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ON THE STABILITY IN BONNET'S THEOREM OF THE SURFACE THEORY

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Dedicated to the memory of Ilia N. Vekua

Abstract. In the space $\mathbb{R}^{n+1}$, n-dimensional surfaces are considered having the parametrizations which are functions of the Sobolev class W_p^2 with $p > n$. The first and the second fundamental tensor are defined. The Peterson–Codazzi equations for such functions are understood in some generalized sense. It is proved that if the first and the second fundamental tensor of one surface are close to the first and, respectively, to the second fundamental tensor of the other surface, then these surfaces will be close up to the motion of the space $\mathbb{R}^{n+1}$. A difference between the fundamental tensors and the nearness of the surfaces are measured with the help of suitable W-norms. The proofs are based on a generalization of Frobenius' theorem about completely integrable systems of the differential equations which was proved by Yu. E. Borovskiĭ. The integral representations of functions by differential operators with complete integrability condition are used, which were elaborated by the author in his other works.

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1. Introduction. In [1], [2] and [3], the problem of a continuous dependence of a surface on its first and second quadratic forms is investigated in the three-dimensional space. The interest of the authors of these papers in the foundations of differential geometry of surfaces is motivated by questions arisen in the elasticity theory.

In [1]–[3], consideration is given to surfaces of the class $\mathcal{C}^3$ and it is shown that if in the topology of the space $\mathcal{C}^l$, $l \geq 2$, the fundamental quadratic forms of surfaces are close, then these surfaces differ little from each other in the sense of the topology of the space $\mathcal{C}^{l+2}$.

In the present paper, it is claimed that the requirements for smoothness in [1]–[3] can be essentially weakened by means of minimal assumptions under which the classical analytic methods of the surface theory can be constructed in differential geometry. We consider n-dimensional surfaces of the class W_p^2, $p > n$, in the space $\mathbb{R}^{n+1}$. Thus we admit surfaces whose parametrization has no third order derivatives.

In the context of the above-said, a natural question arises: how should we understand the basic derivational equations of a surface which demand the differentiation of expressions containing second derivatives? These equations

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(Peterson–Mainardi–Codazzi equations) have the form

$$\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} = R, \tag{1.1}$$

where P and Q are the values containing second derivatives of the surface parametrization. We can answer this question following the ideas stated in the classical works of S. L. Sobolev (see, for instance, [4], [5]). The formulation and proof of derivational equations for smooth surfaces of the class W_2^2 in the space $\mathbb{R}^3$ were given for the first time by I. Ya. Bakel'man in [9]. It was shown there how the basic notions of the surface theory in differential geometry can be extended to the case we are interested in.

Peterson–Mainardi–Codazzi equations yield necessary conditions which must be satisfied by the quadratic forms so that one of these forms be the first and the other be the second quadratic form of a surface. Yu. E. Borovskiĭ [8] established the sufficiency of these conditions for n-dimensional surfaces of the class W_p^2, $p > n$, in $\mathbb{R}^{n+1}$. The proof of the sufficiency rests on the generalization of Frobenius' theorem on completely integrable systems. This generalization was obtained by Yu. E. Borovskiĭ in his papers [6] and [7].

The main result of the present paper is contained in Theorems 1, 2 and 4. Preliminarily, we give some corollaries of Borovskiĭ's theorem on Pfaff differential equations for systems of linear differential equations of the form

$$\frac{\partial z}{\partial x_i} = A_i(x)z, \quad i = 1, 2, \dots, n, \tag{1.2}$$

where $A_i(x)$ are $m \times m$ matrices whose elements are functions of the class L_p, $p > n$. It is assumed that system (1.2) is completely integrable. For functions $z : \Omega \to \mathbb{R}^m$, where Ω is a domain in $\mathbb{R}^n$, we define differential operators

$$L_i z = \frac{\partial z}{\partial x_i} - A_i(x)z, \quad i = 1, 2, \dots, n.$$

An integral representation of a vector function z is constructed through the set of differential operators $L_i z$. This is done by the method previously developed by the author of the paper for the case where matrix functions $A_i(x)$ are sufficiently smooth (see [10], [11]). As will be shown in the sequel, this method with no essential modifications extends also to the case where the elements of functions $A_i(x)$ are functions of the class L_p for appropriate values of p (see Subsection 3).

Using the integral representations obtained by us, we first estimate, in Subsection 4, the norm of a function z in the space W_p^1 by means of L_p-norms of functions $L_i z$ (Theorem 1). Further, in Subsection 5, using the result of Theorem 1 we prove the stability of solutions of a linear, completely integrable system of equations with respect to a change of the coefficients of this system (Theorem 2).

The statement as to the stability in Bonnet's theorem (Subsection 6, Theorem 4) follows from Theorem 2.

In Subsection 6 we also give the formulation of Yu. E. Borovskiĭ's theorem (Theorem 3), where Bonnet's theorem on the recovery of a surface by its first and second quadratic forms extends to the case of surfaces from the class W_p^2 (see [9]).

Analytic techniques used in the present paper were developed by the author in connection with the problem on the stability in Liouville's theorem on conformal mappings of a space. Some authors, in their comments on my works dealing with this problem, note that the results obtained in them have a "qualitative" character, since explicit values of constants in the found estimates cannot be indicated. This observation is formally a correct one. Indeed, we can only prove the existence and finiteness of constants occurring in the obtained estimates of the stability. It should be said that this situation is typical of modern studies in the area of mathematical analysis. Even in the cases where explicit values of constants in the obtained estimates can be indicated, these values are either fantastically large or, on the contrary, extremely small.[1]

V. I. Semenov indicated ([16]) explicit values for individual constants in the estimates obtained by me and presented in [11].

Mention should also be made of yet another important fact. Let $\varepsilon \geq 0$ characterize an extent of violation of the conditions of Liouville's theorem for some mapping. (For $\varepsilon = 0$, the mapping is the Möbius one). The estimates established by me give an exact order, as $\varepsilon \to 0$, of a deviation of the mapping from the Möbius mapping (for details see the author's monograph [11]).

2. Notation and some preliminary results. In the sequel, Ω is a domain, i.e., a connected open set in the space $\mathbb{R}^n$. It is assumed that $n \geq 2$.

The symbol $B(a, r)$ is an open ball in $\mathbb{R}^n$, centered at a and having a radius $r > 0$; $S(a, r)$ denotes a sphere with center a and radius r.

The Banach space of real functions defined almost everywhere and p-th power integrable, $p \geq 1$, over the domain Ω is denoted by $L_p(\Omega)$ or, simply, by L_p. The norm in $L_p(\Omega)$ is defined as usual:

$$\|f\|_{L_p} = \left\{ \int\limits_{\Omega} |f(x)|^p \, dx \right\}^{\frac{1}{p}}. \tag{2.1}$$

The family of all functions defined on the set Ω and having there generalized (in the Sobolev sense) derivatives of order l, which are p-th power integrable, $p \geq 1$, are denoted by the symbol $W_p^l(\Omega)$. The set W_p^l is a Banach space. The norm in this space is defined by the equality

$$\|f\|_{W_p^1} = \|\Pi f\| + \left\{ \int\limits_{\Omega} \Big[\sum_{|\alpha|=l} \frac{l!}{\alpha!} \Big(\frac{\partial^\alpha z}{\partial x^\alpha}(x) \Big)^2 \Big]^{\frac{p}{2}} dx \right\}^{\frac{1}{p}}, \tag{2.2}$$

[1] This is exemplified by the following fact: in the proof of one well-known theorem of the theory of numbers, which was carried out by the methods of mathematical analysis, the validity of the theorem statement was established only for integer numbers $n > C$, where $C = 3^{3^{15}} > 10^{5000000}$.

where Π is the projection mapping of the space $W_p^l(\Omega)$ onto the set of all polynomials of a degree not exceeding $l-1$.

$\mathcal{C}^r(\Omega)$ is the set of all functions defined in Ω, whose all partial derivatives of order r are defined and continuous in the domain Ω. A function $\varphi \in \mathcal{C}_0^\infty(\Omega)$ if φ vanishes outside the compact set contained in Ω, and $\varphi \in \mathcal{C}^r(\Omega)$ for any r.

We also consider vector and matrix functions defined in the domain Ω. Given a function

$$z : x \in \Omega \mapsto (z_1(x), z_2(x), \dots, z_m(x)),$$

we say that $z \in L_p(\Omega)$ ($z \in W_p^l(\Omega)$, $z \in \mathcal{C}^r$) if each of the real functions z_i, $i = 1, 2, \dots, m$, belongs to the class $L_p(\Omega)$ (resp., to the class $W_p^l(\Omega)$, $\mathcal{C}^r$). Analogously, a matrix function $Z(x) = (z_{ij}(x))_{i,j=1,2,\dots,m}$ defined in the domain Ω belongs to the class $L_p(\Omega)$ (to the class $W_p^r(\Omega)$, $\mathcal{C}^r(\Omega)$) of each of the real functions $z_{i,j}$, $i, j = 1, 2, \dots, m$ belongs to the class $L_p(\Omega)$ (resp., to the class $W_p^r(\Omega)$, $\mathcal{C}^r(\Omega)$).

