Georgian Mathematical Journal
Volume 14 (2007), Number 3, 519–531

STAGNATION ZONES OF A-SOLUTIONS

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To the blessed memory of Ilia Vekua

Abstract. We investigate stagnation zones of solutions of partial differential elliptic equations. With the domain width being much less than its length and special boundary conditions, these solutions can be almost constant over large subdomains. Such domains are called stagnation zones (s-zones). We estimate the size, the location of these s-zones and study the behavior of solutions on s-zones.

2000 Mathematics Subject Classification: 28A75, 30C75, 35J25.
Key words and phrases: Stagnation zone, A-harmonic function, Laplace–Beltrami equation, capacity.

1. Introduction. Below we investigate solutions of partial differential elliptic equations on non-smooth surfaces. The investigation is motivated in part by the fact that, with a recent increasing interest in areas such as micro-electrical-mechanical systems (MEMS) and nanoscale physiological processes, there is a greater need to improve our understanding of fluid flows, flows of electric and chemical fields in the micro- or nanocanals, good conductors, cracks etc.

When the width of the band is much smaller than the length, zones inside which the flows are almost stationary, and consequently their potential functions are almost constant, will be of sufficiently large size.

We estimate the size and location of these stagnation zones of solutions. At first sight, the situation seems to be of little interest. However, keeping in mind that a minute change in the potential function value occurs over a very long interval, it is clear that a better understanding of such stagnation zones, which we call s-zones, may allow one to organize calculations in a better way and minimize the amount of computation as much as possible.

First we define the following concept which will be the key one in this article:

Let $f : D \to \mathbb{R}$ be a continuous function. Fix a subdomain $U \subset D$ and a constant $s > 0$. A subdomain U is said to be a *stagnation zone with the deviation* s (s-zone) of f if the oscillation of f on U does not exceed the pre-assigned constant s.

Some estimates of stagnation zones are given in [8], [7], [9, Ch. 7] for solutions of Laplace–Beltrami equations on Lipschitz surfaces.

In the situation where a solution $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ describes the goods barter intensity on some geographical space, the boundary condition $\partial f/\partial \overline{\mathbf{n}} = 0$ describes the borderline which could be crossed from both sides with an economic

ISSN 1072-947X / \$8.00 / © Heldermann Verlag *www.heldermann.de*

gain only in exceptional cases (see F. Braudel [1, Ch. I]). The theorems established in [7] give (under appropriate restrictions on the model chosen) estimates of a geometrical size of the stagnation zone (which adjoins that part of the borderline) in the world economy.

The theorems on stagnation zones are closely connected with the pre-Liouville theorems, i.e., with the solution oscillation estimates which immediately give rise to different versions of the classical Liouville theorem on an entire doubly-periodic function to be identically constant [7].

If an s-zone $U \subset D$ is defined for a deviation $s > 0$, which is the measurement error of f, then the key problem is the search of indirect methods to receive information on f on U without direct measurements. Here we present some results in this direction.

Following I. N. Vekua [10, Chapter 6], we consider equations in metrics of sufficiently general form.

2. Surfaces. Let $D \subset \mathbb{R}^p$ be a domain, $2 \le p \le m$, and let Ω be a p-dimensional surface in $\mathbb{R}^m$ given by a locally Lipschitz vector-function

$$x = f(u) = (f_1(u_1, \ldots, u_p), \ldots, f_m(u_1, \ldots, u_p)) : D \to \mathbb{R}^m \,, \tag{1}$$

where $x = (x_1, \ldots, x_m)$.

In the general case, Ω can have self-intersections. We shall say, that Ω is *imbedded* in $\mathbb{R}^m$, if f realizes a homeomorphic map of D onto $f(D)$ with the metric (and, consequently, topology !) induced by $\mathbb{R}^m$. A surface Ω is *immersed* in $\mathbb{R}^m$ if f is imbedded locally on D.

Since f is locally Lipschitz, by the Rademacher theorem [2, Section 3.1.6], there exists the differential $df(u)$ a.e. on D.

Let $u \in D$ be a point where f has the differential. By

$$f' = \begin{pmatrix} f'_{1u_1}\, f'_{2u_1} \,\cdots\, f'_{mu_1} \\ f'_{1u_2}\, f'_{2u_2} \,\cdots\, f'_{mu_2} \\ \cdots \\ f'_{1u_p}\, f'_{2u_p} \,\cdots\, f'_{mu_p} \end{pmatrix}$$

we denote the derivative of f at a point $u = (u_1, \ldots, u_p)$, where it exists. Using the standard notation

$$g_{ij} = \left\langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \right\rangle , \quad i, j = 1, 2, \ldots, p \,,$$

we define the first quadratic form of Ω on D:

$$ds^2_\Omega = \sum_{i,j=1}^{p} g_{ij}(u)\, du_i\, du_j \,. \tag{2}$$

Below we assume that the following property holds a.e. on D:

$$\operatorname{rank}(df) = p\,. \tag{3}$$

Let $g = \det(g_{ij})$. At each point where (3) holds, the quadratic form (2) is positive and $g > 0$. At such a point, the inverse matrix $(g^{ij}) = (g_{ij})^{-1}$ is defined and

$$g^{ij} = \frac{\operatorname{adj} g_{ij}}{g}\,.$$

We set

$$|\xi|_\Omega = \left(\sum_{i,j=1}^{p} g^{ij}(u)\,\xi_i\xi_j\right)^{1/2}\,.$$

We denote by

$$d\mathcal{H}^p_\Omega = \sqrt{g}\,du_1\cdots du_p$$

the volume element on Ω.

