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ON DOUBLY PERIODIC SOLUTIONS OF NONLINEAR HYPERBOLIC EQUATIONS OF HIGHER ORDER

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Abstract. Unimprovable conditions of the existence and uniqueness of doubly periodic solutions are established for nonlinear hyperbolic equations of higher order with two independent variables.

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1. Statement of the Problem and the Main Results

Consider the nonlinear hyperbolic equation

$$u^{(2m,2n)} = \sum_{k=0}^{n-1}\Big(a_{mk}u^{(2m,2k)} + b_{mk}u^{(2m,2k+1)}\Big) + \sum_{i=0}^{m-1}\Big(a_{in}u^{(2i,2n)} + c_{in}u^{(2i+1,2n)}\Big)$$
$$+ \sum_{i=0}^{m-1}\sum_{k=0}^{n-1}\Big(a_{ik}u^{(2i,2k)} + b_{ik}u^{(2i,2k+1)} + c_{ik}u^{(2i+1,2k)}\Big) + f(x,y,u), \tag{1.1}$$

where a_{ik}, b_{ik}, c_{ik} are real constants, $f : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function, and for any i and k

$$u^{(i,k)}(x,y) = \frac{\partial^{i+k}u(x,y)}{\partial x^i \partial y^k}.$$

A function $u : \mathbb{R}^2 \to \mathbb{R}$ is called a *solution of equation* (1.1) if it is continuous together with all its derivatives $u^{(i,j)}$ $(i = 0, 1, \dots, 2m;\ k = 0, 1, \dots, 2n)$ and satisfies equation (1.1) everywhere in $\mathbb{R}^2$.

Let ω_1 and ω_2 be positive numbers. A solution u of equation (1.1) is called (ω_1, ω_2)-periodic if

$$u(x + \omega_1, y) = u(x, y), \qquad u(x, y + \omega_2) = u(x, y) \quad \text{for} \quad (x, y) \in \mathbb{R}^2.$$

Problems on the existence of (ω_1, ω_2)-periodic solutions to hyperbolic equations of second and fourth orders were studied in [1–4, 8–14], and for higher order equations in [6]. In the present paper new, unimprovable in a sense, conditions of existence and uniqueness of (ω_1, ω_2)-solutions are established.

Naturally, we assume that f is (ω_1, ω_2)-periodic with respect to the first two variables, i.e., the following equalities hold in $\mathbb{R}^3$

$$f(x + \omega_1, y, z) = f(x, y, z), \qquad f(x, y + \omega_2, z) = f(x, y, z).$$

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Theorem 1.1. *Let*

$$(-1)^{m+n+i+k}a_{ik} \geq 0 \quad (i=0,\dots,m;\ k=0,\dots,n;\ i+k<m+n),\quad a_{0n}\neq 0,\quad a_{m0}\neq 0, \tag{1.2}$$

and let there exist positive constants a, b *and* c *such that*

$$\sum_{k=0}^{n-1}\Big(\frac{\omega_2}{2\pi}\Big)^{2n-2k}|a_{mk}| + \sum_{i=0}^{m-1}\Big(\frac{\omega_1}{2\pi}\Big)^{2m-2i}|a_{in}| + \sum_{i=0}^{m-1}\sum_{k=0}^{n-1}\Big(\frac{\omega_1}{2\pi}\Big)^{2m-2i}\Big(\frac{\omega_2}{2\pi}\Big)^{2n-2k}|a_{ik}| + \Big(\frac{\omega_1}{2\pi}\Big)^{m}\Big(\frac{\omega_2}{2\pi}\Big)^{n} b^{\frac{1}{2}} < 1, \tag{1.3}$$

$$a|z| - c \leq (-1)^{m+n} f(x,y,z) \leq b|z| + c \quad \text{for} \quad (x,y,z)\in\mathbb{R}^3. \tag{1.4}$$

Then equation (1.1) *has at least one* (ω_1,ω_2)*-periodic solution.*

Theorem 1.2. *Let inequalities* (1.2) *and* (1.3) *hold, and*

$$a|z_1 - z_2| \leq (-1)^{m+n}\big(f(x,y,z_1) - f(x,y,z_2)\big) \leq b|z_1 - z_2|, \tag{1.5}$$

where a *and* b *are positive constants. Then equation* (1.1) *has one and only one* (ω_1,ω_2)*-periodic solution.*

For higher order ordinary differential equations results similar to Theorems 1.1 and 1.2 were obtained by I. Kiguradze and T. Kusano in [7].

Example 1.1. Let $\varepsilon \in (0,1)$ be an arbitrarily small number and

$$\delta = \frac{1-\sqrt{1-\varepsilon}}{2}.$$

Consider the differential equation

$$u^{(2m,2n)} = \sum_{k=0}^{n-1} a_{mk}u^{(2m,2k)} + \sum_{i=0}^{m-1} a_{in}u^{(2i,2n)} + \sum_{i=1}^{m-1}\sum_{k=1}^{n-1} a_{ik}u^{(2i,2k)} + (-1)^{m+n} b\,u + \sin\frac{2\pi x}{\omega_1}\sin\frac{2\pi y}{\omega_2}, \tag{1.6}$$

where a_{ik} ($i=0,\dots,m;\ k=0,\dots,n;\ 2\leq i+k<m+n$) are constants satisfying inequalities (1.2), and