The norm in $L_p(\Omega)$ of vector functions $z = (z_1(x), z_2(x), \dots, z_m(x))$ of the class $L_p(\Omega)$ (of the class $W_p^l(\Omega)$) is defined as a sum of L_p-norms (resp., of W_p^l-norms) of the components of a vector function z.

For an arbitrary $m \times m$ matrix Z, the symbol $|Z|$ denotes the operator norm of Z, i.e.,

$$|Z| = \sup_{|\xi| \leq 1} |Z\xi|.$$

We apply Sobolev's embedding theorems for functions of classes W_p^1. These theorems are true under the assumption that the domain boundary satisfies certain additional conditions. We say that Ω is a domain of Sobolev type or, briefly, a domain of the class $\mathcal{S}$ if for it there hold the conclusions Sobolev's embedding theorems for the space W_p^1. Any domain in $\mathbb{R}^n$, whose boundary is an $(n-1)$-dimensional manifold of the class $\mathcal{C}^1$, is a domain of the class $\mathcal{S}$.

We will consider the vector functions $z : \Omega \to \mathbb{R}^m$ of the class $W_1^1(\Omega)$ which satisfy a system of differential equations of the form

$$\frac{\partial z}{\partial x_i} = A_i(x) z(x), \quad i = 1, 2, \dots, n, \tag{2.3}$$

where $A_i(x)$ is an $m \times m$ matrix for each $x \in \Omega$. Note that elements of matrix functions $A_i(x)$ are measurable functions in the domain Ω.

Lemma 1. *Let Ω be a bounded domain of the class S in the space $\mathbb{R}^n$ and a function $z : \Omega \to \mathbb{R}^m$ of the class $W_1^1(\Omega)$ be such that for each $i = 1, 2, \dots, n$ and for almost all $x \in \Omega$ the equality*

$$\frac{\partial z}{\partial x_i} = A_i(x, z) \tag{2.4}$$

is fulfilled, where vector functions $A_i(x, z)$ are defined on the set $\Omega \times \mathbb{R}^m$ and are such that

$$\forall i \in \{1, 2, \dots, n\} A_i(x, 0) \in L_p(\Omega), \ \ |A_i(x, z) - A_i(x, \bar{z})| \leq M(x)|z - \bar{z}|, \tag{2.5}$$

where the function M belongs to a class $L_p(\Omega)$, $p > n$. Then the vector function $z : \Omega \to \mathbb{R}^m$ belongs to the class W_p^1. In particular, in that case the function $z(x)$ is continuous in the domain Ω.

Proof. Let us assume that the conditions of the lemma are fulfilled. According to Sobolev's embedding theorem, if $z \in W_r^1$, where $r \leq n$, then $z \in L_s$ for s satisfying the condition $\frac{1}{s} = \frac{1}{r} - \frac{1}{n}$. In particular, if $z \in W_1^1$, then $z \in L_s$, where $s = \frac{n}{n-1} > 1$. The proof of the lemma rests on two propositions.

Proposition A. *Let $z : \Omega \to \mathbb{R}^m$ be a solution of equation* (2.5)*. In that case, if $z \in L_q$ for some $q \geq \frac{n}{n-1}$, then $z \in W_\lambda^1$, where λ is such that $\frac{1}{\lambda} = \frac{1}{p} + \frac{1}{q}$.*

Condition (2.5) obviously implies that for any (x, z) and for each $i = 1, 2, \ldots, n$ the following inequality is valid:

$$|A_i(x, z)| \leq |A_i(x, 0)| + M(x)|z|. \tag{2.6}$$

Applying estimate (2.6) we obtain

$$\left(\int\limits_\Omega |A_i(x, z(x))|^\lambda \, dt\right)^{1/\lambda} \leq \left(\int\limits_\Omega |A_i(x, 0)|^\lambda \, dx\right)^{1/\lambda} + \left(\int\limits_\Omega |M(x)|^\lambda |z(x)|^\lambda \, dt\right)^{1/\lambda}. \tag{2.7}$$

Here λ is defined from the condition $\frac{1}{\lambda} = \frac{1}{p} + \frac{1}{q}$. It is obvious that $1 < \lambda < p$.

The first integral on the right-hand side of inequality (2.7) is finite since $\lambda < p$ and the domain Ω is bounded. To the second integral on the right-hand side of (2.7) we apply the Hölder inequality. Assuming $\sigma = \frac{p+q}{q}$, $\tau = \frac{p+q}{p}$ and $\lambda = \frac{pq}{p+q}$, we have $\frac{1}{\sigma} + \frac{1}{\tau} = 1$, $\lambda\sigma = p$, $\lambda\tau = q$. Now,

$$\int\limits_\Omega \|M(x)\|^\lambda |z(x)|^\lambda \, dx \leq \left\{\int\limits_\Omega \|M(x)\|^{\lambda\sigma} \, dx\right\}^{\frac{1}{\sigma}} \left\{\int\limits_\Omega |z(x)|^{\lambda\tau} \, dt\right\}^{\frac{1}{\tau}}. \tag{2.8}$$

Inequalities (2.7) and (2.8) imply

$$\left\|\frac{\partial z}{\partial x_i}\right\|_{L_\lambda} \leq \|A_i(x, 0)\|_\lambda + \|M\|_{L_p} \|z\|_{L_q}.$$

Thus all partial derivatives of the vector function $z(x)$ are λ-th power integrable, $\lambda = \frac{pq}{p+q}$. Proposition A is proved. □

Proposition B. *If a solution $z(x)$ of system* (2.4) *belongs to the class W_λ^1, $\lambda \leq n$, then $z \in L_q(\Omega)$, where q is such that*

$$\frac{1}{q} = \frac{1}{\lambda} - \frac{1}{n}.$$

If the right-hand side of the latter equality is equal to zero, then $z \in L_q(\Omega)$ for any $q > 1$.

It is obvious that by virtue of Sobolev's embedding theorem the next proposition is also true.

Proposition C. *If a solution z of system* (2.4) *belongs to a class W^1_r, where $r > n$, then $z \in W^1_p$.*

Indeed, if $z \in W^1_r$, where $r > n$, then by virtue of Sobolev's embedding theorem the function $z(x)$ is bounded and continuous in the domain Ω. Hence by virtue of conditions (2.5) it follows that the right-hand side of each equation of system (2.4) belongs to the class $L_p(\Omega)$. Therefore we conclude that all derivatives $\frac{\partial z}{\partial x_i}$ belong to the class $L_p(\Omega)$ and thus $z \in W^1_p(\Omega)$. Proposition C is proved. □

We finish proving the lemma by means of some iteration procedure. By condition, a solution $z(x)$ of system (2.4) belongs to the class $W^1_1(\Omega)$.

Assume that $\alpha = \frac{1}{n} - \frac{1}{p}$. Obviously, $0 < \alpha < \frac{1}{n}$.

Let k be a smallest integer number such that $\frac{1}{n} > 1 - k\alpha$. It is clear that $k \geq 1$. By the definition of k we have $1 - (k-1)\alpha \geq \frac{1}{n}$. Hence it follows that $1 - k\alpha \geq \frac{1}{n} - \alpha = \frac{1}{p} > 0$.

For a nonnegative integer number ν we assume that $r_\nu = \frac{1}{1-\nu\alpha}$. By the condition of the lemma, z belongs to the class $W^1_{r_0}$.

Assume that for some $\nu < k$ it is proved that $z \in W^1_{r_\nu}$. If $r_\nu < n$, then Proposition B allows us to conclude that then $z \in L_q$, where

$$\frac{1}{q} = \frac{1}{r_\nu} - \frac{1}{n}.$$

Hence by virtue of Proposition A it follows that $z \in W^1_r$, where r is defined from the condition

$$\frac{1}{r} = \frac{1}{q} + \frac{1}{p} = 1 - \nu\alpha + \frac{1}{p} - \frac{1}{n} = 1 - (\nu+1)\alpha.$$

We conclude by induction that if $r_\nu \neq n$ for all integer numbers $\nu < k$, then $z \in W^1_{r_k}$. Since $r_k > n$, by virtue of Proposition C the function $z \in W^1_p(\Omega)$.

Let us consider the case where $r_\nu = n$ for some $\nu < k$. It is obvious that this may take place only for $\nu = k-1$.

