{ij})_{i,j=1}^m. \quad (1.15)$$

Let $(u_i)_{i=1}^m$ be a solution of problem (1.1), (1.2). Then by Lemma 1.1, for every $j \in \{1, \dots, m\}$ we have

$$\|u_j\|_{C(I)} \leq \frac{(b-a)^{1/2}}{2} \left(\frac{b-a}{2\pi} \right)^{n_j-1} \left(\ell_j \mu(u_j) + \left(\int_a^b |u_j^{(n_j)}(t)|^2 dt \right)^{1/2} \right),$$

where

$$\ell_j = 2(b-a)^{-1/2} \left(\frac{2\pi}{b-a} \right)^{n_j-1}.$$

Moreover, if $n_j \geq 2$, then

$$\|u_j^{(k-1)}\|_{C(I)} \leq \frac{(b-a)^{1/2}}{2} \left(\frac{b-a}{2\pi} \right)^{n_j-k} \left(\int_a^b |u_j^{(n_j)}(t)|^2 dt \right)^{1/2} \quad (k = 2, \dots, n_j)$$

since

$$\mu(u_j^{(k-1)}) = 0 \quad (k = 2, \dots, n_j).$$

Therefore, if we suppose that

$$x_j = 1 + \ell_j \mu(u_j) + \left(\int_a^b |u_j^{(n_j)}(t)|^2 dt \right)^{1/2} \quad (j = 1, \dots, m), \quad (1.16)$$

then

$$\|u_j^{(k-1)}\|_{C(I)} < \frac{(b-a)^{1/2}}{2} \left(\frac{b-a}{2\pi} \right)^{n_j-k} x_j \quad (k = 1, \dots, n_j; j = 1, \dots, m). \quad (1.17)$$

By virtue of (1.1) and (1.17), the following inequalities

$$\begin{aligned} (-1)^{n_i-1} u_i^{(2n_i)}(t) u_i(t) &\leq \left(\sum_{j=1}^m \sum_{k=1}^{n_j} p_{ijk}(t) \|u_j^{(k-1)}\|_{C(I)} + q(t) \right) \|u_i\|_{C(I)} \\ &\leq \sum_{j=1}^m \left(\frac{b-a}{4} \sum_{k=1}^{n_j} \left(\frac{b-a}{2\pi} \right)^{n_i+n_j-k-1} p_{ijk}(t) \right) x_j x_i + \frac{q(t)}{\ell_i} x_i \quad (i = 1, \dots, m) \end{aligned}$$

are satisfied almost everywhere on I . If we integrate these inequalities from a to b , and take into account notation (1.5), we get

$$\begin{aligned} \sum_{k=1}^{n_i} (-1)^{n_i-k} [u_i^{(2n_i-k)}(b) u_i^{(k-1)}(b) - u_i^{(2n_i-k)}(a) u_i^{(k-1)}(a)] + \int_a^b |u_i^{(n_i)}(t)|^2 dt \\ \leq \left(\sum_{j=1}^m h_{ij} x_j + y_i \right) x_i \quad (i = 1, \dots, m), \end{aligned}$$

where

$$y_i = \ell_i^{-1} \int_a^b q(t) dt \quad (i = 1, \dots, m). \quad (1.18)$$

Hence, by (1.2), it follows that

$$\begin{aligned} \frac{(1 + \ell_i \mu(u_i))^2}{\delta} + \int_a^b |u_i^{(n_i)}(t)|^2 dt &\leq \frac{(1 + \ell_i \mu(u_i))^2}{\delta} - (\phi_{i1}(\mu_i(u)) - \phi_{i1}(0)) \\ &+ \sum_{k=1}^{n_i} |\phi_{ik}(0)| + \left(\sum_{j=1}^m h_{ij} x_j + y_i \right) x_i \quad (i = 1, \dots, m). \end{aligned} \quad (1.19)$$

On the other hand, in view of (1.16) we have

$$\frac{(1 + \ell_i \mu(u_i))^2}{\delta} + \int_a^b |u_i^{(n_i)}(t)|^2 dt \geq x_i^2 / (1 + \delta) \quad (i = 1, \dots, m).$$