$$\sum_{k=0}^{n-1}\Big(\frac{\omega_2}{2\pi}\Big)^{2n-2k}|a_{mk}| + \sum_{i=0}^{m-1}\Big(\frac{\omega_1}{2\pi}\Big)^{2m-2i}|a_{in}| + \sum_{i=0}^{m-1}\sum_{k=0}^{n-1}\Big(\frac{\omega_1}{2\pi}\Big)^{2m-2i}\Big(\frac{\omega_2}{2\pi}\Big)^{2n-2k}|a_{ik}| = 1-\delta^2, \quad b = \delta^2\Big(\frac{2\pi}{\omega_1}\Big)^{2m}\Big(\frac{2\pi}{\omega_2}\Big)^{2n}. \tag{1.7}$$

It is clear that equation (1.6) satisfies all of the conditions of Theorems 1.1 and 1.2 except condition (1.3). Instead of (1.3) it satisfies the condition

$$\sum_{k=0}^{n-1}\left(\frac{\omega_2}{2\pi}\right)^{2n-2k}|a_{mk}|+\sum_{i=0}^{m-1}\left(\frac{\omega_1}{2\pi}\right)^{2m-2i}|a_{in}|$$
$$+\sum_{i=0}^{m-1}\sum_{k=0}^{n-1}\left(\frac{\omega_1}{2\pi}\right)^{2m-2i}\left(\frac{\omega_2}{2\pi}\right)^{2n-2k}|a_{ik}| \;+\left(\frac{\omega_1}{2\pi}\right)^{m}\left(\frac{\omega_2}{2\pi}\right)^{n}b^{\frac{1}{2}}<1+\varepsilon. \tag{1.8}$$

Let us show that equation (1.1) has no (ω_1,ω_2)-periodic solution. Assume the contrary that such a solution exists. Then

$$\int_0^{\omega_1}\int_0^{\omega_2}\sin\frac{2\pi x}{\omega_1}\sin\frac{2\pi y}{\omega_2}u^{(2i,2k)}(x,y)\,dx\,dy$$
$$=(-1)^{i+k}\left(\frac{2\pi}{\omega_1}\right)^{2i}\left(\frac{2\pi}{\omega_2}\right)^{2k}\int_0^{\omega_1}\int_0^{\omega_2}\sin\frac{2\pi x}{\omega_1}\sin\frac{2\pi y}{\omega_2}u(x,y)\,dx\,dy. \tag{1.9}$$

Multiplying both sides of equation (1.6) by $\sin\frac{2\pi x}{\omega_1}\sin\frac{2\pi y}{\omega_2}$, integrating over $[0,\omega_1]\times[0,\omega_2]$ and taking into account (1.7) and (1.9), we get the contradiction

$$\int_0^{\omega_1}\int_0^{\omega_2}\sin^2\frac{2\pi x}{\omega_1}\sin^2\frac{2\pi y}{\omega_2}\,dx\,dy=0.$$

The constructed example shows that in Theorems 1.1 and 1.2 inequality (1.3) cannot be replaced by inequality (1.9) whatever small $\varepsilon>0$ may be.

Theorem 1.3. *Let*

$$(-1)^{m+n+i+k}a_{ik}\le 0 \quad (i=0,\dots,m;\ k=0,\dots,n;\ i+k<m+n),\quad a_{0n}\ne 0,\quad a_{m0}\ne 0, \tag{1.10}$$

and let there exist a positive constant c such that

$$(-1)^{m+n}f(x,y,z)z<0 \quad \text{for} \quad (x,y)\in\mathbb{R}^2,\ |z|>c. \tag{1.11}$$

Then equation (1.1) has at least one (ω_1,ω_2)-periodic solution.

Example 1.2. Consider the differential equation

$$u^{(2m,2n)}=\sum_{k=0}^{n-1}a_{mk}u^{(2m,2k)}+\sum_{i=0}^{m-1}a_{in}u^{(2i,2n)}$$
$$+\sum_{i=1}^{m-1}\sum_{k=1}^{n-1}a_{ik}u^{(2i,2k)}+\frac{1}{1+u^2}, \tag{1.12}$$

where a_{ik} $(i=0,\dots,m;\ k=0,\dots,n;\ 2\le i+k<m+n)$ are the constants satisfying inequalities (1.10). Equation (1.12) satisfies all the conditions of

Theorem 1.3 except condition (1.11). Instead of (1.11) it satisfies the condition

$$\limsup_{|z|\to+\infty}(-1)^{m+n}f(x,y,z)\le 0 \quad \text{for} \quad (x,y)\in\mathbb{R}^2. \tag{1.13}$$

On the other hand, it is clear that equation (1.12) has no (ω_1,ω_2)-periodic solution for any positive ω_1 and ω_2. This example shows that in Theorem 1.3 condition (1.11) is optimal and cannot be replaced by condition (1.13).