Thus, assuming that $z \in W^1_n$ is proved, by virtue of Sobolev's embedding theorem $z \in L_q$ for any $q \geq 1$. Proposition A allows us now to conclude that $z \in W^1_r$ for $r = r(q) = \frac{pq}{p+q}$ for any $q \geq 1$.

Furthermore, we have $\lim\limits_{q\to\infty} r(q) = p > n$. Thus there exists a value q such that $r(q) > n$. Therefore $z \in W^1_r$ for some $r > n$ and thus, by Proposition C, $z \in W^1_p$. The lemma is proved. □

Corollary. *Assume that Ω is a domain of the class S and the matrix functions A_i in system* (2.3) *belong to to the class $L_p(\Omega)$ for some $p > n$. In that case, if a vector function $z : \Omega \to \mathbb{R}^m$ of the class W^1_1 satisfies system* (2.3), *then*

z belongs to the class W_p^1. In particular, the function $z(x)$ is then continuous in the domain Ω.

The proof is obvious.

3. Some information on completely integrable systems of equations. Let a system of differential equations of the form

$$\frac{\partial z}{\partial x_i} = A_i(x)z(x), \quad i = 1, 2, \ldots, n, \tag{3.1}$$

be given in a domain Ω of the space $\mathbb{R}^n$. Here $A_i(x)$ for each $x \in \Omega$ is an $m \times m$ matrix whose elements are functions of a class $L_p(\Omega)$, where $p > n$. According to Lemma 1, any solution of system (3.1) is a function of the class $W_p^1(\Omega)$ and therefore is a bounded continuous function in the domain Ω.

Let us first recall some of Yu. Borovskiĭ's results from [6], [7], where a general system of equations of the form

$$\frac{\partial z}{\partial x_i} = A_i(x, z) \tag{3.2}$$

is considered, assuming that for each $i = 1, 2, \ldots, m$ the vector function $A_i(x, z)$ satisfies the conditions

$$A_i(x, 0) \in L_2(\Omega), \quad \|A_i(x, z) - A_i(x, \bar{z})\| \leq M(x)|z - \bar{z}'|, \tag{3.3}$$

where the function M belongs to a class $L_p(\Omega)$, $p > n$.

In [6], the author introduces the notion of a complete system of differential equation of form (3.2). Let z^λ, $\lambda = 1, 2, \ldots, m$, be the coordinates of a vector z in the space $\mathbb{R}^m$. System (3.3) can be rewritten in the form

$$dz^\lambda = \omega^\lambda = \sum_{i=1}^{n} A_i^\lambda[x, z(x)]dx_i.$$

The notion of a generalized differential is introduced for external forms. A form θ with degree 2 is a generalized differential of form ω with degree 1 if the equality

$$\int_\Omega \theta \wedge \varphi = \int_\Omega \omega \wedge d\varphi$$

is fulfilled for any external form φ with degree $n - 2$ whose coefficients are functions of the class $C_0^\infty(\Omega)$.

As shown in [6], in the particular case of systems of form (3.1) (we are interested only in this case) the condition of system completeness can be represented in the following equivalent form. For vector functions $A_i(x, u)$ we have the equalities

$$\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} = A_i A_j - A_j A_i. \tag{3.4}$$

The derivatives are meant in the sense of the theory of generalized functions. Namely, equality (3.4) is equivalent to the following: for any function $\varphi \in \mathcal{C}^\infty(\Omega)$ with a compact support including in the domain Ω the equality

$$\int\limits_\Omega \left\{ \frac{\partial \varphi}{\partial x_i} A_j - \frac{\partial \varphi}{\partial x_j} A_i \right\} dx = \int\limits_\Omega \varphi [A_i A_j - A_j A_i] \, dx \tag{3.5}$$

holds.

By the definition given in [6], system (3.1) is completely integrable if there exists a homeomorphism $\varkappa$ of the space $\mathbb{R}^m$ into the space $W^1_{p,m}$ of functions of class W^1_p its value in $\mathbb{R}^m$ such that a) for each $\beta \in \mathbb{R}^m$ the function $u(x, \beta) = \varkappa(\beta)$ is a solution of system (3.1) and b) for almost all $x \in \Omega$ the mapping $\beta \mapsto u(x, \beta) \in \mathbb{R}^m$ is a one-to-one mapping onto. It is shown in [6] that a system of equations, which is complete in the sense of [5], is completely integrable.

Let reformulate the definition of complete integrability so as to make it sound like a more customary assertion of the solvability of the Cauchy problem.

Let $E \subset \Omega$ be the set of measure zero, consisting of those points x for which $\beta \mapsto u(x, \beta)$ is not a homeomorphism of the space $\mathbb{R}^n$ onto.

Let $x_0 \notin E$. Then for any vector $\xi \in \mathbb{R}^n$ there exists a unique solution $u(x)$ of system (1.1) such that $u(x_0) = \xi$. Indeed, for this x_0 $\beta \mapsto (u(x_0, \beta)$ there is a homeomorphism of the space $\mathbb{R}^n$ onto and therefore there exists $\beta_0 \in \mathbb{R}^n$ such that $u(x_0, \beta_0) = \xi$. It is obvious that $u(x) = u(x, \beta_0)$ is the desired solution of system (1.1).

Thus we obtain that if the system of differential equations (1.1) is complete in the sense of the definition given in [5], then there exists a set $E \subset \Omega$ such that if $x \notin E$, then for any vector $\xi \in \mathbb{R}^m$ there exists a unique solution z of system (1.1) that satisfies the Cauchy condition $z(x_0) = \xi$.

Let us show that the set E is empty. For this we need some simple formalism that is well known in the regular case [15].

Along with equation (3.1) we will also consider the following system of equations with respect to matrix functions

$$\frac{\partial X}{\partial x_i}(x) = A_i(x) X(x), \tag{3.6}$$

where $X(x)$ is a $m \times m$ matrix.

System (3.6) is completely integrable in the sense that for any point $x_0 \in \Omega \backslash E$ and for any $m \times m$ matrix H there exists a solution X of this system that satisfies the Cauchy condition $X(x_0) = H$.

Indeed, let the vector $\mathbf{a}_i$ be the i-th column of the matrix H and $\mathbf{x}^{(i)}$ be a solution of system (3.1). Then the matrix function $X(x)$, whose i-th column is $\mathbf{x}^{(i)}$ for each $i = 1, 2, \ldots, m$, is a solution of system (3.6) for which the Cauchy condition $X(x_0) = H$ is fulfilled.

Consider the system of equations

$$\frac{\partial v}{\partial x_i}(x) = -A_i^*(x) v(x), \quad i = 1, 2, \ldots, m. \tag{3.7}$$

Here $*$ denotes the operation of matrix transposition. The system of equations (3.7) is called conjugate to system (3.1).

Let us show that system (3.7) also satisfies the condition of completeness. Indeed, applying the operation of transposition to the left- and right-hand sides of equality (3.5), we obtain that for any function $\varphi \in \mathcal{C}_0^\infty(\Omega)$ there holds the equality

$$\int\limits_{\Omega} \left\{ \frac{\partial \varphi}{\partial x_i}(A_j)^* - \frac{\partial \varphi}{\partial x_j}(A_i)^* \right\} dx = \int\limits_{\Omega} \varphi[(A_i A_j)^* - (A_j A_i)^*]\, dx. \tag{3.8}$$

For any matrices P and Q we have $(PQ)^* = Q^* P^*$. Assume that $B_i = -A_i^*$. After obvious transformations, equality (3.8) yields

$$\int\limits_{\Omega} \left\{ \frac{\partial \varphi}{\partial x_i} B_j - \frac{\partial \varphi}{\partial x_j} B_i \right\} dx = \int\limits_{\Omega} \varphi[B_i B_j - B_j B_i]\, dx$$

for any function $\varphi \in \mathcal{C}_0^\infty(\Omega)$. Hence it follows that system (3.7) is complete in the sense of Borovskiĭ [6].

By what has been proved above there exists a set $E' \subset \Omega$ of measure zero such that for any $x_0 \in \Omega \setminus E'$ and any vector $\xi \in \mathbb{R}^m$ there exists a solution $v(x)$ of system (3.5) such that $v(x_0) = \xi$.

Along with system (3.7) we will also consider the following system of equations with respect to matrix functions

$$\frac{\partial Y}{\partial x_i}(x) = -A_i^*(x) Y(x). \tag{3.9}$$

We have $Y(x_0) = H$. Replacing A_i by $-A_i^*$ in the above reasoning, we obtain that for every $m \times m$ matrix H and for any point $x_0 \notin E'$ there exists a solution of system (3.9) for the Cauchy condition $Y(x_0) = H$ is fulfilled.