If, moreover, we take into account notation (1.4) and (1.16), then from (1.19) we find

$$x_i \leq (1 + \delta) \sum_{j=1}^m h_{ij} x_j + z_i \quad (i = 1, \dots, m),$$

where

$$z_i = \frac{1 + \delta}{\varepsilon} (1 + \nu_i(u_i; \varepsilon)) + (1 + \delta) y_i \quad (i = 1, \dots, m) \quad (1.20)$$

and

$$\varepsilon = \min \left\{ \frac{\delta}{2 + 2\ell_i^2} : i = 1, \dots, m \right\}.$$

Hence, according to (1.15), we obtain

$$\mathbf{x} \leq H_\delta \mathbf{x} + \mathbf{z}, \quad (1.21)$$

where

$$\mathbf{x} = (x_i)_{i=1}^n, \quad \mathbf{z} = (z_i)_{i=1}^n.$$

By virtue of condition (1.14) and the non-negativeness of the matrix H_δ , inequality (1.21) yields

$$\mathbf{x} \leq (E - H_\delta)^{-1} \mathbf{z},$$

where E is the $m \times m$ unit matrix. Consequently,

$$\sum_{i=1}^m x_i \leq \|(E - H_\delta)^{-1}\| \sum_{i=1}^m z_i. \quad (1.22)$$

On the other hand, from (1.17) we find

$$\sum_{i=1}^m \|u_i\|_{C^{n_i-1}(I)} \leq \rho_1 \sum_{i=1}^m x_i, \quad (1.23)$$

where

$$\rho_1 = \frac{(b-a)^{1/2}}{2} \sum_{i=1}^m \sum_{k=1}^{n_i} \left(\frac{b-a}{2\pi} \right)^{n_i-k}.$$

If we take into account the fact that

$$\ell_i^{-1} \leq \rho_1 \quad (i = 1, \dots, m),$$

then from (1.18) and (1.20) we get

$$\sum_{i=1}^m z_i \leq m(1 + \delta)(\varepsilon^{-1} + \rho_1) \left(1 + \int_a^b q(t) dt + \sum_{i=1}^m \nu_i(u_i; \varepsilon) \right). \quad (1.24)$$

Inequalities (1.22)–(1.24) result in (1.7), where

$$\rho = m\rho_1(1 + \delta)(\varepsilon^{-1} + \rho_1) \|(E - H_\delta)^{-1}\|;$$

at that, the constants ε and ρ do not depend on q , ϕ_{ik} ($k = 1, \dots, n_i; i = 1, \dots, m$), and $(u_i)_{i=1}^m$. $\square$

Corollary 1.1. *Let inequality (1.6) hold. Then there exist positive constants ε and ρ_0 , independent of the functions q and ϕ_{ik} ($k = 1, \dots, n_i; i = 1, \dots, m$), such that if for some $x_0 > 0$ the inequalities*

$$\varepsilon(\phi_{i1}(x) - \phi_{i1}(0)) \geq x^2 \quad \text{for } x > x_0 \quad (i = 1, \dots, m) \quad (1.25)$$

are satisfied, then an arbitrary solution $(u_i)_{i=1}^m$ of problem (1.1), (1.2) admits the estimate

$$\sum_{i=1}^m \|u_i\|_{C^{n_i-1}(I)} \leq \rho_0 \left(1 + x_0^2 + \int_a^b q(t) dt + \sum_{i=1}^m \sum_{k=1}^{n_i} |\phi_{ik}(0)| \right). \quad (1.26)$$

Proof. Due to (1.4) and (1.25), we have

$$\nu_i(u_i; \varepsilon) \leq \sum_{k=1}^{n_i} |\phi_{ik}(0)| \quad \text{for } \mu(u_i) \geq x_0^2,$$

and

$$\nu_i(u_i; \varepsilon) \leq x_0^2 + \sum_{k=1}^{n_i} |\phi_{ik}(0)| \quad \text{for } \mu(u_i) \leq x_0^2.$$

Therefore (1.7) yields estimate (1.26), where $\rho_0 = m\rho$. $\square$

If $\phi_{ik}(x) \equiv 0$ ($k = 1, \dots, n_i; i = 1, \dots, m$), then (1.3) implies (1.2). Therefore Theorem 1.1 gives rise to

Corollary 1.2. *If inequality (1.6) holds, then there exists a positive constant ρ , independent of the function q , such that an arbitrary solution $(u_i)_{i=1}^m$ of problem (1.1), (1.3) admits the estimate*

$$\sum_{i=1}^m \|u_i\|_{C^{n_i-1}(I)} \leq \rho \left(1 + \int_a^b q(t) dt + \sum_{i=1}^m \mu^2(u_i) \right).$$