Theorem 1.4. *Let all of the conditions of Theorem* 1.3 *hold and*

$$(-1)^{m+n}\big(f(x,y,z_1)-f(x,y,z_2)\big)(z_1-z_2)<0 \ \text{ for } (x,y)\in\mathbb{R}^2,\ z_1\ne z_2. \tag{1.14}$$

Then equation (1.1) *has one and only one* (ω_1,ω_2)*-periodic solution.*

In contrast to Theorems 1.1 and 1.2, Theorems 1.3 and 1.4 do not restrict the growth order of the function f with respect to the third argument. For example, the function

$$f(x,y,z)=(-1)^{m+n+1}p_0(x,y)\exp(p_1(x,y)z^2)z^{2l-1}+q(x,y),$$

where $p_0:\mathbb{R}^2\to(0,+\infty)$, $p_1:\mathbb{R}^2\to[0,+\infty)$ and $q:\mathbb{R}^2\to\mathbb{R}$ are arbitrary continuous (ω_1,ω_2)-periodic functions and l is an arbitrary natural number, satisfies conditions (1.11) and (1.14).

Example 1.3. Let inequalities (1.10) hold and

$$f(x,y,z)=(-1)^{m+n+1}p_0(x,y)f_0(z),$$

where

$$f_0(z)=\begin{cases}0 \quad \text{for} \quad |z|\le\delta,\\ z-\delta\,\operatorname{sgn} z \quad \text{for} \quad |z|>\delta,\end{cases}$$

δ is a positive constant, and $p_0:\mathbb{R}^2\to(0,+\infty)$ is a continuous (ω_1,ω_2)-periodic function. Then it is clear that f satisfies condition (1.11), where $c>\delta$. However, instead of (1.14) f satisfies the condition

$$(-1)^{m+n}\big(f(x,y,z_1)-f(x,y,z_2)\big)(z_1-z_2)\le 0 \quad \text{for} \quad (x,y)\in\mathbb{R}^2. \tag{1.15}$$

On the other hand, it is clear that for any $\gamma\in[-\delta,\delta]$ the constant function $u(x,y)=\gamma$ is a (ω_1,ω_2)-periodic solution of equation (1.1). Thus we have shown that in Theorem 1.4 condition (1.14) cannot be replaced by (1.15).

The equation

$$\begin{aligned}u^{(2m,2n)}&=\sum_{k=0}^{n-1}\Big(a_{mk}u^{(2m,2k)}+b_{mk}u^{(2m,2k+1)}\Big)+\sum_{i=0}^{m-1}\Big(a_{in}u^{(2i,2n)}+c_{in}u^{(2i+1,2n)}\Big)\\ &+\sum_{i=0}^{m-1}\sum_{k=0}^{n-1}\Big(a_{ik}u^{(2i,2k)}+b_{ik}u^{(2i,2k+1)}+c_{ik}u^{(2i+1,2k)}\Big)+p(x,y)u+q(x,y)\end{aligned} \tag{1.16}$$

is a particular case of equation (1.1), where p and $q:\mathbb{R}^2\to\mathbb{R}$ are continuous (ω_1,ω_2)-periodic functions. Theorems 1.2 and 1.4, respectively, imply the following Corollaries 1.1 and 1.2.

Corollary 1.1. *Let inequalities* (1.2) *and* (1.3) *hold, and*

$$a \le (-1)^{m+n} p(x,y) \le b \quad \textit{for} \quad (x,y) \in \mathbb{R}^2,$$

where a and b are positive constants. Then equation (1.16) *has one and only one (ω_1, ω_2)-periodic solution.*

Corollary 1.2. *If*

$$(-1)^{m+n} p(x,y) < 0 \quad \textit{for} \quad (x,y) \in \mathbb{R}^2$$

and inequalities (1.10) *hold, then equation* (1.16) *has one and only one (ω_1, ω_2)-periodic solution.*

2. Auxiliary Statements

2.1. Lemmas on a priori estimates. Denote by $C^{k,l}_{\omega_1\omega_2}$ the Banach space of continuous (ω_1, ω_2)-periodic functions u having continuous partial derivatives $u^{(i,j)}$ $(i = 0, \dots, k;\ j = 0, \dots, l)$, with the norm

$$\|u\|_{C^{k,l}_{\omega_1\omega_2}} = \max\bigg\{ \sum_{i=0}^{k} \sum_{j=0}^{l} |u^{(i,k)}(x,y)| : (x,y) \in \Omega \bigg\}.$$

Besides, we will use the notation $C^{0,0}_{\omega_1\omega_2} = C_{\omega_1\omega_2}$ and

$$\|u\|_{L^2_{\omega_1\omega_2}} = \Big(\int_0^{\omega_1} \int_0^{\omega_2} u^2(s,t)\, ds\, dt \Big)^{\frac{1}{2}}.$$

Set

$$\begin{aligned} \mathcal{L}(u)(x,y) = u^{(2m,2n)}(x,y) &- \sum_{k=0}^{n-1} \Big(a_{mk} u^{(2m,2k)}(x,y) + b_{mk} u^{(2m,2k+1)}(x,y) \Big) \\ &+ \sum_{i=0}^{m-1} \Big(a_{in} u^{(2i,2n)}(x,y) + c_{in} u^{(2i+1,2n)}(x,y) \Big) \\ + \sum_{i=0}^{m-1} \sum_{k=0}^{n-1} \Big(a_{ik} u^{(2i,2k)}(x,y) &+ b_{ik} u^{(2i,2k+1)}(x,y) + c_{ik} u^{(2i+1,2k)}(x,y) \Big), \end{aligned} \tag{2.1}$$