Let $\Omega\subset\mathbb{R}^m$ be a locally bi-Lipschitz surface. Since the metric on $f(D)$ is induced by $\mathbb{R}^m$, the p-dimensional Hausdorff measure $\mathcal{H}^p_\Omega(E)$, $E\subset f(D)$ is defined. Moreover, a tangent plane $T_y(\Omega)$ exists a.e. on Ω and consequently a.e. on Ω we can define the gradient ∇_Ω.

By $W^{1,p}_{\mathrm{loc}}(\Omega)$ we denote the set of functions $\varphi:\Omega\to\mathbb{R}$ of the class $W^{1,p}(U)$ on every open subset $U\Subset\Omega$.[1]

Let D be a subdomain of Ω, homeomorphic to the ball $B^p(0,1)$ in $\mathbb{R}^p$ and let $w: D\to B^p(0,1)$ be a homeomorphism from D onto the ball. We fix a closed set $S\subset\partial B^p(0,1)$ and denote by $\mathcal{S}$ the set of all sequences $\{u_k\}$ lying on D and such that $w(u_k)\to S$. We call the sets of such sequences $\mathcal{S}$ *boundary sets* of D.

A subdomain $U\subset D$ *adjoins* a boundary set $\mathcal{S}$, if there exists a sequence $\{u_k\}\in\mathcal{S}$ lying in U.

Let P, $Q\subset\partial B^p(0,1)$ be non-overlapping sets and let $\mathcal{P}$, $\mathcal{Q}$ denote the sets of $\{u_k\}$ lying in D and such that $w(u_k)\to P$, $w(u_k)\to Q$, respectively. A triple of the form $(\mathcal{P},\mathcal{Q};D)$ is called a *condenser*.

3. Structure conditions. Let $D\Subset\Omega$ be a domain on the surface Ω and let

$$A:\bigwedge^p(T(D))\to\bigwedge^p(T(D))$$

be a map defined a.e. on the foliation $\bigwedge^p(T(D))$ of tangent p-covectors. We suppose that for a.e. $y\in D$ the map A is defined on the space $\bigwedge^p(T_y(D))$ of p-covectors, i.e., for a.e. $y\in D$, the map

$$A(y,\ .\ ):\xi\in\bigwedge^p(T_y(D))\to\bigwedge^p(T_y(D))$$

is defined and continuous. We assume that the map

$$y\mapsto A(y,\xi)$$

[1] $U\Subset\Omega$ means that a subdomain U is bounded and its closure $\overline{U}\subset\Omega$.

is measurable in the Lebesgue sense for all measurable p-covector fields ξ and

$$A(y,\lambda\xi)=\lambda\,|\lambda|^{\alpha-2}\,A(y,\xi)\,,\qquad \lambda\in\mathbb{R}^1\,,\qquad \alpha\ge 1. \tag{4}$$

Suppose that for a.e. $y\in D$ and all $\xi\in\bigwedge^p(T_y(D))$ the following properties hold:

$$\nu_1\,|\xi|_\Omega^\alpha\le\langle\xi,A(y,\xi)\rangle_\Omega,\quad |A(x,\xi)|_\Omega\le\nu_2\,|\xi|_\Omega^{\alpha-1} \tag{5}$$

with $\alpha\ge 1$ and some constants ν_1, $\nu_2>0$. (Here and below, the subscript Ω means that the corresponding quantity is calculated with respect to the metric of Ω.)

We consider the equation

$$\operatorname{div}_\Omega A(y,\nabla f)=0\,. \tag{6}$$

Recall that the divergence of a vector field A on Ω is defined by

$$\operatorname{div}_\Omega A=\sum_{i=1}^{p}\langle\nabla_{E_i}A,E_i\rangle_\Omega\,.$$

Here the summation is taken over an arbitrary orthonormal basis $\{E_i\}_{i=1}^p$ of the tangent space $T_y(\Omega)$.

We note a special case of (6)

$$\Delta_\Omega h=\frac{1}{\sqrt{g}}\sum_{i=1}^{p}\frac{\partial}{\partial u_i}\left(\sqrt{g}\sum_{j=1}^{p}g^{ij}(u)\frac{\partial h}{\partial u_j}\right)=0\,. \tag{7}$$

Equation (7) is called the *Laplace–Beltrami equation* with respect to the metric of Ω.

4. Capacity. Let D be a subdomain of $\mathbb{R}^p$, $p\ge 2$, and let Ω be a surface in $\mathbb{R}^m$ given by the locally Lipschitz vector-function (1) with (3).