2. SYSTEMS WITH BOUNDED SOLUTIONS

On the interval $\mathbb{R}_+$, we consider the system of differential inequalities

$$|u_i^{(n_i)}(t)| \leq \sum_{j=1}^m \sum_{k=1}^{n_i-1} p_{ijk} \|u_j^{(k)}\|_{C([0,t])}^{\frac{n_j}{k}} + q \left(\sum_{j=1}^m \sum_{k=1}^{n_i-1} \|u_j^{(k)}\|_{C([0,t])}^{\frac{n_j}{k}} \right) \quad (2.1)$$

$(i = 1, \dots, m)$

with the conditions

$$|u_i(t)| \leq \ell_i \quad \text{for } t \in \mathbb{R}_+ \quad (i = 1, \dots, m). \quad (2.2)$$

Here $n_i \geq 2$ ($i = 1, \dots, m$), $p_{ijk} \geq 0$, $\ell_i > 0$ ($k = 1, \dots, n_j$; $i, j = 1, \dots, m$), and $q : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a non-decreasing function.

A vector function $(u_i)_{i=1}^m : \mathbb{R}_+ \rightarrow \mathbb{R}^m$ is said to be a **solution of system (2.1)** if

$$u_i \in \tilde{C}_{loc}^{n_i-1}(\mathbb{R}_+) \quad (i = 1, \dots, m),$$

and almost everywhere on $\mathbb{R}_+$ inequalities (2.1) are satisfied. A solution of system (2.1) that satisfies conditions (2.2) is said to be a **solution of problem (2.1), (2.2)**.

Put

$$h_{ij} = \sum_{k=1}^{n_i-1} 2^{(n_j-k)n_j} \ell_j^{\frac{n_j-k}{k}} p_{ijk}, \quad H = (h_{ij})_{i,j=1}^m. \quad (2.3)$$

Theorem 2.1. *If*

$$r(H) < 1 \quad (2.4)$$

and

$$\lim_{x \rightarrow +\infty} \frac{q(x)}{x} = 0, \quad (2.5)$$

then there exists a positive constant ρ such that an arbitrary solution $(u_i)_{i=1}^m$ of problem (2.1), (2.2) admits the estimate

$$|u_i^{(k)}(t)| \leq \rho \quad \text{for } t \in \mathbb{R}_+ \quad (k = 1, \dots, n_i - 1; \quad i = 1, \dots, m). \quad (2.6)$$

Proof. Let $i \in \{1, \dots, m\}$ be fixed arbitrarily. For any $t \in \mathbb{R}_+$ we set

$$v_k(t) = \min\{|u_i^{(k)}(s)| : 0 \leq s \leq t\}, \quad w_k(t) = \|u_i^{(k)}\|_{C([0,t])} \quad (k = 1, \dots, n_i - 1),$$

and

$$w_{n_i}(t) = \text{ess sup}\{|u_i^{(n_i)}(s)| : 0 \leq s \leq t\}, \quad w_0(t) = \ell_i.$$

First we suppose that

$$w_k(t) \not\equiv 0 \quad (k = 1, \dots, n_i - 1). \quad (2.7)$$

Then there exist $t_0 > 0$ and $\delta \in (0, \ell_i)$ such that

$$w_k(t) \geq \delta \quad \text{for } t \geq t_0 \quad (k = 1, \dots, n_i - 1). \quad (2.8)$$

On the other hand, in view of (2.2) we have

$$\lim_{t \rightarrow +\infty} v_k(t) = 0 \quad (k = 1, \dots, n_i - 1). \quad (2.9)$$

According to (2.8), for any $k \in \{1, \dots, n_i - 1\}$ and $t \geq t_0$, there exist s_{kt} and $s_{kt}^* \in [0, t]$ such that

$$|u_i^{(k)}(s_{kt})| = v_k(t), \quad |u_i^{(k)}(s_{kt}^*)| = w_k(t),$$

and

$$u_i^{(k)}(s) \neq 0 \quad \text{for} \quad \min\{s_{kt}, s_{kt}^*\} \leq s \leq \max\{s_{kt}, s_{kt}^*\}.$$