and consider the differential inequalities

$$a|u(x,y)| - c \le (-1)^{m+n} \mathcal{L}(u)(x,y) \operatorname{sgn} u(x,y) \le b|u(x,y)| + c \tag{2.2}$$

and

$$(-1)^{m+n} \mathcal{L}(u)(x,y) u(x,y) \le -g(x,y,u(x,y)), \tag{2.3}$$

where $a > 0$, $b \ge a$, $c \ge 0$ are constants and $g : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function such that

$$g(x+\omega_1, y, z) = g(x,y,z), \quad g(x, y+\omega_2, z) = g(x,y,z) \quad \text{for} \quad (x,y,z) \in \mathbb{R}^3 \tag{2.4}$$

and

$$g(x,y,z) > 0 \quad \text{for} \quad (x,y) \in \mathbb{R}^2,\ |z| \ge c. \tag{2.5}$$

By a (ω_1,ω_2)-periodic solution of the differential inequality (2.2) (differential inequality (2.3)) we understand a function $u\in C^{2m,2n}_{\omega_1\omega_2}$ satisfying this inequality everywhere in $\mathbb{R}^2$.

Lemma 2.1. *If conditions* (1.2) *and* (1.3) *hold, then there exists a positive constant r independent of c such that an arbitrary (ω_1,ω_2)-periodic solution u of the differential inequality* (2.2) *admits the estimate*

$$\|u\|_{C_{\omega_1\omega_2}}\le r\,c. \tag{2.6}$$

To prove the lemma we will need Lemmas 2.2–2.6 formulated below.

Lemma 2.2. *Let $u\in C^{1,1}_{\omega_1\omega_2}$ and*

$$|u(x_0,y_0)|=\min\{|u(x,y)|:(x,y)\in\mathbb{R}^2\}.$$

Then

$$|u(x_0,y_0)|\le(\omega_1\omega_2)^{-\frac12}\|u\|_{L^2_{\omega_1\omega_2}}$$

and

$$\begin{aligned}\|u\|_{C_{\omega_1\omega_2}}\le&|u(x_0,y_0)|+\Big(\frac{\omega_2}{\omega_1}\Big)^{\frac12}\|u^{(0,1)}\|_{L^2_{\omega_1\omega_2}}\\&+\Big(\frac{\omega_1}{\omega_2}\Big)^{\frac12}\|u^{(1,0)}\|_{L^2_{\omega_1\omega_2}}+2(\omega_1\omega_2)^{\frac12}\|u^{(1,1)}\|_{L^2_{\omega_1\omega_2}}.\end{aligned}$$

The proof of the lemma is in [6].

Lemma 2.3. *If $u:\mathbb{R}\to\mathbb{R}$ is a k-times continuously differentiable ω-periodic function, then*

$$\int_0^\omega|u^{(i)}(s)|^2\,ds\le\Big(\frac{\omega}{2\pi}\Big)^{2k-2i}\int_0^\omega|u^{(k)}(s)|^2\,ds\quad(i=1,\dots,k).$$

This is Wirtinger's lemma and one can find its proof in [5] (see also [7]).
Lemma 2.3 immediately implies

Lemma 2.4. *If $u\in C^{k,l}_{\omega_1\omega_2}$, then*

$$\|u^{(i,0)}\|_{L^2_{\omega_1\omega_2}}\le\Big(\frac{\omega_1}{2\pi}\Big)^{k-i}\|u^{(k,0)}\|_{L^2_{\omega_1\omega_2}},\quad\|u^{(0,j)}\|_{L^2_{\omega_1\omega_2}}\le\Big(\frac{\omega_2}{2\pi}\Big)^{l-j}\|u^{(0,l)}\|_{L^2_{\omega_1\omega_2}},$$

$$\|u^{(i,j)}\|_{L^2_{\omega_1\omega_2}}\le\Big(\frac{\omega_1}{2\pi}\Big)^{k-i}\Big(\frac{\omega_2}{2\pi}\Big)^{l-j}\|u^{(k,l)}\|_{L^2_{\omega_1\omega_2}}\quad(i=1,\dots,k;\ j=1,\dots,l).$$

Lemma 2.5. *Let ε be a positive constant and u be a (ω_1,ω_2)-periodic solution of the differential inequality* (2.2)*. Then the following inequalities hold in $\mathbb{R}^2$*

$$u^2(x,y)\le\frac{2}{a}|\mathcal{L}(u)(x,y)u(x,y)|+\frac{c^2}{a^2}, \tag{2.7}$$

$$(-1)^{m+n}\mathcal{L}(u)(x,y)u(x,y)\ge|\mathcal{L}(u)(x,y)u(x,y)|-\frac{c^2}{a}, \tag{2.8}$$

$$\mathcal{L}^2(u)(x,y)<(b+\varepsilon)|\mathcal{L}(u)(x,y)u(x,y)|+\gamma^2c^2, \tag{2.9}$$

where

$$\gamma = (2a\varepsilon)^{-\frac{1}{2}}(b+\varepsilon). \tag{2.10}$$

Proof. From (2.2) we have

$$(-1)^{m+n}\mathcal{L}(u)(x,y)u(x,y) \geq au^2(x,y) - c\,|u(x,y)| \geq \frac{a}{2}u^2(x,y) - \frac{c^2}{2a}$$

which implies inequalities (2.7) and (2.8).