Fix a constant $\alpha\ge 1$ and a condenser $(\mathcal{P},\mathcal{Q};D)$. Let

$$\operatorname{cap}_{\alpha,\Omega}(\mathcal{P},\mathcal{Q};D)=\inf\int\limits_D|\nabla_\Omega\varphi|_\Omega^\alpha\,d\mathcal{H}_\Omega^p\,. \tag{8}$$

Here the infimum is taken over all locally Lipschitz functions $\varphi:D\to(0,+\infty)$ with properties:

$$\lim_{\{u_k\}\in\mathcal{P}}\varphi(u_k)=0\,,\qquad \lim_{\{u_k\}\in\mathcal{Q}}\varphi(u_k)=1 \tag{9}$$

and such that for every subdomain $D'\Subset D$

$$0<\operatorname*{ess\,inf}_{D'}|\nabla_\Omega\varphi(x)|_\Omega\le\operatorname*{ess\,sup}_{D'}|\nabla_\Omega\varphi(x)|_\Omega<\infty\,. \tag{10}$$

The quantity $\operatorname{cap}_{\alpha,\Omega}(\mathcal{P},\mathcal{Q};D)$ is called the *α-capacity of a condenser* with respect to the metric of Ω.

Also, we shall need a capacity of a more generalized form. Consider a condenser $(\mathcal{P}, \mathcal{Q}; D)$ on the surface Ω. Fix a vector field $A(y,\xi)$ and define the A-capacity of $(\mathcal{P}, \mathcal{Q}; D)$ by

$$\operatorname{cap}_A(\mathcal{P}, \mathcal{Q}; D) = \inf \int\limits_D \langle A(y, \nabla_\Omega \varphi), \nabla_\Omega \varphi\rangle_\Omega \, d\mathcal{H}^p_\Omega \,, \tag{11}$$

where the infimum is taken over all functions φ with (9), (10).

In the case, where $A(y,\xi) = |\xi|^{p-2}\xi$ and $p = 2$, we have the standard harmonic capacity of the condenser on the surface. For $p = n$, we obtain the conformal capacity.

5. A-solutions. Let $\varphi : D \subset \Omega \to \mathbb{R}$ be a locally Lipschitz function. We denote by $D_b(\varphi)$ the set of all points $a \in D$ at which φ has no differential. We denote by $D_b(\Omega)$ the set of $y \in \Omega$, in which Ω has no tangent plane.

Let $U \subset D$ be a subset and let $\partial' U = \partial U \setminus \partial D$ be its boundary with respect to D. If $\partial' U$ is $(\mathcal{H}^{n-1}, n-1)$-rectifiable, then it has a locally finite perimeter in the sense of De Giorgi and $\mathcal{H}^{n-1}$-almost everywhere on ∂U, a unit normal vector $\mathbf{n}$ exists [2, Sections 3.2.14, 3.2.15].

Let $D \subset \Omega$ be a domain and let $\mathcal{S}$ be a boundary set of D. Define the concept of a generalized solution of (6) with zero boundary Neumann condition on $\partial D \setminus \mathcal{S}$. A subset $U \subset D$ is called *admissible*, if U does not adjoin $\mathcal{S}$ and has a $(\mathcal{H}^{n-1}, n-1)$-rectifiable boundary with respect to D.

We denote by $\tilde{\partial} D$ the boundary of D with respect to the extended space $\mathbb{R}^p \cup \{\infty\}$. Let $G \subset \tilde{\partial} D$ be a set closed in $\mathbb{R}^p \cup \{\infty\}$ (the case where $G = \varnothing$ is possible). We consider the set (G, D) of all subdomains $U \subset D$ with $\tilde{\partial} U \subset (D \cup G)$ and $(\mathcal{H}^{p-1}, p-1)$-rectifiable boundaries $\partial' U = \tilde{\partial} U \setminus \tilde{\partial} D$.

Definition 1. We say that a boundary set $G \subset \tilde{\partial} D$ is *absolutely nontransparent for a locally Lipschitz solution* $f : D \to \mathbb{R}$ of (6), if for every subdomain $U \in (G, D)$,

$$\mathcal{H}^{p-1}\left[\partial' U \cap D_b(f)\right] = 0 \,, \tag{12}$$

and for every locally Lipschitz function $\varphi : \overline{U} \setminus G \to \mathbb{R}$ the following property holds

$$\int\limits_{\partial' U} \varphi \, \langle A(u, \nabla_\Omega f) \,, \overline{\mathbf{n}}\rangle_\Omega \, d\mathcal{H}^{p-1}_\Omega = \int\limits_U \langle A(u, \nabla_\Omega f) \,, \nabla_\Omega \varphi\rangle_\Omega \, d\mathcal{H}^p_\Omega \,. \tag{13}$$

Here $\overline{\mathbf{n}}$ is a unit inner normal vector to $\partial' U$ and $d\mathcal{H}^p_\Omega$ is a volume element on Ω.

Following [3, Chapter 6], we call solutions of (6) *A-solutions*. However we should note that our definition of generalized solutions is slightly different from the definition in [3].

In the case of a smooth surface Ω, a smooth boundary ∂D, smooth A_i ($i = 1, \ldots, p$) and $f \in C^2$, relation (13) implies (6) with

$$\langle A(y, \nabla_\Omega h), \overline{\mathbf{n}}\rangle_\Omega = 0 \tag{14}$$

everywhere on G (see [9, §7.2.1]).

