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ON SOME PROPERTIES OF SOLUTIONS OF POLYHARMONIC EQUATION IN POLYHEDRAL ANGLES

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Abstract. For a higher order differential equation with the polyharmonic operator, the Dirichlet and Riquier boundary value problems are studied in some polyhedral angles. Uniqueness theorems for solutions with a bounded "energy integral" of the corresponding BVPs are proved. Recurrent formulas are constructed for representation of fundamental solutions and Green's functions. The asymptotic behavior of solutions at infinity is studied.

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1. Introduction

The Dirichlet problem in a bounded domain for a polyharmonic equation was studied by S. L. Sobolev [17]–[19]. Dirichlet, Neumann, Riquier and some other BVPs were investigated by I.Vekua [23]–[24] in bounded and unbounded domains for harmonic, biharmonic and metaharmonic functions. At various times and by different methods, many authors studied analogous problems in more general cases (see, e.g., [1]–[15], [20]–[21]. In the present paper, Dirichlet and Riquier problems are studied in some unbounded domains for the higher order elliptic differential equation

$$\Delta^m u(x) = f(x) \quad (f \in C_0^\infty) \tag{1.1}$$

with the polyharmonic operator on the left-hand side (the first short report about one special case can be found in [20]). In particular, the relation between the asymptotic behavior of solutions of the corresponding BVPs and the right-hand side of equations is investigated.

Throughout the paper the following notation is used:

$\mathbb{N}$ is the set of natural numbers and n is a space dimension;

Δ^m is the polyharmonic operator, where Δ is the Laplace operator;

$m \in \mathbb{N}$ is the order of a polyharmonic operator;

the Greek letters $\alpha, \beta, \dots$ are multiindexes (e.g., $\alpha \equiv (\alpha_1, \alpha_2, \dots, \alpha_n)$ where every $\alpha_i \in \mathbb{N} \bigcup \{0\}$);

$D_i u(x) \equiv u_{,i}(x) \equiv \frac{\partial u(x)}{\partial x_i}, D_{ik} u(x) \equiv u_{,ik}(x) \equiv \frac{\partial^2 u(x)}{\partial x_i \partial x_k}$ for every $i, k = \overline{1, n}$;

$D^\alpha \equiv D_1^{\alpha_1} \dots D_n^{\alpha_n}$;

$|\alpha| \equiv \alpha_1 + \dots + \alpha_n$ is the module of a multiindex α;

$\alpha! \equiv \alpha_1! \alpha_2! \cdots \alpha_n!$ is the factorial of a multiindex α;

the repeated indexes mean that summation is performed over them;

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$\Re^n_l \equiv \{x \equiv (x_1, x_2, \ldots, x_n) \in \Re^n : x_i \geq 0,\ i = n-l, n-l+1, \ldots, n\}, l \in \{0, 1, \ldots, n-1\}$, are polyhedral angles;

$\Re^n_0 \equiv \Re^n_+ \equiv \{x \equiv (x_1, x_2, \ldots, x_n) : x_n > 0\}$ is the half-space;

$E_m(u, v) \equiv \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha v D^\alpha u$ for every nonnegative integer m and for all functions $u(x)$ and $v(x)$;

$E_m(u) \equiv E_m(u, u)$;

$|x| \equiv \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$

$Q_\rho \equiv \Re^n_l \cap \{x \in \Re^n_l : |x| < \rho\}$; for $l = 0$, Q_ρ is a semiball of radius ρ centered at the origin;

$\delta(x)$ is the Dirac function;

$\Lambda_{n|m}(x)$ is a fundamental solution of the polyharmonic equation of order m, n is a space dimension;

$G_{n|m}(x, y)$ is the Green's function for the Dirichlet problem for a polyharmonic equation of order m, n is a space dimension.