Let $Y(x)$ be a solution of the matrix system of equations (3.9) and

$$\frac{\partial Y}{\partial x_i}(x) = -A_i^*(x) Y(x)$$

for each $i = 1, 2, \ldots, n$. Applying the operation of matrix transposition to both sides of this equality, we obtain

$$\frac{\partial Y^*}{\partial x_i}(x) = -[Y(x)]^* A_i(x) \tag{3.10}$$

for each $i = 1, 2, \ldots, n$.

If the matrix function X is a solution of equation (3.6) and Y is a solution of equation (3.9), then the matrix $Y^* X$ in the domain Ω is constant. Indeed, by equality (3.10) we obtain

$$\frac{\partial}{\partial x_i}(Y^* X) = \frac{\partial Y^*}{\partial x_i} X + Y^* \frac{\partial X}{\partial x_i} = -Y^* A_i X + Y^* A_i X = 0.$$

Thus the derivatives of the matrix function $Z = Y^* X$ in the domain Ω are constant, whence it follows this function is constant in the domain Ω.

Let us choose arbitrarily a point $x_0 \in \Omega$ not belonging to anyone of the sets E and E'. Let the matrix functions $P(x)$ and $Q(x)$ be solutions of systems (3.6) and (3.9) with Cauchy conditions $P(x_0) = I$ and $Q(x_0) = I$, respectively. (Here I denotes the unit $m \times m$ matrix.) Such solutions exist since $x_0 \notin E \cup E'$. Then $[Q(x)]^* P(x) = I$ for $x = x_0$ and therefore $[Q(x)]^* P(x) = I$ for all $x \in \Omega$.

The matrix $[Q(x)]^*$ is thus inverse to $P(x)$ at each point $x \in \Omega$.

We put

$$\Theta(x, y) = P(x)[Q(y)]^*, \quad \widetilde{\Theta}(x, y) = Q(x)[P(y)]^*. \tag{3.11}$$

For any vectors ξ, $\eta \in \mathbb{R}^m$ the vector function $z(x) = \Theta(x, y)\xi$ is a solution of system (3.1), while $v(x) = \widetilde{\Theta}(x, y)\eta$ is a solution of system (3.9). Also, $z(y) = P(y)[Q(y)]^*\xi = \xi$ and, exactly in the same way, $v(y) = Q(y)[P(y)]^*\xi = \eta$, i.e., the functions $z(x)$ and $v(x)$ satisfy the Cauchy conditions $z(y) = \xi$, $v(y) = \eta$.

The point $y \in \Omega$ has been chosen arbitrarily and therefore we obtain that for any point $y \in \Omega$ and for any vector $\xi \in \mathbb{R}^m$ there exists a solution of system (3.1) such that $z(y) = \xi$.

Simultaneously, we obtain that for any $\eta \in \mathbb{R}^m$ system (3.6) has a solution $v(x)$ for which $v(y) = \eta$. This means that the sets E and E', consisting of those points $y \in \Omega$ for which the Cauchy problems $z(y) = \xi$ and $v(x) = \eta$ are not always solvable, are empty.

The matrix function $\Theta(x, y) = P(x)[Q(y)]^* = P(x)[P(y)]^{-1}$ is called *the fundamental matrix of system* (3.1). Analogously, $\widetilde{\Theta}(x, y)$ is called *the fundamental matrix of system* (3.6).

Further, we have

$$\widetilde{\Theta}(x, y) = [\Theta(y, x)]^*.$$

Let L_i be the differential operator defined by the equality $(L_i z)(x) = \frac{\partial z}{\partial x_i}(x) - A_i(x)z(x)$, where $A_i(x)$ are the matrices contained in system (3.1), while $P(x)$ and $Q(x)$ are the matrix function define above. Then we have the equality

$$L_i = P \circ \frac{\partial}{\partial x_i} \circ Q^* \tag{3.12}$$

or, in the explicit form,

$$L_i z(x) = P(x) \frac{\partial}{\partial x_i} \big\{ [Q(x)]^* z(x) \big\}. \tag{3.13}$$

Let $z(x)$ and $v(x)$ be solutions of systems (3.1) and (3.7), respectively, in the domain Ω. Then the scalar product $\langle z(x), v(x) \rangle$ is a constant function in the domain Ω. Indeed, if we choose arbitrarily a point $y \in \Omega$ and put $z(y) = \xi$, $v(y) = \eta$, then we have

$$\begin{aligned} \langle z(x), v(x) \rangle &= \langle \Theta(x, y)\xi, \widetilde{\Theta}(x, y)\eta \rangle = \langle \Theta(x, y)\xi, [\Theta(y, x)]^*\eta \rangle \\ &= \langle \Theta(y, x)\Theta(x, y)\xi, \eta \rangle = \langle \xi, \eta \rangle. \end{aligned}$$

4. Estimates for operators under the condition of complete integrability. A domain Ω in the space $\mathbb{R}^n$ is called *star-shaped with respect to a ball* $B(a,r)\subset\Omega$ if for any point $x\in\Omega$ and for any point $y\in B(a,r)$ the rectilinear segment connecting the points x and y (i.e., the set of all points $u=\lambda x+(1-\lambda)y$, where $0\le\lambda\le 1$) is contained in the domain Ω. A domain which is star-shaped with respect to a ball belongs to the class $\mathcal{S}$.

The main results are further formulated for the case of star-shaped domains with respect to a ball. Transition to a more general case is performed by purely technical means as was done, for instance, in the monograph [9].

Assume that we are given the linear differential operators

$$L_i=\frac{\partial}{\partial x_i}-A_i(x),\quad i=1,3,\dots,n, \tag{4.1}$$

defined in the domain Ω and being such that the system of equations

$$L_i z=0,\quad i=1,2,\dots,n, \tag{4.2}$$

satisfies all conditions discussed in Subsection 3.

Assume that Ω is a star-shaped domain of the space $\mathbb{R}^n$ with respect to a ball $B(a,r)$. Then any function $z\in W_p^1$, $p\ge 1$, admits an integral representation of the form

$$z(x)=\int\limits_{\Omega} z(y)\varphi(y)\,dy+\int\limits_{\Omega}\sum_{k=1}^{n}K(x,y)(x_k-y_k)\frac{\partial z}{\partial x_i}(y)dy, \tag{4.3}$$

where φ is a nonnegative function of the class $\mathcal{C}^\infty$ whose carrier is contained in the ball $B(a,r)$. Assuming $z\equiv 1$ in equality (4.3), we obtain that the integral of the function φ with respect to the domain Ω is equal to one. The function $K(x,y)$ admits a representation $K(x,y)=\lambda(x,x-y)-\mu(x,y)$, where

$$\lambda(x,z)=\int\limits_0^\infty\varphi(x-zt)t^{n-1}\,dt,\quad \mu(x,y)=\int\limits_0^1\varphi[x+(y-x)t]t^{n-1}\,dt. \tag{4.4}$$

The function $\lambda(x,z)$ is of positively homogeneous degree $-n$ and belongs to the class $\mathcal{C}^\infty$ on the set $\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})$, $\mu(x,y)$ is a function of the class $\mathcal{C}^\infty$ on the set $\mathbb{R}^n\times\mathbb{R}^n$. Applying equality (3.13), from (4.4) we can obtain an integral representation of the function z through the values of differential operators L_i, $i=1,2,\dots,n$, satisfying the condition of complete integrability.

Let the matrix functions $P(x)$ and $Q(x)$ be defined as indicated in Subsection 3, i.e., $P(x)$ is a solution of equation (3.6), $Q(x)$ is a solution of equation (3.9), where $Q(y)=I$ for some $y\in\omega$ and $P(y)=I$. Replacing in (4.4) the function z by $[Q(x)]^*z(x)$ and left-multiplying both parts of the resulting equality by the matrix $P(x)$, after obvious transformations and by means of equalities (3.13) and (3.11) we come to the integral representation

$$z(x) = \int\limits_{\Omega} \Theta(x,y) z(y) \varphi(y)\, dy + \int\limits_{\Omega} \Theta(x,y) \sum_{k=1}^{n} K(x,y)(x_k - y_k) L_k z(y)\, dy. \tag{4.5}$$

Assume that

$$(\Pi z)(x) = \int\limits_{\Omega} \Theta(x,y) z(y) \varphi(y)\, dy, \tag{4.6}$$

where φ is the function contained in equalities (4.3) and (4.5).

Substituting an arbitrary solution $z(x)$ of system (4.2) into equality (4.5), we see that the equality $z(x) \equiv (\Pi z)(x)$ is fulfilled for this $z(x)$.

For an arbitrary function $z \in W_p^1(\Omega)$ assume that

$$\|Lz\|_{L_p(\Omega)} = \sum_{i=1}^{n} \|L_i z\|_{L_p(\Omega)}.$$

In the sequel we will need the following proposition whose proof can be found, for instance, in the monograph [14].