The surface integral exists by (12). Indeed, this assumption guarantees that $\nabla_\Omega f(u)$ exists $\mathcal{H}^{p-1}$-a.e. on $\partial' U$. The assumption that $U \in (G, U)$ implies the existence of a normal vector $\overline{\mathbf{n}}$ for $\mathcal{H}^{p-1}$-a.e. points on $\partial' U$ [2, Chapter 2 §3.2]. Thus the scalar product $\langle A(u, \nabla_\Omega f)\, , \overline{\mathbf{n}}\rangle_\Omega$ is defined and finite a.e. on $\partial' U$.

Monotonicity close to the boundary. Let $D \subset \mathbb{R}^p$ be a domain and let $2 \le p < \infty$.

Definition 2. A function $f : D \to \mathbb{R}$ is called *monotone up to a boundary set* $G \subset \tilde{\partial} D$, if for every subdomain $U \subset D$ with $(\partial U \setminus G) \subset D$ the following property holds

$$\operatorname{osc}(f, U) \le \operatorname{osc}(f, \partial U \setminus G)\,. \tag{15}$$

For functions on subdomains of $\mathbb{R}^2$, this property was studied in [4], [5]. If $G = \varnothing$, then we have the well-known class of functions monotone in the Lebesgue sense. If $G = \tilde{\partial} D$, then it is easy to see that every monotone up to the boundary function $f \equiv \text{const}$.

Theorem 1. *If a boundary set $G \subset \tilde{\partial} D$ is absolutely non-transparent for a locally Lipschitz solution $f : D \to \mathbb{R}$ of* (6) *with* (4) *and* (5)*, then f is monotone up to G.*

Proof. The idea of the proof is close to the proof of the corresponding result in [5] and so we shall restrict ourselves to its main points. We fix a subdomain U of D such that $(\partial U \setminus G) \subset D$. First we prove that

$$\sup_U f(u) = \sup_{\partial' U} f(u). \tag{16}$$

Suppose that the contrary holds. There exists a point $u_0 \in U$ where

$$f(u_0) > \sup_{\partial' U} f(u) = M.$$

We choose $\epsilon > M$ such that $f(u_0) > \epsilon$. By Theorem 3.2.22 [2] for a.e. $\epsilon > 0$, sets $\{x \in D : f(x) = \epsilon\}$ are $(\mathcal{H}^{p-1}, p-1)$-rectifiable and, consequently, have a locally finite perimeter. In particular, a.e. there exists a normal to these sets.

We fix a component U, $u_0 \in U$, of $\{x \in U : f(x) > \epsilon\}$. Without loss of generality, we may assume that the boundary $\partial' U$ is locally $(\mathcal{H}^{p-1}, p-1)$-rectifiable. Using (13) with $\varphi = f(x) - \epsilon$, we write

$$\int\limits_U \langle \nabla_\Omega f\, , A(u, \nabla_\Omega f)\rangle_\Omega \, d\mathcal{H}^p_\Omega = \int\limits_{\partial' U} (f - \epsilon)\langle A(u, \nabla_\Omega f), \overline{\mathbf{n}}\rangle_\Omega \, d\mathcal{H}^{p-1}_\Omega = 0.$$

By (5), we have

$$\nabla_\Omega f(u) = 0 \quad \text{a.e. on} \quad U,$$

and $f \equiv \text{const}$ on U. This contradicts the definition of a component U, $x_0 \in U$. Thus (16) is proved.

The function $-f$ is a solution of this equation. Hence (16) implies

$$\inf_U f(u) = \inf_{\partial' U} f(u). \tag{17}$$

Relations (16) and (17) guarantee the monotonicity of f up to G. □

7. Stagnation zones. Let D be a domain in $\mathbf{R}^p$, $p \geq 2$. Let Ω be a p-dimensional locally Lipschitz surface in $\mathbb{R}^m$, $2 \leq p < m < \infty$, defined by the vector function (1) with (3).

Theorem 2. *Assume that a boundary set $G \subset \tilde{\partial} D$ is absolutely non-transparent for a locally Lipschitz solution $f : D \to \mathbb{R}$ of (6) with (4) and (5). If subdomains $U_1 \subset U_2 \subset U_3$ of D belong to the class (G, D), then*

$$\int\limits_{U_1} |\nabla_\Omega f|^\alpha_\Omega \, d\mathcal{H}^p_\Omega \leq C \operatorname{cap}^\alpha_{\alpha,\Omega}(U_1, U_3 \setminus U_2; U_3) \,. \tag{18}$$

Here

$$C = k\, \alpha^\alpha \sup{}^\alpha_{U_3} |f(u)| \,, \quad k = \frac{\nu_2^\alpha}{\nu_1^{\alpha-1}} \,.$$

Proof. Fix a locally Lipschitz function $\psi \geq 0$ on U_3 such that

$$\psi|_{U_3 \setminus U_2} = 0 \,, \quad \psi_{U_1}|_{U_1} = 1 \,.$$

Let $\varphi = f\, \psi^\alpha$. By (13) we may write

$$\int\limits_{U_3} \langle A(u, \nabla f) \,, \nabla(f \psi^\alpha) \rangle_\Omega \, d\mathcal{H}^p_\Omega = \int\limits_{\partial' U_3} f \psi^\alpha \, \langle A(u, \nabla f) \,, \overline{\mathbf{n}} \rangle_\Omega \, d\mathcal{H}^{p-1}_\Omega = 0 \,.$$