Let ω be an open set in $\Re^n_l$ and γ be a subset of its boundary $\gamma \subset \partial\omega$. If ω is a bounded domain, denote by $H_2(\omega, \gamma)$ the Sobolev space obtained by completion of the set of m times continuously differentiable functions u on $\overline{\omega}$ that equal zero in a neighborhood of γ with respect to the norm

$$\|u\|_m \equiv \left[\int\limits_\omega \sum_{|\alpha| \leq m} (D^\alpha u)^2 dx \right]^{\frac{1}{2}}.$$

But if ω is an unbounded domain, then the definition of $H_2(\omega, \gamma)$ applies with $u(x) = 0$ also at the intersection of ω with some neighborhood of the point at infinity.

Definition 1.1. We say that a function $u(x)$ is a **generalized solution** of the equation (1.1) in $\Re^n_l$ with homogeneous Dirichlet boundary conditions if $u \in H_m(\Re^n_l)$ and if it satisfies the integral identity

$$(-1)^m \int\limits_{Q_\rho} E_m(u, v) dx = \int\limits_{Q_\rho} f(x) v(x) dx \tag{1.2}$$

for any $\rho > 0$ and any function $v \in H_m(Q_\rho, \partial Q_\rho)$, where $f \in L_2(\Re^n_l)$.

In this paper we prove the uniqueness theorems based on the "generalized Hardy's inequality". The formulation and proof of this inequality can be found, e.g., in the paper by V. A. Kondratiev and O. A. Oleĭnik [10].

Lemma 1.1 (Generalized Hardy's inequality)**.** *Let the numbers j, n and $p \in (0, \infty)$ be such that $j + n - p \neq 0$. If for a sufficiently smooth function $g(x)$ in a cone $V \subset \Re^n$ with vertex at the origin, the following condition*

$$\int\limits_V |x^j| | \bigtriangledown g(x)|^p dx < \infty \tag{1.3}$$

is fulfilled, where $\bigtriangledown \equiv (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$ *is the gradient vector, then there exists some constant* M *such that*

$$\int\limits_V |x^{j-p}||g(x) - M|^p dx < K \int\limits_V |x^j|| \bigtriangledown g(x)|^p dx\,, \tag{1.4}$$

where the constant K *is independent of the function* $g(x)$. *If, in addition,* $g(0) = 0$, *then the constant* $M = 0$.

2. On the Dirichlet and Riquier Problems for Biharmonic Equation in the Half-Space

We consider, in the half-space $\Re^n_+$ Sophie Germain's equation (the particular case of equation (1.1) when $m = 2$)

$$\Delta^2 u(x) = f(x) \;\; (f \in C_0^\infty(\Re^n_+)) \tag{2.1}$$

with the homogeneous boundary conditions

$$u\Big|_{x_n=0} = 0, \quad \frac{\partial u}{\partial x_n}\Big|_{x_n=0} = 0 \quad \text{(Dirichlet problem)}, \tag{2.2}$$

or

$$u\Big|_{x_n=0} = 0, \quad \frac{\partial^2 u}{\partial x_n^2}\Big|_{x_n=0} = 0 \quad \text{(Riquier problem)}. \tag{2.3}$$

We assume that all solutions of both problems have a finite "energy integral" (i.e., for all solutions of both BVPs the condition

$$\int\limits_{\Re^n_+} E_2(u) dx < \infty \tag{2.4}$$

holds and sometimes an additional condition

$$\int\limits_{\Re^n_+} E_1(u) dx < \infty \tag{2.5}$$

is assumed to be fulfilled).

Remark 2.1. It is easy to check that every classical (smooth) solution of the problem (2.1)–(2.2) is also a generalized solution. It is likewise easy to see that the classical solution of the Riquier problem (2.1)–(2.3) satisfies the integral identity (1.2) for any semiball Q_ρ and for any function $v \in H_2(Q_\rho, \partial Q_\rho)$.

We begin our discussion by proving the uniqueness theorems of solutions of the corresponding BVPs.

Theorem 2.1 (Uniqueness of a Generalized solution of the Dirichlet BVP)**.** *Let the function* $u(x)$ *be a generalized solution of the homogeneous equation* (2.1) (*i.e.,* $f(x) \equiv 0$) *with the boundary conditions* (2.2). *Then* $u(x) \equiv 0$ *in* $\Re^n_+$ *for any* $n > 2$.