Let $(x,y) \in \mathbb{R}^2$ be an arbitrarily fixed point. If $u(x,y) = 0$, then in view of (2.2) we have $|\mathcal{L}(u)(x,y)| \leq c$ and, consequently, inequality (2.9). Therefore it remains to consider the case, where $u(x,y) \neq 0$. Setting

$$\eta = \big((-1)^{m+n}\mathcal{L}(u)(x,y)\operatorname{sgn} u(x,y) - a|u(x,y)| + c\big)\big((b-a)|u(x,y)| + 2c\big)^{-1},$$
$$p = a + (b-a)\eta, \quad q = (2\eta - 1)c \operatorname{sgn} u(x,y),$$

and taking into account (2.2) we get

$$a \leq p \leq b, \quad |q| \leq c. \tag{2.11}$$

On the other hand, it is clear that

$$(-1)^{m+n}\mathcal{L}(u)(x,y) = pu(x,y) + q.$$

Therefore

$$\begin{aligned}\mathcal{L}^2(u)(x,y) &= (-1)^{m+n}p\mathcal{L}(u)(x,y)u(x,y) + pqu(x,y) + q^2 \\ &\leq p|\mathcal{L}(u)(x,y)u(x,y)| + \frac{a}{2}\varepsilon u^2(x,y) + \frac{p^2q^2}{2a\varepsilon} + q^2.\end{aligned}$$

Hence according to (2.7), (2.10) and (2.11) we get inequality (2.9). □

The following lemma is an immediate consequence of the formula of integration by parts.

Lemma 2.6. *If*

$$u \in C^{2m,2n}_{\omega_1\omega_2},$$

then for arbitrary $k \in \{0,\dots,m\}$, $l \in \{0,\dots,n\}$, $i \in \{0,\dots,2m-2k\}$ *and* $j \in \{0,\dots,2n-2l\}$

$$\int_0^{\omega_1}\int_0^{\omega_2} u^{(i,j)}(x,y)u^{(i+2k,j+2l)}(x,y)\,dx\,dy = (-1)^{k+l}\|u^{(i+k,j+l)}\|^2_{L^2_{\omega_1\omega_2}},$$

$$\int_0^{\omega_1}\int_0^{\omega_2} u^{(i,j)}(x,y)u^{(i+2k-1,j+2l)}(x,y)\,dx\,dy = 0,$$

$$\int_0^{\omega_1}\int_0^{\omega_2} u^{(i,j)}(x,y)u^{(i+2k,j+2l-1)}(x,y)\,dx\,dy = 0.$$

Proof of Lemma 2.1. By (1.3) there exist numbers $\varepsilon > 0$ and $\delta \in (0,1)$ such that

$$\sum_{k=0}^{n-1}\Big(\frac{\omega_2}{2\pi}\Big)^{2n-2k}|a_{mk}| + \sum_{i=0}^{m-1}\Big(\frac{\omega_1}{2\pi}\Big)^{2m-2i}|a_{in}|$$

$$+\sum_{i=0}^{m-1}\sum_{k=0}^{n-1}\Big(\frac{\omega_1}{2\pi}\Big)^{2m-2i}\Big(\frac{\omega_2}{2\pi}\Big)^{2n-2k}|a_{ik}| + \Big(\frac{\omega_1}{2\pi}\Big)^{m}\Big(\frac{\omega_2}{2\pi}\Big)^{n}(b+\varepsilon)^{\frac{1}{2}} < 1-\delta. \quad (2.12)$$

Let u be an arbitrary (ω_1,ω_2)-periodic solution of equation (2.2). Then by Lemma 2.5, inequalities (2.7)–(2.9) hold, where γ is the number given by (2.10).

By Lemma 2.6 and conditions (1.2), (2.1), we have

$$(-1)^{m+n}\int_0^{\omega_1}\int_0^{\omega_2}\mathcal{L}(u)(x,y)u(x,y)\,dx\,dy = \|u^{(m,n)}\|^2_{L^2_{\omega_1\omega_2}} - \sum_{i=0}^{m-1}|a_{in}|\,\|u^{(i,n)}\|^2_{L^2_{\omega_1\omega_2}}$$

$$-\sum_{k=0}^{n-1}|a_{mk}|\,\|u^{(m,k)}\|^2_{L^2_{\omega_1\omega_2}} - \sum_{i=0}^{m-1}\sum_{k=0}^{n-1}|a_{ik}|\,\|u^{(i,k)}\|^2_{L^2_{\omega_1\omega_2}},$$

$$\int_0^{\omega_1}\int_0^{\omega_2}\mathcal{L}(u)(x,y)u^{(2m,2n)}(x,y)\,dx\,dy = \mu^2 - \sum_{i=0}^{m-1}|a_{in}|\,\|u^{(m+i,2n)}\|^2_{L^2_{\omega_1\omega_2}}$$

$$-\sum_{k=0}^{n-1}|a_{mk}|\,\|u^{(2m,n+k)}\|^2_{L^2_{\omega_1\omega_2}} - \sum_{i=0}^{m-1}\sum_{k=0}^{n-1}|a_{ik}|\,\|u^{(m+i,n+k)}\|^2_{L^2_{\omega_1\omega_2}},$$