If along with this we take into account inequalities (2.2) and (2.8), then from the equality

$$|u_i^{(k)}(s_{kt}^*)|^2 = |u_i^{(k)}(s_{kt})|^2 + 2 \int_{s_{kt}}^{s_{kt}^*} u_i^{(k)}(s) u_i^{(k+1)}(s) ds$$

we obtain

$$\begin{aligned} w_k^2(t) &\leq v_k^2(t) + 2w_{k+1}(t) \left| \int_{s_{kt}}^{s_{kt}^*} u^{(k)}(s) ds \right| \\ &= v_k^2(t) + 2w_{k+1}(t) |u^{(k-1)}(s_{kt}^*) - u^{(k-1)}(s_{kt})| \leq v_k^2(t) + 4w_{k+1}(t)w_{k-1}(t) \\ &\leq \left(4 + \frac{v_k^2(t)}{\delta^2}\right) w_{k+1}(t)w_{k-1}(t) \quad \text{for } t \geq t_0 \quad (k = 1, \dots, n_i - 1). \end{aligned}$$

Hence, by (2.9), it follows that

$$w_k(t) \leq \gamma_i(t) w_k^{1/2}(t) w_{k-1}^{1/2}(t) \quad \text{for } t \geq t_0 \quad (k = 1, \dots, n_i - 1), \quad (2.10)$$

where

$$\gamma_i(t) = 2 + \delta^{-1} \sum_{k=1}^{n_i-1} v_k(t) \rightarrow 2 \quad \text{as } t \rightarrow +\infty. \quad (2.11)$$

By induction, from (2.10) we find

$$w_k(t) \leq \gamma_i^k(t) w_0^{\frac{1}{k+1}}(t) w_{k+1}^{\frac{k}{k+1}}(t) = \gamma_i^k(t) \ell_i^{\frac{1}{k+1}} w_{k+1}^{\frac{k}{k+1}}(t) \quad \text{for } t \geq t_0 \quad (k = 1, \dots, n_i - 1)$$

and

$$w_k(t) \leq \gamma_i^{k(n_i-k)}(t) \ell_i^{\frac{n_i-k}{n_i}} w_{n_i}^{\frac{k}{n_i}}(t) \quad \text{for } t \geq t_0 \quad (k = 1, \dots, n_i - 1). \quad (2.12)$$

If (2.7) is violated, i.e., $w_{k_0}(t) \equiv 0$ for some $k_0 \in \{1, \dots, n_i - 1\}$, then in view of (2.2) it is obvious that $u_i(t) \equiv \text{const}$, and

$$v_k(t) \equiv w_k(t) \equiv 0 \quad (k = 1, \dots, n_i - 1).$$

Consequently, inequalities (2.11) are satisfied in this case, too.

By virtue of the inequality $r(H) < 1$, there exists $\varepsilon > 0$ such that

$$r(H_\varepsilon) < 1, \quad \text{where } H_\varepsilon = ((1 + \varepsilon)h_{ij})_{i,j=1}^m. \quad (2.13)$$

On the other hand, due to (2.11) and (2.12), there exists $t^* > t_0$ such that

$$\begin{aligned} \|u_i^{(k)}\|_{C([0,t])} &\leq (1 + \varepsilon)^{\frac{1}{n}} 2^{k(n_i-k)} \ell_i^{\frac{n_i-k}{n_i}} y_i^{\frac{k}{n_i}}(t) \quad \text{for } t \geq t^* \\ &\quad (k = 1, \dots, n_i - 1; \quad i = 1, \dots, m), \end{aligned} \quad (2.14)$$

where $n = \max\{n_1, \dots, n_m\}$, and

$$y_i(t) = \text{ess sup}\{\|u_i^{(n_i)}(s)\| : 0 \leq s \leq t\} \quad (i = 1, \dots, m).$$

By virtue of (2.3) and (2.14), from (2.1) we obtain

$$y_i(t) \leq (1 + \varepsilon) \sum_{j=1}^m h_{ij} y_j(t) + q(\ell_0 y_0(t)) \quad \text{for } t \geq t^* \quad (i = 1, \dots, m),$$

where

$$y_0(t) = \sum_{i=1}^m y_i(t), \quad \ell_0 = (1 + \varepsilon) \sum_{j=1}^m \sum_{k=1}^{n_j-1} 2^{(n_j-k)n_j} \ell_j^{\frac{n_j-k}{k}}.$$