Theorem A (on integrals with a weak singularity). *Let Ω be the bounded domain in the space $\mathbb{R}^n$, $\omega(x,z)$, the function of variables $x \in \Omega$ and $z \in (\mathbb{R}^n \setminus 0)$, be homogeneous of zero degree with respect to the variable z. Assume that the function ω has all partial derivatives of order not higher than l on the set $\Omega \times (\mathbb{R}^n \setminus 0)$, these derivatives being bounded on the set $\Omega \times S(0,1)$. Let a number $p > 1$ and a function $f \in L_p(\Omega)$ be given. Assume that*

$$Hf(x) = \int\limits_{\Omega} \frac{\omega(x, x-y)}{|x-y|^{n-l}}\, dy. \tag{4.7}$$

The value $Hf(x)$ is well-defined and finite for almost all $x \in \Omega$ and the thus defined function $Hf(x)$ belongs to the class $W_p^l(\Omega)$. Then there exists a number $C < \infty$ not depending on a choice of f and being such that the inequality

$$\|Hf\|_{W_p^l(\Omega)} \le C \|f\|_{L_p(\Omega)} \tag{4.8}$$

is fulfilled.

Theorem 1. *Let L_i, $i = 1, 2, \ldots, n$, be the set of differential operators of form (4.1) defined in the domain Ω and being such that the matrix functions A_i belong to the class $L_p(\Omega)$ and the system of differential equations (4.2) is completely integrable. Then there exists a constant $< \infty$ such that for any function $z : \Omega \to \mathbb{R}^m$ of the class $W_p^1(\Omega)$ the inequality*

$$\|z - \Pi z\|_{W_p^1(\Omega)} \le C \|Lz\|_{L_p(\Omega)} \tag{4.9}$$

is fulfilled, where Π is the integral operator defined by equality (4.6).

Proof. Let $z : \Omega \to \mathbb{R}^m$ be an arbitrary function of the class $W_p^1(\Omega)$, where as usual it is assumed that $p > n$. From the integral representation (4.5) we conclude that

$$z(x) - (\Pi z)(x) = \int\limits_{\Omega} \Theta(x, y) \sum_{k=1}^{n} K(x, y)(x_k - y_k) L_k z(y)\, dy. \tag{4.10}$$

The matrix $\Theta(x, y)$ is representable in the form $\Theta(x, y) = P(x)[P(y)]^{-1}$, while the matrix function $P(x)$ is a solution of system (3.6) and $[P(x)]^{-1} = [Q(x)]^*$, where $Q(x)$ is a solution of system (3.9). Also, $P(y) = Q(y) = I$ for some point $y \in \Omega$. In that case, the point y can be chosen arbitrarily. Assume that $P(x_0) = Q(x_0) = I$. Put $L_i u = v_i$. The matrix functions P and Q belong to the class $W_p^1(\Omega)$ and therefore are bounded and continuous on the set Ω. Let $M < \infty$ be such that the inequalities $|P(x)| \le M$, $|[P(x)]^{-1}| \le M$ are fulfilled for all $x \in \Omega$.

Let us consider individual summands on the right-hand side of equality (4.10). Assume that

$$R_k v_i(x) = P(x) \int\limits_{\Omega} (x_k - y_k) K(x, y)[P(y)]^{-1} v_i(y)\, dy.$$

We have $R_k v_i(x) = S_k v_i(x) - T_k v_i(x)$, where

$$S_k v_i(x) = \int\limits_{\Omega} (x_k - y_k)\lambda(x, x - y) v_i(y)\, dy, \ T_k v_i(x) = \int\limits_{\Omega} (x_k - y_k)\mu(x, y) v_i(y)\, dy,$$

while the functions $\lambda(x, z)$ and $\mu(x, y)$ are defined by equalities (4.4). Assuming that in the theorem on integrals with a weak singularity $\omega(x, z) = z_k |z|^{n-1} \lambda(x, z)$, we obtain $\|S_k v_i\|_{W_p^1(\Omega)} \le C\|v_i\|_{L_p(\Omega)} \le MC\|u_i\|_{L_p(\Omega)}$.

The function $\mu(x, y)$ has continuous derivatives of any order with respect to each of the variables x and y everywhere in $\mathbb{R}^n$. In the domain Ω these derivatives are bounded. Hence it follows that the functions $T_k v_i(x)$ have continuous derivatives, which can be found if in the formula representing $T_k v_i(x)$ we replace the integrand by the corresponding derivative. Now it is not difficult to conclude that derivatives of the function $T_k v_i(x)$ on the set Ω are bounded and continuous and therefore $T_k v_i(x)$ belongs to the class $W_p^1(\Omega)$. From the integral formula by means of which derivatives of the function $T_k v_i(x)$ are expressed it immediately follows that $\|T_k v_i\|_{W_p^1(\Omega)} \le C\|v_i\|_{L_p(\Omega)} \le MC\|u_i\|_{L_p(\Omega)}$.

Thus the functions $R_k v_i = S_k v_i - T_k v_i$ belong to the class $W_p^1(\Omega)$ and the estimate

$$\|R_k v_i\|_{W_p^1(\Omega)} \le C\|u_i\|_{L_p(\Omega)}$$

is valid.

Let $u : \Omega \to \mathbb{R}^m$ be a vector function of the class $W_p^1(\Omega)$, $p > n$, and $P(x)$ be the matrix function defined above. Then $u \in W_1^1(\Omega)$ and the equality

$$\frac{\partial (Pu)}{\partial x_i} = \frac{\partial P}{\partial x_i} u + P \frac{\partial u}{\partial x_i}$$

is fulfilled for each $i = 1, 2, \ldots, n$. The validity of this statement can be easily established if we approximate the functions $P(x)$ and $u(x)$ by Sobolev averaged functions. Then we obtain

$$\left\|\frac{\partial Pu}{\partial x_i}\right\|_{L_p(\Omega)} \leq \sup_{x\in\Omega}|u(x)|\left\|\frac{\partial P}{\partial x_i}\right\|_{L_p(\Omega)} + \sup_{x\in\Omega}|P(x)|\left\|\frac{\partial u}{\partial x_i}\right\|_{L_p(\Omega)}. \tag{4.11}$$

Since $p > n$, the value $\sup_{x\in\Omega}|u(x)|$ is finite. Also, $\sup_{x\in\Omega}|u(x)| \leq C\|u\|_{W_p^1(\Omega)}$.

Let us find a constant $M < \infty$ such that $\max_{x\in\Omega}|P(x)| \leq M$ and

$$\left\|\frac{\partial P}{\partial x_i}\right\|_{L_p(\Omega)} \leq M$$

for each $i = 1, 2, \ldots, n$. Then from inequality (4.11) it follows that

$$\|Pu\|_{W_p^1(\Omega)} \leq C\|u\|_{W_p^1(\Omega)},$$

where $C < \infty$.

The obtained estimates obviously imply the assertion of the theorem. □

5. Stability of solutions of completely integrable systems. Let Ω be a bounded domain in the space $\mathbb{R}^n$, star-shaped with respect to the ball $B(x_0, r) \subset \Omega$. It is assumed that in the domain Ω we have the completely integrable systems of differential equations

$$\frac{\partial z}{\partial x_i}(x) - A_i(x)z(x) = 0, \tag{5.1}$$

$$\frac{\partial z}{\partial x_i}(x) - B_i(x)z(x) = 0, \tag{5.2}$$

where $A_i(x)$ and $B_i(x)$ are $m \times m$ matrices, $i = 1, 2, \ldots, n$. Also, it is assumed that the matrix functions A_i and B_i belong to the class $L_p(\Omega)$ for some $p > n$. We put

$$\delta_p(A, B) = \left\{\sum_{i=1}^n \Big\||A_i - B_i|\Big\|_{L_p}^p\right\}^{1/p}.$$

Theorem 2. *Let $z : \Omega \to \mathbb{R}^m$ and $\overline{z} : \Omega \to \mathbb{R}^m$ be solutions of systems* (5.1) *and* (5.2)*, respectively. Assume that these systems satisfy all the conditions given above and the equality $z(x_0) = \xi$, $\overline{z}(x_0) = \overline{\xi}$ $\xi, \overline{\xi} \in \mathbb{R}^m$, is fulfilled. Then there exist constants $\varepsilon_0 > 0$, $C < \infty$ and $M < \infty$ such that if $\delta_p(A, B) < \varepsilon_0$, then the inequality*

$$\|\overline{z} - z\|_{W_p^1(\Omega)} \leq C\delta_p(A, B)|\overline{\xi}| + M|\xi - \overline{\xi}|$$

is fulfilled.