We have

$$\int\limits_{U_3} \psi^\alpha \langle \nabla f, \, A(u, \nabla f) \rangle_\Omega \, d\mathcal{H}^p_\Omega = -\alpha \int\limits_{U_3} f \psi^{\alpha - 1} \langle \nabla \psi, \, A(u, \nabla f) \rangle_\Omega \, d\mathcal{H}^p_\Omega \,.$$

Assumption (5) for A implies that

$$\langle \nabla \psi, \, A(u, \nabla f) \rangle^\alpha_\Omega \leq k \, |\nabla_\Omega \psi|^\alpha_\Omega \, \langle \nabla f, \, A(u, \nabla f) \rangle^{\alpha-1}_\Omega \,. \tag{19}$$

Indeed, we have

$$|\langle \nabla \psi, \, A(u, \nabla f) \rangle_\Omega| \leq |\nabla \psi|_\Omega \, |A(u, \nabla f)|_\Omega$$

and

$$|A(u, \, \nabla f)|_\Omega \leq \nu_2 \, |\nabla f|^{\alpha-1}_\Omega =$$

$$= \nu_2 \, \left(|\nabla f|^\alpha_\Omega\right)^{(\alpha-1)/\alpha} \leq \nu_2 \left(\frac{1}{\nu_1} \langle \nabla f, \, A(u, \nabla f) \rangle_\Omega \right)^{(\alpha-1)/\alpha} \,.$$

Thus we obtain

$$|A(u, \nabla f)|^\alpha_\Omega \leq \frac{\nu_2^\alpha}{\nu_1^{\alpha-1}} \, \langle \nabla f, \, A(u, \nabla f) \rangle^{\alpha-1}_\Omega$$

and (19) follows easily.

By (19) we have

$$\int\limits_{U_3} \psi^\alpha \langle \nabla f, \, A(u, \nabla f) \rangle_\Omega \, d\mathcal{H}^p_\Omega \leq c_1 \int\limits_{U_3} \psi^{\alpha-1} |\nabla \psi|_\Omega \langle \nabla f, \, A(u, \nabla f) \rangle^{\frac{\alpha-1}{\alpha}}_\Omega \, d\mathcal{H}^p_\Omega$$

with

$$c_1 = k^{\frac{1}{\alpha}}\,\alpha\,\sup_{U_3} |f(u)|\,.$$

The Hölder inequality implies

$$\int\limits_{U_3} \psi^{\alpha-1} |\nabla\psi|_\Omega \,\langle \nabla f,\, A(u,\nabla f)\rangle_\Omega^{\frac{\alpha-1}{\alpha}}\, d\mathcal{H}^p_\Omega$$

$$\le \left(\int\limits_{U_3} |\nabla\psi|^\alpha_\Omega \, d\mathcal{H}^p_\Omega\right)^{\frac{1}{\alpha}} \left(\int\limits_{U_3} \psi^\alpha \langle \nabla f,\, A(u,\nabla f)\rangle_\Omega \, d\mathcal{H}^p_\Omega\right)^{\frac{\alpha-1}{\alpha}}.$$

Thus we come to the inequality

$$\int\limits_{U_3} \psi^\alpha \langle \nabla f,\, A(u,\nabla f)\rangle_\Omega \, d\mathcal{H}^p_\Omega \le c_1^\alpha \int\limits_{U_3} |\nabla\psi|^\alpha_\Omega \, d\mathcal{H}^p_\Omega\,.$$

Since $\psi \equiv 1$ on U_1, we obtain

$$\int\limits_{U_1} \langle \nabla f,\, A(u,\nabla f)\rangle_\Omega \, d\mathcal{H}^p_\Omega \le c_1^\alpha \int\limits_{U_3\setminus U_2} |\nabla\psi|^\alpha_\Omega \, d\mathcal{H}^p_\Omega\,.$$

Taking the infimum over all admissible functions ψ, we come to (18). □

Now, let $h(u)$ be a locally Lipschitz function such that

$$\lim_{u\to G} h(u) = 0\,.$$

We may choose $h(u)$, for example, as a distance function from the boundary set G to $u \in D$ at the metric ds_Ω. For an arbitrary $t > 0$ we set

$$\Sigma_h(t) = \{u \in D : h(u) = t\}\,, \quad B_h(t) = \{u \in D : h(u) > t\}\,.$$

Since h is locally Lipschitz by Theorem 3.2.15 [2], a.e. h-spheres $\Sigma_h(t)$ are countably $(\mathcal{H}^{p-1}, p-1)$-rectifiable. Let $\alpha \ge 1$ be a constant. For every open subset $V \subset \Sigma_h(t)$, we introduce the following numerical characteristic

$$\mu_{\alpha,A}(V) = \inf \left[\frac{\int\limits_V \langle \nabla_\Sigma\varphi,\, A(u,\nabla_\Sigma\varphi)\rangle_\Sigma |\nabla_\Omega h|^{-1}_\Omega d\mathcal{H}^{p-1}_\Sigma}{\int\limits_V \varphi^\alpha |\nabla_\Sigma h|_\Sigma \,|\nabla_\Omega h|^{-1}_\Omega d\mathcal{H}^{p-1}_\Sigma}\right]^{1/\alpha}. \tag{20}$$