where $\mu = \|u^{(2m,2n)}\|_{L^2_{\omega_1\omega_2}}$. Hence by Lemma 2.4 and inequality (2.8) it follows that

$$\int_0^{\omega_1}\int_0^{\omega_2}|\mathcal{L}(u)(x,y)u(x,y)|\,dx\,dy + |a_{0n}|\,\|u^{(0,n)}\|^2_{L^2_{\omega_1\omega_2}} + |a_{m0}|\,\|u^{(m,0)}\|^2_{L^2_{\omega_1\omega_2}}$$

$$\le \Big(\frac{\omega_1}{2\pi}\Big)^{2m}\Big(\frac{\omega_2}{2\pi}\Big)^{2n}\mu^2 + \frac{\omega_1\omega_2}{a}c^2, \quad (2.13)$$

$$\mu^2 \le \Bigg(\sum_{i=0}^{m-1}\sum_{k=0}^{n-1}\Big(\frac{\omega_1}{2\pi}\Big)^{2m-2i}\Big(\frac{\omega_2}{2\pi}\Big)^{2n-2k}|a_{ik}| + \sum_{k=0}^{n-1}\Big(\frac{\omega_2}{2\pi}\Big)^{2n-2k}|a_{mk}|$$

$$+\sum_{i=0}^{m-1}\Big(\frac{\omega_1}{2\pi}\Big)^{2m-2i}|a_{in}|\Bigg)\mu^2 + \int_0^{\omega_1}\int_0^{\omega_2}\mathcal{L}(u)(x,y)u^{(2m,2n)}(x,y)\,dx\,dy. \quad (2.14)$$

On the other hand, by Schwartz's inequality and inequality (2.9) we have

$$\int_0^{\omega_1}\int_0^{\omega_2}\mathcal{L}(u)(x,y)u^{(2m,2n)}(x,y)\,dx\,dy$$

$$\leq \mu(b+\varepsilon)^{\frac{1}{2}}\bigg(\int_0^{\omega_1}\int_0^{\omega_2} |\mathcal{L}(u)(x,y)u(x,y)|\,dx\,dy\bigg)^{\frac{1}{2}} + \gamma(\omega_1\omega_2)^{\frac{1}{2}}\mu c.$$

If along with this we take into account inequalities (2.12) and (2.13), then from (2.14) we get

$$\mu^2 = (1-\delta)\mu^2 + \delta r_0 c, \quad \text{where} \quad r_0 = \delta^{-1}\Big((b+\varepsilon)^{\frac{1}{2}}\Big(\frac{\omega_1\omega_2}{a}\Big)^{\frac{1}{2}} + \gamma(\omega_1\omega_2)^{\frac{1}{2}}\Big)$$

and, consequently,

$$\mu \leq r_0 c. \tag{2.15}$$

Setting

$$r_1 = \Big(\frac{\omega_1}{2\pi}\Big)^m\Big(\frac{\omega_2}{2\pi}\Big)^n r_0 + \Big(\frac{\omega_1\omega_2}{a}\Big)^{\frac{1}{2}}$$

and applying Lemma 2.4 again, from (2.7), (2.13) and (2.15) we obtain

$$\|u^{(i,j)}\|_{L^2_{\omega_1\omega_2}} \leq r_{ij}c \quad (i,j=0,1), \tag{2.16}$$

where

$$r_{00} = \Big(\frac{2}{a}\Big)^{\frac{1}{2}} r_1 + a^{-1}(\omega_1\omega_2)^{\frac{1}{2}}, \qquad r_{01} = |a_{0n}|^{-\frac{1}{2}}\Big(\frac{\omega_2}{2\pi}\Big)^{n-1} r_1,$$
$$r_{10} = |a_{m0}|^{-\frac{1}{2}}\Big(\frac{\omega_1}{2\pi}\Big)^{m-1} r_1, \qquad r_{11} = \Big(\frac{\omega_1}{2\pi}\Big)^{m-1}\Big(\frac{\omega_2}{2\pi}\Big)^{n-1} r_0.$$

By Lemma 2.2, estimate (2.6) follows from (2.16), where

$$r = (\omega_1\omega_2)^{-\frac{1}{2}} r_{00} + \Big(\frac{\omega_2}{\omega_1}\Big)^{\frac{1}{2}} r_{01} + \Big(\frac{\omega_1}{\omega_2}\Big)^{\frac{1}{2}} r_{10} + 2(\omega_1\omega_2)^{\frac{1}{2}} r_{11}$$

is a positive constant independent of u and c. □

Lemma 2.7. *Let conditions* (1.10), (2.4) *and* (2.5) *hold. Then there exists a positive constant r independent of c such that an arbitrary (ω_1,ω_2)-periodic solution u of the differential inequality* (2.3) *admits the estimate*

$$\|u\|_{C_{\omega_1\omega_2}} \leq r g^*(c), \tag{2.17}$$

where

$$g^*(c) = c + \max\{|g(x,y,z)|^{\frac{1}{2}} : (x,y)\in\mathbb{R}^2,\ |z|\leq c\}. \tag{2.18}$$