Proof. Let z_0 be a solution of system (5.1) that satisfies the Cauchy condition $z_0(x_0) = \overline{\xi}$. We assume that $\zeta = \overline{z} - z_0$. From equality (5.2) we have

$$\frac{\partial \overline{z}}{\partial x_i}(x) - A_i(x)\overline{z}(x) = -[A_i(x) - B_i(x)]\overline{z}, \quad i = 1, 2, \ldots, n. \tag{5.3}$$

Replacing in equality (5.1) z by z_0 and subtracting termwise the resulting equality from (5.3), we obtain

$$\frac{\partial(\overline{z}-z_0)}{\partial x_i}(x) - A_i(x)[\overline{z}(x) - z_0(x)] = -[A_i(x) - B_i(x)]\overline{z}. \tag{5.4}$$

Hence it follows that

$$\frac{\partial\zeta}{\partial x_i}(x) - A_i(x)\zeta(x) = -[A_i(x) - B_i(x)]z_0(x) - [A_i(x) - B_i(x)]\zeta(x). \tag{5.5}$$

Let $\Theta(x, y)$ be the fundamental matrix of system (5.1). The matrix function $P(x) = \Theta(x, x_0)$ belongs to the class W_p^1 and therefore by virtue of Sobolev's embedding theorem

$$\|P\|_{L_\infty(\Omega)} = \sup_{x\in\Omega} |P(x)| = K < \infty.$$

Thus we have $z_0(x) = P(x)\overline{\xi}$. Hence we conclude that $|z_0(x)| \leq K|\overline{\xi}|$ for all $x \in \Omega$.

By virtue of Lemma 1, the function ζ belongs to the class $W_p^1(\Omega)$. By the embedding theorem this implies that $\|\zeta\|_{L_\infty(\Omega)} \leq C\|\zeta\|_{W_p^1(\Omega)}$. By virtue of Theorem 1, from equalities (5.5) it follows that

$$\|\zeta\|_{W_p^1(\Omega)} \leq C\delta_p(A, B)|z|_{L_\infty(\Omega)} + C\delta_p(A, B)|\zeta|_{L_\infty(\Omega)}.$$

Hence we obtain

$$\|\zeta\|_{W_p^1(\Omega)} \leq C_1\delta_p(A, B)|\overline{\xi}| + C_2\delta_p(A, B)|\zeta|_{W_p^1(\Omega)},$$

where C_1 and C_2 are constants.

Let us define $\varepsilon_0 > 0$ by the condition $C_2\varepsilon_0 \leq \frac{1}{2}$. Then for $\delta_p(A, B) \leq \varepsilon_0$ we have the inequality

$$\|\zeta\|_{W_p^1(\Omega)} \leq C\delta_p(A, B)|\overline{\xi}|, \tag{5.6}$$

where $C = 2C_1$.

We have $z(x) = P(x)\xi$, $z_0(x) = P(x)\overline{\xi}$, whence

$$\|z - z_0\|_{W_p^1(\Omega)} \leq \|P\|_{W_p^1(\Omega)}|\xi - \overline{\xi}| = M|\xi - \overline{\xi}|. \tag{5.7}$$

Inequalities (5.6) and (5.7) obviously imply

$$\|\overline{z} - z\|_{W_p^1(\Omega)} \leq C\delta_p(A, B)|\overline{\xi}| + M|\xi - \overline{\xi}|.$$

The theorem is proved. □

6. Stability in Bonnet's theorem of the surface theory. We will use the terms and notation from differential geometry without specifying them specially.

Let Ω be a bounded domain of the class S in the space $\mathbb{R}^n$, and $\mathbf{r} : \Omega \to \mathbb{R}^{n+1}$ be a vector function of the class $W_p^2(\Omega)$, $p > n$. Assume $\mathbf{r}_i = \frac{\partial \mathbf{r}}{\partial x^i}$. The functions $\mathbf{r}_i$, $i = 1, 2, \ldots, n$, belong to the class W_p^1. They are continuous and bounded in the domain Ω since $p > n$.

We say that the mapping $\mathbf{r}$ defines the hypersurface V of the class W_p^2 in the space $\mathbb{R}^{n+1}$ if the vectors $\mathbf{r}_1(x), \mathbf{r}_2(x), \ldots, \mathbf{r}_n(x)$ are linearly independent for any $x \in \Omega$.

Further, $\langle \cdot, \cdot \rangle$ denotes a scalar product in $\mathbb{R}^{n+1}$.

For any point $x \in \Omega$, let $\mathbf{n}(x)$ be the normal unit vector to the n-dimensional surface V in the point $\mathbf{r}(x)$, i.e., the vector orthogonal to n-dimensional plane "stretched" onto the vectors $\mathbf{r}_i(x)$, $i=1, 2, \dots, n$, and being such that $|\mathbf{n}(x)|=1$.

The vectors $\mathbf{r}_1(x), \dots, \mathbf{r}_n(x), \mathbf{n}(x)$ are linearly independent at each point $x \in \Omega$. The direction of the normal is defined under the assumption that the orientation of the frame $\{\mathbf{r}_1(x), \dots, \mathbf{r}_n(x), \mathbf{n}(x)\}$ is the same as that of the base frame of the space $\mathbb{R}^{n+1}$. A collection of vectors $\mathbf{r}_1(x), \mathbf{r}_2(x), \dots, \mathbf{r}_n(x), \mathbf{n}(x)$ is called *a moving frame of the parametrization* $\mathbf{r}$ *of the surface* V *at a point* x.

Let $R(x)$ be an $(n+1) \times (n+1)$-matrix such that for $1 \leq i \leq n$ the i-th row of this matrix is the vector $\mathbf{r}_i$, while the row with number $n+1$ is the vector $\mathbf{n}$. We say that $R(x)$ is a matrix representation of a moving frame of the parametrization $\mathbf{r}$ of the hypersurface V at a point x.

Note that $\det R(x) > 0$ for any $x \in \Omega$.

The metric tensor $g_{ij} = \langle \mathbf{r}_i, \mathbf{r}_j \rangle$ of the hypersurface V is defined in a usual manner. We also call it it the first fundamental tensor of the hypersurface V. The components of this tensor g_{ij} are functions of the class W^1_p. These functions are continuous since, by assumption, $p > n$.

Let $G(x)$ be the matrix $(g_{ij})_{i,j=1,2,\dots,n}$ whose elements are the components of the metric tensor of the surface, $g(x) = \det G(x) > 0$. The function $g(x)$ is continuous. Hence we conclude that for any domain Σ whose closure is compact and contained in Ω there exists a constant γ_Σ such that $g(x) \geq \gamma_\Sigma$ for all $x \in \Sigma$.

Let $g^{ij}(x)$ be the contravariant form of the metric tensor of the hypersurface V. For all $x \in \Omega$ we have $g^{i\alpha}(x) g_{j\alpha}(x) = \delta^i_j$, where δ^i_j is the Kronecker symbol. The matrix formed by the components of the tensor $g^{ij}(x)$ is inverse to the matrix $G(x)$. Since the function $g(x)$ is continuous and, moreover, $g(x) > 0$ for all $x \in \Omega$, the functions g^{ij} belong to the class $W^1_{p,\mathrm{loc}}(\Omega)$.

The Christoffel symbols Γ^i_{jk} of the hypersurface V are defined as it is done for surfaces satisfying the standard requirements of differential geometry for regularity. Namely,

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\alpha} \left\{ \frac{\partial g_{j\alpha}}{\partial x^k} + \frac{\partial g_{k\alpha}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\alpha} \right\},$$

where the derivatives are understood in the Sobolev sense. As follows from the previous assumptions, the functions Γ^i_{jk} belong to the class $L_{p,\mathrm{loc}}(\Omega)$.

We further assume $b_{ij} = \left\langle \frac{\partial^2 \mathbf{r}}{\partial x^i \partial x^j}, \mathbf{n} \right\rangle$. The tensor b_{ij} thus defined is called *the second fundamental tensor of the hypersurface* V. The equalities

$$\frac{\partial \mathbf{r}_i}{\partial x^j} = \Gamma^\alpha_{ij} \mathbf{r}_\alpha + b_{ij} \mathbf{n}, \qquad \frac{\partial \mathbf{n}}{\partial x^i} = -b^\alpha_i \mathbf{r}_\alpha \tag{6.1}$$

are valid. We call them *derivational formulas for the hypersurface* V.