Here the infimum is taken over all Lipschitz functions $\varphi : \overline{V} \to \mathbb{R}$ such that

$$\int\limits_V \varphi \, d\mathcal{H}^{p-1}_\Sigma = 0\,. \tag{21}$$

Theorem 3. *Let D be a domain in $\mathbb{R}^p$, $p \geq 2$. Assume that a non-empty boundary set $G \subset \tilde{\partial} D$ is absolutely non-transparent for a locally Lipschitz solution $f : D \to \mathbb{R}$ of $A\,[f] = 0$ with (5).*

If the subdomains $B_h(t_2) \subset B_h(t_1)$, $t_1 < t_2$, belong to (G, D), then

$$I(t_1) \leq I(t_2) \exp \left\{ -k^\alpha \int\limits_{t_1}^{t_2} \mu(\Sigma_h(t))\, dt \right\} \tag{22}$$

for

$$I(t) = \int\limits_{B_h(t)} \langle \nabla f,\, A(u, \nabla f) \rangle_\Omega \, d\mathcal{H}^p_\Omega$$

and

$$\mu(\Sigma_h(t)) = \mu_{\alpha,\mathcal{A}}(\Sigma_h(t))\,.$$

Moreover, if $s > 0$, $U_i = B_h(t_i)$, $i = 1, 2$, and

$$k\, \alpha^\alpha \sup\nolimits^\alpha_D |f(u)| \operatorname{cap}^\alpha_{\alpha,\Omega}(U_1, D \setminus U_2; D) \exp \left\{ -k^\alpha \int\limits_{t_1}^{t_2} \mu(\Sigma_h(t))\, dt \right\} < s\,, \tag{23}$$

then U_1 is an s-zone of f.

Proof. We assume that $B_h(t)$ is an h-ball with a locally $(\mathcal{H}^{p-1}, p-1)$-rectifiable boundary h-sphere $\Sigma_h(t)$. Since $t_1 < t < t_2$ and $B_h(t_1)$, $B_h(t_2)$ belong to the class (G, D), $B_h(t)$ also belongs to (G, D). We choose a constant c_0 such that the function $\varphi = f - c_0$ satisfies (21). By (13), we may write that

$$I(t) = \int\limits_{B_h(t)} \langle A(u, \nabla_\Omega f)\,, \nabla_\Omega f \rangle_\Omega \, d\mathcal{H}^p_\Omega = \int\limits_{\Sigma_h(t)} (f - c_0)\, \langle A_i(u, \nabla_\Omega f),\, \overline{\mathbf{n}} \rangle_\Omega \, d\mathcal{H}^{p-1}_\Sigma\,.$$

Here $\overline{\mathbf{n}}$ is a unit normal to $\Sigma_h(t)$.

For $\mathcal{H}^{p-1}_\Sigma$-a.e. points on $\Sigma_h(t)$, we have

$$\overline{\mathbf{n}} = \nabla_\Omega h / \left| \nabla_\Omega h \right|_\Omega,$$

and hence for a.e. t we have

$$I(t) = \int\limits_{\Sigma_h(t)} (f - c_0)\, \langle A(u, \nabla_\Omega f),\, \frac{\nabla_\Omega h}{|\nabla_\Omega h|_\Omega} \rangle_\Omega \, d\mathcal{H}^{p-1}_\Sigma\,.$$

Let $a \in \Sigma_h(t)$ be a point where the h-sphere has a tangent plane. Using (5), we have

$$|\langle A(u, \nabla_\Omega f),\, \nabla_\Omega h \rangle_\Omega|^\alpha \leq k\; |\nabla_\Omega h|^\alpha_\Omega \, |\langle \nabla_\Omega f,\, A(a, \nabla_\Omega f) \rangle|^{\alpha - 1}\,.$$

Since this h-sphere is locally $(\mathcal{H}^{p-1}, p-1)$-rectifiable, we find

$$I(t) \le \int\limits_{\Sigma_h(t)} |f - c_0| \left| \langle A(u, \nabla_\Omega f) \frac{\nabla_\Omega h}{|\nabla_\Omega h|} \rangle_\Omega \right| d\mathcal{H}_\Sigma^{p-1}$$
$$\le k^{\frac{1}{\alpha}} \int\limits_{\Sigma_h(t)} |f - c_0| \, |\nabla_\Omega h|_\Omega \langle \nabla_\Omega f, \, A(u, \nabla_\Omega f) \rangle_\Omega^{\frac{\alpha-1}{\alpha}} \frac{d\mathcal{H}_\Sigma^{p-1}}{|\nabla_\Omega h|_\Omega} \, .$$