Proof. If we integrate inequality (2.3) over $[0,\omega_1]\times[0,\omega_2]$, then in view of Lemma 2.6 and inequality (2.11) we get

$$\|u^{(m,n)}\|^2_{L^2_{\omega_1\omega_2}} + \sum_{i=0}^{m-1}|a_{in}|\,\|u^{(i,n)}\|^2_{L^2_{\omega_1\omega_2}} + \sum_{k=0}^{n-1}|a_{mk}|\,\|u^{(m,k)}\|^2_{L^2_{\omega_1\omega_2}}$$
$$+\sum_{i=0}^{m-1}\sum_{k=0}^{n-1}|a_{ik}|\,\|u^{(i,k)}\|^2_{L^2_{\omega_1\omega_2}} + \int_0^{\omega_2}\int_0^{\omega_1} g(x,y,u(x,y))\,dx\,dy \leq 0.$$

Hence by (2.5) and (2.18) it follows that

$$\min\{|u(x,y)| : (x,y)\in\mathbb{R}^2\} \leq c, \tag{2.19}$$

$$\|u^{(m,n)}\|_{L^2_{\omega_1\omega_2}} \leq (\omega_1\omega_2)^{\frac{1}{2}} g^*(c), \quad \|u^{(0,n)}\|_{L^2_{\omega_1\omega_2}} \leq |a_{0n}|^{-1} (\omega_1\omega_2)^{\frac{1}{2}} g^*(c),$$
$$\|u^{(m,0)}\|_{L^2_{\omega_1\omega_2}} \leq |a_{m0}|^{-1} (\omega_1\omega_2)^{\frac{1}{2}} g^*(c).$$

Hence Lemma 2.4 yields

$$\|u^{(i,j)}\|_{L^2_{\omega_1\omega_2}} \leq r_{ij} g^*(c) \qquad (i,j=0,1), \tag{2.20}$$

where

$$r_{10} = |a_{m0}|(\omega_1\omega_2)^{\frac{1}{2}}\Big(\frac{\omega_1}{2\pi}\Big)^{m-1}, \qquad r_{01} = |a_{0n}|(\omega_1\omega_2)^{\frac{1}{2}}\Big(\frac{\omega_2}{2\pi}\Big)^{n-1},$$
$$r_{11} = (\omega_1\omega_2)^{\frac{1}{2}}\Big(\frac{\omega_1}{2\pi}\Big)^{m}\Big(\frac{\omega_2}{2\pi}\Big)^{n}.$$

By Lemma 2.2, (2.19) and (2.20) imply estimate (2.6), where

$$r = 1 + \Big(\frac{\omega_2}{\omega_1}\Big)^{\frac{1}{2}} r_{01} + \Big(\frac{\omega_1}{\omega_2}\Big)^{\frac{1}{2}} r_{10} + 2\big(\omega_1\omega_2\big)^{\frac{1}{2}} r_{11}$$

is a constant independent of u and c. □

2.2. Lemmas on the solvability of linear and nonlinear periodic problems. Consider the linear nonhomogeneous and homogeneous hyperbolic equations

$$\mathcal{L}(u) = (-1)^{m+n} bu + q(x,y), \tag{2.21}$$
$$\mathcal{L}(u) = (-1)^{m+n} bu \tag{2.22}$$

and the linear homogeneous ordinary differential equations

$$v^{(2m)} = \sum_{i=0}^{m-1} (a_{in} v^{(2i)} + c_{in} v^{(2i+1)}), \tag{2.23}$$
$$w^{(2n)} = \sum_{k=0}^{n-1} (a_{mk} w^{(2k)} + b_{mk} W^{(2k+1)}), \tag{2.24}$$

where $\mathcal{L}$ is the differential operator given by equality (2.1) and b is some constant different from zero.

Lemma 2.8. *Let either $b > 0$ and inequalities (1.2) and (1.3) or $b < 0$ and inequalities (1.10) hold. Then there exists a linear bounded operator $\mathcal{G} : C_{\omega_1\omega_2} \to C^{2m,2n}_{\omega_1\omega_2}$ such that for any $q \in C_{\omega_1\omega_2}$ equation (2.21) has a unique (ω_1,ω_2)-periodic solution*

$$u(x,y) = \mathcal{G}(q)(x,y) \quad \textit{for} \quad (x,y) \in \mathbb{R}^2. \tag{2.25}$$

Proof. By Theorem 1.1 from [6], to prove the lemma it is sufficient to show that equation (2.22) has only a trivial (ω_1,ω_2)-periodic solution, and equation (2.23) (equation (2.24)) has only a trivial ω_1-periodic (ω_2-periodic) solution.

Let u, v and w be, respectively, a (ω_1,ω_2)-periodic, a ω_1-periodic and a ω_2-periodic solution of equations (2.22), (2.23) and (2.24). Our goal is to prove that $u(x,y) \equiv 0$, $v(t) \equiv 0$, $w(t) \equiv 0$.