Using the notation introduced above, equations (6.1) can be rewritten as

$$\frac{\partial R}{\partial x_j}(x) = A_j(x) R(x), \tag{6.2}$$

where $A_j(x)$ an $n+1) \times (n+1)$-matrix,

$$A_j(x) = \begin{pmatrix} \Gamma^1_{1j} & \dots & \Gamma^n_{1j} & b_{1j} \\ \vdots & \ddots & \vdots & \vdots \\ \Gamma^1_{nj} & \dots & \Gamma^n_{nj} & b_{nj} \\ -b^i_j & \dots & -b^n_j & 0 \end{pmatrix}. \tag{6.3}$$

Lemma 2. *Let a mapping* $\mathbf{r} : \Omega \to \mathbb{R}^{n+1}$ *define the hypersurface* V *of the class* W^2_p *in the space* $\mathbb{R}^{n+1}$. *Assume that the matrices* $A_i(x)$, $i = 1, 2, \dots, n$, *and* $R_{\mathbf{r}}(x)$ *are defined by* $\mathbf{r}$ *as described above. Then for any* $i, j = 1, 2, \dots, n$, *the relation* (*Peterson–Codazzi identity*)

$$\frac{\partial A_i}{\partial x^j}(x) - \frac{\partial A_j}{\partial x^i}(x) = A_i(x)A_j(x) - A_j(x)A_i(x) \tag{6.4}$$

is fulfilled, where the derivatives are understood in the sense of the theory of generalized functions. This means that the equality

$$\int\limits_{\Omega\times\mathbb{R}^n} \left\{ \frac{\partial \varphi}{\partial x^i} A_j - \frac{\partial \varphi}{\partial x^j} A_i \right\} dx = \int\limits_{\Omega\times\mathbb{R}^n} \varphi[A_iA_j - A_jA_i]\, dx \tag{6.5}$$

holds for any function $\varphi \in \mathcal{C}^\infty_0(\Omega)$.

Remark. For the case $n = 2$ the validity of this assertion was established by I. Ya. Bakel'man in [9], where equality (6.4) is treated in a manner different from that indicated in the lemma formulation. However the treatment in [9] is equivalent to the one discussed here, but the paper [9] deals with a more general case, namely, it is assumed that the function $\mathbf{r}$ belongs to the class W^1_2 and its derivatives $\mathbf{r}_i = \frac{\partial \mathbf{r}}{\partial x_i}$, $i = 1, 2$, are continuous and linearly independent at each point $x \in \Omega$.

Proof of Lemma 2. In the case, where $\mathbf{r}$ satisfies the standard requirements of differential geometry for regularity, equalities (6.4) are the well-known Peterson–Codazzi equations. The form in which these equations are written here can be found, for instance, in [12], where surfaces are considered in the three-dimensional Euclidean space. For the classical version of Peterson–Codazzi equations see, for instance, the monograph [13]. Here we give the proof of relations (6.4) for the regular case.

Let $\mathbf{r} : \Omega \to \mathbb{R}^{n+1}$ be a surface of the class $\mathcal{C}^\infty$. We have the equalities

$$\frac{\partial R}{\partial x^j}(x) = A_j(x)R(x),$$
$$\frac{\partial R}{\partial x_i}(x) = A_i(x)R(x).$$

Differentiating the first of these equalities termwise with respect to x^i and the second with respect to x^j, and subtracting the resulting equalities we see that the relation

$$\left\{ \frac{\partial A_i}{\partial x^j}(x) - \frac{\partial A_j}{\partial x^i}(x) - [A_i(x)A_j(x) - A_j(x)A_i(x)] \right\} R(x) = 0$$

is valid for all $x \in \Omega$.

Since $\det R(x) \neq 0$ for all $x \in \Omega$, it follows that

$$\frac{\partial A_i}{\partial x^j}(x) - \frac{\partial A_j}{\partial x^i}(x) - [A_i(x)A_j(x) - A_j(x)A_i(x)] = 0 \tag{6.6}$$

for all $x \in \Omega$. The general case can be reduced to the case considered here if the vector function $\mathbf{r}$ is approximated by the corresponding Sobolev averaged function $\mathbf{r}^h$ and if we pass to the limit as $h \to 0$. We do not give the relevant details because of obviousness (see also [8]). □

The next assertion is valid. We omit its proof.

Theorem 3 (Yu. E. Borovskiĭ [9])**.** *Let Ω be a simply connected domain in the space $\mathbb{R}^n$; g_{ij} and b_{ij} be the tensor fields defined in the domain Ω; $g_{ij} \in W_p^1(\Omega)$, for all $x \in \Omega$ $g_{ij}(x)\xi^i\xi^j$ is positively defined, $b_{ij} \in L_p(\Omega)$, where $p > n$. Let the matrix functions $A_i(x)$, $i = 1, 2, \dots, n$, be defined by the functions g_ij and b_ij by equalities* (6.3)*. In that case, if equalities* (6.4) *are fulfilled for the matrix functions $A_i(x)$, then there exists a hypersurface $\mathbf{r} : \Omega \to \mathbb{R}^{n+1}$ of the class $W_p^2(\Omega)$, for which g_{ij} is the first and b_{ij} is the second fundamental tensor.*

This assertion is a simple corollary of Borovskiĭ's theorem on the solvability of linear, completely integrable equations (see [6], [7]). Conditions (6.4) imply that the system of equations

$$\frac{\partial z}{\partial x_i}(x) = A_i(x)z(x), \quad i = 1, 2, \dots, n,$$

is completely integrable. Along with system, we define a completely integrable system of equations with respect to matrix functions. By choosing an appropriate solution of the matrix system it is not difficult to show that it defines some moving frame of the surface sought for. This second half of the proof is carried out by the arguments repeating word for word the arguments usually given manuals on differential geometry. The vector function $\mathbf{r}$ is recovered by the moving frame of integration.

Let the hypersurfaces $\mathbf{r} : \Omega \to \mathbb{R}^{n+1}$ and $\bar{\mathbf{r}} : \Omega \to \mathbb{R}^{n+1}$ of a class W_p^2, $p > n$, be given in the space $\mathbb{R}^{n+1}$, g_{ij}; $\overline{g}_{ij}$ be the metric functions of these hypersurfaces. Assume that there exists a constant $\gamma > 0$ such that the discriminant of each of these tensors is bounded from below by the number γ^2 at each point of the domain Ω. Let b_{ij} and $\overline{b}_{ij}$ be the second fundamental tensors of the considered hypersurfaces. We assume

$$\Delta_p(\mathbf{r}, \bar{\mathbf{r}}) = \sum_{1 \le i,j \le n} \|\overline{g}_{ij} - g_{ij}\|_{W_p^1(\Omega)} + \sum_{1 \le i,j \le n} \|\overline{b}_{ij} - b_{ij}\|_{L_p(\Omega)}.$$

The value characterizes an extent of the closeness of the fundamental tensors of the surfaces. Let $A_i(x)$ and $\overline{A}_i(x)$, $i = 1, 2, \dots, n$, be the matrix functions defined for these surfaces by equalities (6.3). The following estimate is valid:

$$\delta_p(A, \overline{A}) \le C\Delta_p(\mathbf{r}, \bar{\mathbf{r}}), \tag{6.7}$$

where C is a constant,

We will further need the following elementary proposition.

Lemma 3. *Let* $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$, $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ *be two arbitrary frames of the space* $\mathbb{R}^n$. *Assume that* $g_{ij} = \langle \mathbf{u}_i, \mathbf{u}_j \rangle$ *and* $h_{ij} = \langle \mathbf{v}_i, \mathbf{v}_j \rangle$. *Let* $\alpha > 0$ *and* $\beta > 0$ *be such that*

$$\alpha g_{ij}\xi^i\xi^j \leq h_{ij}\xi^i\xi^j \leq \beta g_{ij}\xi^i\xi^j$$

for any $\xi = (\xi^1, \xi^2, \dots, \xi^n) \in \mathbb{R}^n$. *Then there exists an orthogonal transformation* Φ *of the space* $\mathbb{R}^n$ *such that the inequality*

$$|\Phi(\mathbf{u}_i) - \mathbf{v}_i| < \left(\sqrt{\frac{\beta}{\alpha}} - 1\right)|\mathbf{u}_i|$$

is fulfilled for any $i = 1, 2, \dots, n$.

Proof. Let F be a linear mapping of the space $\mathbb{R}^n$ onto such that $F(\mathbf{u}_i) = \mathbf{v}_i$ for each $i = 1, 2, \dots, n$. Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$, and ξ^i be its coordinates with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$. We have $\mathbf{x} = \mathbf{u}_i\xi^i$, $F[\mathbf{x}] = \mathbf{v}_i\xi^i$, $|\mathbf{x}|^2 = g_{ij}\xi^i\xi^j$, $|F(\mathbf{x})| = h_{ij}\xi^i\xi^j$.

Hence it follows that the inequalities

$$\sqrt{\alpha} \leq \frac{|F(\mathbf{x})|}{|\mathbf{x}|} \leq \sqrt{\beta} \tag{6.8}$$

hold for any nonzero vector $\mathbf{x}$.

Let $\mathbb{S}$ be the sphere of unit radius with center at point 0 and $\mathbb{E} = F[\mathbb{S}]$. The set E is an ellipsoid. By virtue of inequality (6.8), the inequalities $\sqrt{\alpha} \leq |y| \leq \sqrt{\beta}$ are fulfilled for any point $y \in \mathbb{E}$.