We have

$$I(t) \le k^{\frac{1}{\alpha}} \int\limits_{\Sigma_h(t)} |f - c_0| \, |\nabla_\Omega h|_\Omega \langle \nabla_\Omega f, \, A(u, \nabla_\Omega f) \rangle_\Omega^{\frac{\alpha-1}{\alpha}} \frac{d\mathcal{H}_\Sigma^{p-1}}{|\nabla_\Omega h|_\Omega}$$
$$\le k^{\frac{1}{\alpha}} \left(\int\limits_{\Sigma_h(t)} |f - c_0|^\alpha \, |\nabla_\Omega h|_\Omega \frac{d\mathcal{H}_\Sigma^{p-1}}{|\nabla_\Omega h|} \right)^{\frac{1}{\alpha}}$$
$$\times \left(\int\limits_{\Sigma_h(t)} \langle \nabla_\Omega f, \, A_i(u, \nabla_\Omega f) \rangle_\Omega \frac{d\mathcal{H}_\Sigma^{p-1}}{|\nabla_\Omega h|_\Omega} \right)^{\frac{\alpha-1}{\alpha}} .$$

We use the characteristic μ. By (20) we have

$$\left(\int\limits_{\Sigma_h(t)} |f - c_0|^\alpha \, |\nabla_\Omega h|_\Omega \frac{d\mathcal{H}_\Sigma^{p-1}}{|\nabla_\Omega h|} \right)^{\frac{1}{\alpha}}$$
$$\le \mu^{-1}(\Sigma_h(t)) \left(\int\limits_{\Sigma_h(t)} \langle \nabla_\Omega f, \, A(u, \nabla_\Omega f) \rangle_\Omega \frac{d\mathcal{H}_\Sigma^{p-1}}{|\nabla_\Omega h|_\Omega} \right)^{\frac{1}{\alpha}}$$

and hence

$$I(t) \le k^{\frac{1}{\alpha}} \mu^{-1}(\Sigma_h(t)) \int\limits_{\Sigma_h(t)} \langle \nabla_\Omega f, \, A(u, \nabla_\Omega f) \rangle_\Omega \frac{d\mathcal{H}_\Sigma^{p-1}}{|\nabla_\Omega h|_\Omega} \, .$$

Observing that by the co-area formula

$$I'(t) = \int\limits_{\Sigma_h(t)} \langle \nabla_\Omega f, \, A(u, \nabla_\Omega f) \rangle_\Omega \frac{d\mathcal{H}_\Sigma^{p-1}}{|\nabla_\Omega h|_\Omega} \, ,$$

we come to the differential inequality

$$I(t) \le I'(t) \, k^{\frac{1}{\alpha}} \mu^{-1}(\Sigma_h(t)) \, .$$

Solving it, we obtain

$$\int\limits_{t_1}^{t_2} \mu(\Sigma_h(t)) \, dt \le k^{\frac{1}{\alpha}} \, \log \frac{I(t_2)}{I(t_1)} \, ,$$

and (22) follows easily. □

8. A-solutions on a stagnation zone. The following theorem describes the behavior of A-solutions on a stagnation zone.

Theorem 4. *Let $\Omega \subset \mathbb{R}^m$ be a p-dimensional locally bi-Lipschitz surface and let $(\mathcal{P}, \mathcal{Q}; D)$ be a condenser on Ω. If f is a locally Lipschitz solution of (6) with the boundary condition (14) on $\partial D \setminus (\mathcal{P} \cup \mathcal{Q})$, then for every pair of non-overlapping $(p-1)$-dimensional surfaces Σ_1 and Σ_2 lying in D and separable $\mathcal{P}$, $\mathcal{Q}$, the following property holds*

$$\left(\frac{\mathcal{I}}{\operatorname{cap}_A(\Sigma_1, \Sigma_2; U)}\right)^{1/p} \leq \sup\{a \in \Sigma_1,\ b \in \Sigma_2 :\ |f(a) - f(b)|\}\,. \qquad (24)$$

Here U is the subdomain of D contained between the cross-sections Σ_1, Σ_2 and

$$\mathcal{I} = \left|\int\limits_{\Sigma} \langle A(y, \nabla_\Omega f), \overline{\mathbf{n}}\rangle_\Omega \, d\mathcal{H}^{n-1}_\Sigma\right|$$

is a constant independent of the cross-section Σ separating the boundary sets $\mathcal{P}$ and $\mathcal{Q}$ in D.

Proof. First we observe that (13) implies

$$\int\limits_{\Sigma_1} \langle A(y, \nabla_\Omega f), \overline{\mathbf{n}}\rangle_\Omega \, d\mathcal{H}^{n-1}_\Sigma = \int\limits_{\Sigma_2} \langle A(y, \nabla_\Omega f), \overline{\mathbf{n}}\rangle_\Omega \, d\mathcal{H}^{n-1}_\Sigma\,.$$

Thus the quantity $\mathcal{I}$ is independent of the surface Σ separating $\mathcal{P}$ and $\mathcal{Q}$ in D.