Consequently,

$$\mathbf{y}(t) \leq H_\delta \mathbf{y}(t) + \bar{\mathbf{q}}(t) \quad \text{for } t \geq t_0,$$

where

$$\mathbf{y}(t) = (y_i(t))_{i=1}^m, \quad \bar{\mathbf{q}}(t) = (\bar{q}_i(t))_{i=1}^m, \quad \bar{q}_i(t) = q(\ell_0 y_0(t)) \quad (i = 1, \dots, m).$$

Hence, due to (2.13), we find

$$\mathbf{y}(t) \leq (E - H_\varepsilon)^{-1} \bar{\mathbf{q}}(t) \quad \text{for } t \geq t^*.$$

Consequently,

$$y_0(t) \leq \|(E - H_\varepsilon)^{-1}\| q(\ell_0 y_0(t)) \quad \text{for } t \geq t^*. \quad (2.15)$$

Due to (2.5), there exists $x_0 > 0$ such that

$$\|(E - H_\varepsilon)^{-1}\| q(\ell_0 x) < x \quad \text{for } x \geq x_0.$$

Thus (2.15) yields

$$y_0(t) \leq x_0 \quad \text{for } t \geq t^*.$$

In view of the last inequality, (2.14) results in (2.6), where

$$\rho = (1 + \varepsilon) \sum_{i=1}^m \sum_{k=1}^{n_i-1} 2^{k(n_i-k)} \ell_i^{\frac{n_i-k}{n_i}} x_0^{\frac{k}{n_i}}$$

is a constant independent of $(u_i)_{i=1}^m$. □

As an example we consider the functional-differential equation

$$u^{(n)}(t) = f(t, u(\tau_1(t)), \dots, u^{(n-1)}(\tau_n(t))) \quad (2.16)$$

with the condition

$$|u(t)| \leq \ell \quad \text{for } t \in \mathbb{R}_+, \quad (2.17)$$

where $n \geq 2$, $\ell > 0$, the function $f : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ is measurable in the first argument and continuous in the last n arguments, and $\tau_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ ($i = 1, \dots, n$) are measurable functions. Moreover,

$$\tau_i(t) \leq t \quad \text{for } t \in \mathbb{R}_+ \quad (i = 2, \dots, n). \quad (2.18)$$

Corollary 2.1. *Let inequalities (2.18) be satisfied and there exist a non-decreasing function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and a number $\alpha \in (0, 1)$ such that*

$$|f(t, x_1, \dots, x_n)| \leq g(|x_1|) \left(1 + \sum_{k=2}^n |x_k|^{\frac{n\alpha}{k-1}} \right). \quad (2.19)$$

Then there exists a positive constant ρ such that an arbitrary solution u of problem (2.16), (2.17) admits the estimates

$$|u^{(k)}(t)| \leq \rho \quad \text{for } t \in \mathbb{R}_+ \quad (k = 1, \dots, n-1). \quad (2.20)$$

Proof. In view of (2.18) and (2.19), an arbitrary solution of problem (2.16), (2.17) is a solution of the differential inequality

$$|u^{(n)}(t)| \leq q \left(\sum_{k=1}^{n-1} \|u^{(k)}\|_{C([0,t])}^{\frac{n}{k}} \right), \quad (2.21)$$

too, where

$$q(x) = g(\ell)(1 + nx^\alpha)$$

and

$$\lim_{x \rightarrow +\infty} \frac{q(x)}{x} = 0. \quad (2.22)$$

However, by Theorem 2.1, if condition (2.22) is fulfilled, then there exists $\rho > 0$ such that an arbitrary solution of problem (2.21), (2.17) admits estimates (2.20). $\square$

If

$$f(t, x_1, \dots, x_n) = \sum_{k=1}^n p_k(t)x_k + p_0(t),$$

where $p_k : \mathbb{R}_+ \rightarrow \mathbb{R}$ ($k = 0, \dots, n$) are essentially bounded measurable functions, then for $\alpha = \frac{n-1}{n}$, on $\mathbb{R}_+ \times \mathbb{R}^n$ inequality (2.19) is satisfied, where

$$g(x) = n(1+x) \operatorname{ess\,sup} \left\{ 1 + \sum_{k=0}^n |p_k(t)| : t \in \mathbb{R}_+ \right\}.$$

Thus from Corollary 2.1 there follows the well-known Esclangon–Landau theorem [13] on the boundedness of the derivatives of an arbitrary bounded solution of the differential equation

$$u^{(n)} = \sum_{k=1}^n p_k(t)u^{(k-1)} + p_0(t).$$

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