Let $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ be an orthonormal frame in $\mathbb{R}^n$ chosen so that in the system of coordinates with this frame the ellipsoid $\mathbb{E}$ is given by the equation

$$\frac{x_1^2}{r_1^2} + \frac{x_2^2}{r_2^2} + \dots + \frac{x_n^2}{r_n^2} = 1.$$

Here $r_1, r_2, \dots, r_n$ are the semi-axes of the ellipsoid E. We have $\sqrt{\alpha} \leq r_i \leq \sqrt{\beta}$ for each $i = 1, 2, \dots, n$.

Let H be the linear transformation which in the system of coordinates with with basis $\mathbf{w} = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ is given by the formula $H(\mathbf{x}) = \sum\limits_{i=1}^{n} \frac{x_i}{r_i}\mathbf{c}_i$. Here $x_1, x_2, \dots, x_n$ are the coordinates of the vector $\mathbf{x}$ with respect to the basis $\mathbf{w}$, while $y_i = \frac{x_i}{r_i}$ are the coordinates with to the same basis of the vector $y = H(\mathbf{x})$. If $\mathbf{x} \in \mathbb{E}$, then $H(\mathbf{x})$ belongs to the unit sphere $\mathbb{S}$.

The mapping $\Phi = H \circ F$ transforms the sphere $\mathbb{S}$ into and therefore is an orthogonal transformation. For any vector $\mathbf{x} = \sum\limits_{i=1}^{n} x_i\mathbf{w}_i$ we have the equality

$$|H(\mathbf{x}) - \mathbf{x}| = \sqrt{\sum_{i=1}^{n}\left(\frac{1}{r_i} - 1\right)^2 x_i^2} \leq \left(\frac{1}{\sqrt{\alpha}} - \frac{1}{\sqrt{\beta}}\right)|x|.$$

Assuming here that $\mathbf{x} = F[\mathbf{u}_i]$ and taking into account that $|\mathbf{v}_i| \leq \sqrt{\beta}|\mathbf{u}_i|$, we obtain the desired inequality. $\square$

Theorem 4. *Let $\Omega \subset \mathbb{R}^n$ be a star-shaped domain in the space $\mathbb{R}^n$ with respect to a ball $B(a,r)$, and the functions $\mathbf{r} : \Omega \to \mathbb{R}^{n+1}$ and $\overline{\mathbf{r}} : \Omega \to \mathbb{R}^{n+1}$ give some hypersurfaces of the class W_p^2, $p > n$. Let $g(x)$ and $\overline{g}(x)$ be the discriminants of the first fundamental tensors g_{ij} and $\overline{g}_{ij}$ of these surfaces. Assume that there exists a constant $\delta > 0$ such that $g(x) > \delta$ and $\overline{g}(x) > \delta$ for all $x \in \Omega$. Then there are constants $\varepsilon_0 > 0$ and $C < \infty$ such that if $\Delta_p(\mathbf{r}, \overline{\mathbf{r}}) < \varepsilon$, then there exists a motion Φ of the space $\mathbb{R}^{n+1}$ such that*

$$\|\Phi \circ \overline{\mathbf{r}} - \mathbf{r}\|_{W_p^2(\Omega)} < C\Delta_p(\mathbf{r}, \overline{\mathbf{r}}).$$

Proof. For the sake of brevity, we introduce the notation $\rho = \Delta_p(\mathbf{r}, \overline{\mathbf{r}})$. Then $\|g_{ij} - \overline{g}_{ij}\|_{W_p^1} \leq \rho$. By virtue of the embedding theorem, hence it follows that $|g_{ij}(a) - \overline{g}_{ij}(a)| < C\rho$, where a is the center of the ball $B(a,r)$, with respect to which the domain Ω is star-shaped.

Let τ be an arbitrary motion of the space $\mathbb{R}^n$ that makes the point $\overline{\mathbf{r}}(a)$ coincide with the point $\mathbf{r}(a)$ and the vector $\overline{\mathbf{n}}(a)$ with the vector $\mathbf{n}(a)$. To simplify the notation, we use the same symbol $\overline{\mathbf{r}}(a)$ to denote the surface obtained as a result of displacement of the surface $\overline{\mathbf{r}}(a)$. Note that the vectors $\mathbf{v}_i = \overline{\mathbf{r}}_i(a)$ lie in the same n-dimensional plane as the vectors $\mathbf{u}_i = \overline{\mathbf{r}}_i(a)$. We have

$$|g_{ij}(a)\xi^i\xi^j - \overline{g}_{ij}(a)\xi^i\xi^j| \leq C\rho \left(\sum_{i=1}^n |\xi^i|\right)^2 \leq C\rho\sqrt{n}|x|^2 \leq C_1\rho g_{ij}(a)\xi^i\xi^j.$$

Assume that $\varepsilon_0 = \frac{1}{2C_1}$. Then for $\rho < \varepsilon_0$ we have the inequalities

$$(1 - C_1\rho)\overline{g}_{ij}(a)\xi^i\xi^j \leq \overline{g}_{ij}(a)\xi^i\xi^j \leq (1 + C_1\rho)\overline{g}_{ij}(a)\xi^i\xi^j.$$

By virtue of Lemma 3 we now see that by rotating the surface V about the normal at the point $\mathbf{r}(a)$ we can succeed in fulfilling the inequalities

$$|\mathbf{r}_i(a) - \overline{\mathbf{r}}_i(a)| < \eta,$$

where $\eta = \left(\sqrt{\frac{1+C_1\rho}{1-C_1\rho}} - 1\right) \max\limits_{1\leq i\leq n} |\mathbf{r}_i(a)|$. Taking into account that $C_1\rho < \frac{1}{2}$, after simple calculations we obtain $\eta < C_1\rho$.

From what has been proved above it follows that there exists a motion Φ of the space $\mathbb{R}^{n+1}$ such that for the vector function $\mathbf{z} = \Phi[\overline{\mathbf{r}}]$ the conditions

$$\mathbf{z}(a) = \mathbf{r}(a),$$
$$|\mathbf{z}_i(a) - \mathbf{r}_i(a)| < C_1\rho$$

are fulfilled and the normals of the surfaces $\mathbf{z}$ and $\mathbf{r}$ coincide at the point $\mathbf{z}(a) = \mathbf{r}(a)$. The first fundamental tensor of the surface $\mathbf{z}$ is $\overline{g}_{ij}$, while its second fundamental tensor is $\overline{b}_{ij}$.

Let $U(x)$ be the matrix presentation of the moving frame of the surface $\mathbf{z} = \Phi(\overline{\mathbf{r}})$. Theorem 2 allows us to estimate the difference $R(x) - U(x)$.

Theorem 2 is a certain assertion about vector-functions, while Peterson–Codazzi equations concern matrix functions. To consider the case of vector functions we should proceed as follows. Let z_i be the i-th column of the matrix $R(x)$ and u_i be the i-th column of the matrix $U(x)$. Then by virtue of the Peterson–Codazzi theorem the following equalities are valid:

$$\frac{\partial z_i}{\partial x_j}(x) = A_j(x)z(x),$$
$$\frac{\partial u_i}{\partial x_j}(x) = \overline{A}_j(x)z(x), \quad j = 1, 2, \dots, n.$$

We have $|R(a) - U(a)| \leq C_2\rho$ and therefore $|z_i(a) - u_i(a)| < C_2\rho$. By virtue of Theorem 2 hence it follows that there exists $\varepsilon_0 > 0$ such that if $\delta_p(A_i, \overline{A}_i) < \varepsilon_0$, then the inequality $|z_i(a) - u_i(a)|_{W^1_p(\Omega)} \leq C\rho$ is fulfilled for any $i = 1, 2, \dots, n$. This, in view of inequality (6.7) means that there is $\varepsilon_1 > 0$ such that for $\Delta_p(\mathbf{r}, \overline{\mathbf{r}}) < \varepsilon_1$ the estimate (6.7) holds for any $i = 1, 2, \dots, n$. Thus we conclude that under this condition we have $\|R - U\|_{W^1_p} < C\rho$, which in particular implies that the inequalities

$$\|z_i - u_i\|_{W^1_p} < C\rho = \Delta_p(\mathbf{r}, \overline{\mathbf{r}})$$

are fulfilled for the derivatives of the vector functions $\mathbf{r}$ and $\overline{\mathbf{r}}$. From these inequalities we obtain the desired estimate $\|\Phi \circ \overline{\mathbf{r}} - \mathbf{r}\|_{W^2_p(\Omega)} \leq C\Delta_p(\mathbf{r}, \overline{\mathbf{r}})$.

The theorem is proved. □

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