First consider the case where $b > 0$ and conditions (1.2) and (1.3) hold. Then u is a solution of the differential inequality (2.2), where $a = b$ and $c = 0$. Hence by Lemma 2.1, we get that $u(x, y) \equiv 0$. Multiplying (2.23) and (2.24), respectively, by $v^{(2m)}(t)$ and $w^{(2n)}(t)$ and integrating over $[0, \omega_1]$ and $[0, \omega_2]$, we get

$$\int_0^{\omega_1} |v^{(2m)}(t)|^2\, dt = \sum_{i=0}^{m-1} (-1)^{m+i} a_{in} \int_0^{\omega_1} |v^{(m+i)}(t)|^2\, dt, \tag{2.26}$$

$$\int_0^{\omega_2} |w^{(2n)}(t)|^2\, dt = \sum_{i=0}^{n-1} (-1)^{n+k} a_{mk} \int_0^{\omega_2} |w^{(n+k)}(t)|^2\, dt. \tag{2.27}$$

Hence by Lemma 2.3 and inequality (1.3) it follows that

$$\int_0^{\omega_1} |v^{(2m)}(t)|^2\, dt \le \alpha \int_0^{\omega_1} |v^{(2m)}(t)|^2\, dt, \quad \int_0^{\omega_2} |w^{(2n)}(t)|^2\, dt \le \beta \int_0^{\omega_2} |w^{(2n)}(t)|^2\, dt,$$

where

$$\alpha = \sum_{i=0}^{m-1} \left(\frac{\omega_1}{2\pi}\right)^{2m-2i} |a_{in}| < 1, \quad \beta = \sum_{i=0}^{n-1} \left(\frac{\omega_2}{2\pi}\right)^{2n-2k} |a_{mk}| < 1.$$

Therefore it is clear that $v^{(2m)}(t) \equiv 0$, $w^{(2n)}(t) \equiv 0$ and, consequently, $v(t) \equiv$ const and $w(t) \equiv$ const. Taking into account the fact that $a_{0n} \neq 0$ and $a_{m0} \neq 0$, from (2.23) and (2.24) we conclude that $v(t) \equiv 0$ and $w(t) \equiv 0$.

Now consider the case, where $b < 0$ and inequalities (1.10) hold. Then u is a solution of the differential inequality (2.3), where $g(x, y, z) \equiv |b|z^2$. Hence by Lemma 2.7, it follows that $u(x, y) \equiv 0$. On the other hand, using inequalities (1.10), from (2.23) and (2.26) ((2.4) and (2.27)) we obtain that $v(t) \equiv 0$ ($w(t) \equiv 0$). □

Lemma 2.9. *Let either $b > 0$ and inequalities* (1.2) *and* (1.3) *or $b < 0$ and inequalities* (1.10) *hold. Moreover, let there exist a positive constant ρ such that for any $\lambda \in (0, 1)$ every (ω_1, ω_2)-periodic solution of the differential equation*

$$\mathcal{L}(u) = (-1)^{m+n}(1-\lambda)bu + \lambda f(x, y, u) \tag{2.28}$$

admits the estimate

$$\|u\|_{C^{2m-1,2n-1}_{\omega_1\omega_2}} \le \rho. \tag{2.29}$$

Then equation (1.1) *has at least one (ω_1, ω_2)-periodic solution.*

This lemma immediately follows from Lemma 2.8 and Theorem 2.1 from [6].

3. Proofs of the Main Results

Proof of Theorem 1.1. Let r and $\mathcal{G}$ be the number and the operator appearing in Lemmas 2.1 and 2.8, and let $\|\mathcal{G}\|$ be the norm of the operator $\mathcal{G}$. Set

$$\rho_0 = \max\{b\, r\, c + |f(x, y, z)| : (x, y) \in \mathbb{R}^2,\ |z| \le r\, c\}, \quad \rho = \|\mathcal{G}\|\rho_0. \tag{3.1}$$

By Lemma 2.9 and conditions (1.2), (1.3), to prove the theorem it is sufficient to show that for any $\lambda \in (0,1)$ an arbitrary (ω_1,ω_2)-periodic solution u of equation (2.28) admits estimate (2.29).

According to (1.4) u is a solution of the differential inequality (2.2). Hence by Lemma 2.1 we get estimate (2.6). On the other hand, by Lemma 2.8 we have the representation

$$u(x,y) = \mathcal{G}\big(\lambda f(\cdot,\cdot,u) - (-1)^{m+n}\lambda u\big)(x,y).$$

By (2.6) and (3.1), the latter representation immediately implies estimate (2.29). □

Proof of Theorem 1.2. Inequality (1.4) follows from (1.5), where

$$c = \max\{|f(x,y,0)| : (x,y) \in \mathbb{R}^2\}.$$

However, by Theorem 1.1, this inequality along with conditions (1.2) and (1.3) guarantees the existence of (ω_1,ω_2)-periodic solution of equation (1.1). It remains to prove that (1.1) has at most one (ω_1,ω_2)-periodic solution. Let u_1 and u_2 be arbitrary (ω_1,ω_2)-periodic solutions of that equation and

$$u(x,y) = u_1(x,y) - u_2(x,y).$$

Then in view of (1.5) the function u is a (ω_1,ω_2)-periodic solution of the differential inequality (2.2), where $c = 0$. Hence according to (1.2), (1.3) and Lemma 2.1 it follows that $u(x,y) \equiv 0$, i.e., $u_1(x,y) \equiv u_2(x,y)$. □

We omit the proof of Theorem 1.3, since it can be proved in much the same way as Theorem 1.1. The only difference is that the constant b should be an arbitrary *negative* number, and instead of Lemma 2.1 one should use Lemma 2.7, where

$$g(x,y,z) = \min\{|b|\,z^2, (-1)^{m+n+1} f(x,y,z)\}.$$

Theorem 1.4 immediately follows from Theorem 1.3 and Lemma 2.7.

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