We set

$$\mu = \sup\{a \in \Sigma_1,\ b \in \Sigma_2 :\ |f(a) - f(b)|\}\,.$$

Now we assume that

$$f(y)|_{\Sigma_1} \equiv M\,, \quad f(y)|_{\Sigma_2} \equiv m \quad \text{and} \quad M - m = \mu\,.$$

The function

$$f^*(y) = \frac{f(y) - m}{\mu}$$

is admissible for calculation of the A-capacity of the condenser $(\Sigma_1, \Sigma_2; U)$ and is extremal. Hence

$$\operatorname{cap}_A(\Sigma_1, \Sigma_2; U) = \int\limits_U \langle A(y, \nabla_\Omega f^*), \nabla_\Omega f^*\rangle_\Omega \, d\mathcal{H}^p_\Omega$$

and by (4),

$$\operatorname{cap}_A(\Sigma_1, \Sigma_2; U) = \frac{1}{\mu^p} \int\limits_U \langle A(y, \nabla_\Omega f), \nabla_\Omega f\rangle_\Omega \, d\mathcal{H}^p_\Omega\,.$$

Thus we obtain

$$\operatorname{cap}_A(\Sigma_1, \Sigma_2; U) = \frac{\mathcal{I}}{\mu^p}\,. \qquad (25)$$

In the general case we find the level surfaces $\Sigma_1' \subset \{y : f(y) = M\}$ and $\Sigma_2' \subset \{y : f(y) = m\}$. Let U' be a subdomain of D lying between Σ_1' and Σ_2'. Then

$$\operatorname{cap}_A(\Sigma_1', \Sigma_2'; U') \le \operatorname{cap}_A(\Sigma_1, \Sigma_2; U)$$

and, by (25), for the condenser $(\Sigma_1, \Sigma_2; U)$ we have

$$\operatorname{cap}_A(\Sigma_1, \Sigma_2; U) \ge \frac{\mathcal{I}}{|M-m|^p},$$

which implies (24). □

9. Corollaries. Now we shall formulate some corollaries. We consider a condenser $(\mathcal{P}, \mathcal{Q}; D)$. Let $\rho : D \to \mathbb{R}$ be a locally Lipschitz function satisfying 9) and (10).

Let Σ_1, Σ_2 be a pair of non-overlapping hyper-surfaces in D separating $\mathcal{P}$ and $\mathcal{Q}$ such that under the motion from $\mathcal{P}$ to $\mathcal{Q}$ along the Jordan arc, we first reach Σ_1. We set

$$\rho_1 = \sup_{y \in \Sigma_1} \rho(y), \quad \rho_2 = \inf_{y \in \Sigma_2} \rho(y).$$

Corollary 1. *Let $\Omega \subset \mathbb{R}^m$ be an n-dimensional locally bi-Lipschitz surface and let $(\mathcal{P}, \mathcal{Q}; D)$ be a condenser on Ω. If f is a locally Lipschitz solution of* (6) *with* (14) *on $\partial D \setminus (\mathcal{P} \cup \mathcal{Q})$, then for a pair of non-overlapping $(p-1)$-dimensional surfaces Σ_1 and Σ_2 lying in D, separating $\mathcal{P}$, $\mathcal{Q}$ and such that $\rho_2 > \rho_1$, the following estimate holds:*

$$(\rho_2 - \rho_1) \left(\frac{\mathcal{I}}{\nu_2 \int\limits_U |\nabla_\Omega \rho|_\Omega^p \, d\Omega} \right)^{1/p} \le \sup\{x \in \Sigma_1,\ y \in \Sigma_2 : |f(x) - f(y)|\}. \tag{26}$$

Let us consider another special case. Let Δ be a domain in $\mathbb{R}^{n-1}$ and let $D = \Delta \times \mathbb{R}^1$ be a cylindrical domain in $\mathbb{R}^n$. We assume that a special coordinate system $y = (y', y_n)$, $y' = (y_1, \dots, y_{n-1})$ in $\mathbb{R}^n$ introduced so that the domain Δ lies on the hyperplane of variables $y' = (y_1, \dots, y_{n-1})$, $y_n = 0$.

Denote by $\mathcal{P}$ and $\mathcal{Q}$ the sets of sequences $\{y_k\}$ of points $y_k = (y_k', y_{k,n}) \in D$ with coordinates $y_k' \in \Delta$ and $y_{k,n}$ tending to $-\infty$ and $+\infty$, respectively.

We set

$$\rho(y) = \frac{2}{\pi} \left(\arctan y_{n+1} + \frac{\pi}{2} \right).$$

This function satisfies (9), (10) and, by Corollary 1, we come to the following statement.

Corollary 2. *Let $D \subset \mathbb{R}^n$ be a cylindrical domain of the described form. If f is a locally Lipschitz solution of* (6) *with the boundary conditions* (14) *on $\partial D \setminus (\mathcal{P} \cup \mathcal{Q})$, then for a pair of non-overlapping $(n-1)$-dimensional surfaces*

Σ_1 and Σ_2 lying in D, separating $\mathcal{P}$, $\mathcal{Q}$ and such that $\rho_2 > \rho_1$, the following estimate holds

$$(\rho_2 - \rho_1)\left(\pi\,\mathcal{I}\Big/ 2\,\nu_2 \int\limits_U \frac{dx_1 \dots dx_n}{(1+x_n^2)^{p/2}}\right)^{1/p} \\ \le \sup\{a \in \Sigma_1,\ b \in \Sigma_2 :\ |f(a) - f(b)|\}\,. \qquad (27)$$

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(Received 5.03.2007)

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