Proof. Let us consider an auxiliary function

$$\Theta(s) \equiv \begin{cases} 1, & 0 < s < 1, \\ \theta(s), & 1 < s < 2, \quad 0 < \theta(s) < 1, \quad \Theta \in C_0^\infty(0,\infty), \\ 0, & s > 2 \end{cases} \tag{2.6}$$

such that

$$|\Theta'(s)|^2 \le K_0 \Theta(s), \quad K_0 = \text{const} > 0. \tag{2.7}$$

For any $R > 0$ let

$$\Theta_R(x) \equiv \Theta\left(\frac{|x|}{R}\right). \tag{2.8}$$

Note that for any numbers $i, k = \overline{1,n}$ the following relations hold:

$$\Theta_{R,i}(x) \equiv \frac{\partial \Theta_R(x)}{\partial x_i} = \Theta'(|x|) \cdot \frac{x_i}{R|x|}, \tag{2.9}$$

$$\Theta_{R,ik}(x) = \Theta''(|x|) \cdot \frac{x_i \cdot x_k}{R^2|x|^2} + \Theta'(|x|) \cdot \frac{|x|^2 \cdot \epsilon_{ik} - x_i \cdot x_k}{R|x|^3}, \tag{2.10}$$

where ϵ_{ik} is the Kronecker symbol.

Consider now $v(x) \equiv u(x) \cdot \Theta_R(x)$. Note that, by the boundary conditions (2.2), due to (2.8) we have

$$v \in H_2(Q_{2R}, \partial Q_{2R}).$$

Substituting this function into the integral identity (1.2), we obtain

$$\int\limits_{Q_{2R}} u_{,ik} u_{,ik} \Theta_R(x) dx + 2 \int\limits_{Q_{2R} \setminus Q_R} u_{,ik} u_{,i} \Theta_{R,k}(x) dx + \int\limits_{Q_{2R} \setminus Q_R} u_{,ik} u \Theta_{R,ik}(x) dx = 0.$$

According to our notation and formulas (2.7)–(2.10) this relation can be rewritten as

$$\int\limits_{Q_{2R}} E_2(u) \cdot \Theta_R(x) dx = \left| 2 \int\limits_{Q_{2R} \setminus Q_R} u_{,ik} u_{,i}\, \Theta'(|x|) \frac{x_k}{R|x|} dx + \int\limits_{Q_{2R} \setminus Q_R} u_{,ik} u \Theta_{R,ik}(x) dx \right|.$$

Using a simple inequality

$$ab \le \frac{\varepsilon \cdot a^2}{2} + \frac{b^2}{2\varepsilon} \tag{2.11}$$

which holds for every $\varepsilon = \text{const} > 0$ and for any numbers $a, b \in \Re^1$, and taking into account (2.7)–(2.10) we obtain

$$\int\limits_{Q_{2R}} E_2(u) \cdot \Theta_R(x) dx \le (\varepsilon + \varepsilon_1) \int\limits_{Q_{2R} \setminus Q_R} E_2(u) \cdot \Theta_R dx$$
$$+ \frac{K_1}{\varepsilon} \int\limits_{Q_{2R} \setminus Q_R} \frac{u_{,i} u_{,i}}{|x|^2} dx + \frac{K_2}{\varepsilon_1} \int\limits_{Q_{2R} \setminus Q_R} \frac{u^2}{|x|^4} dx, \tag{2.12}$$

here the constants K_1 and K_2 are independent of the function $u(x)$ and of radius R, while ε and ε_1 are arbitrary constants from relation (2.11). Applying Hardy's inequality (1.4) to the third term on the right-hand side of (2.12) when $p = 2$ and $j = -2$, we have

$$\int\limits_{Q_{2R}} E_2(u)\cdot\Theta_R(x)dx \le (\varepsilon+\varepsilon_1) \int\limits_{Q_{2R}\setminus Q_R} E_2(u)\cdot\Theta_R dx + \left[\frac{K_1}{\varepsilon} + \frac{K_2^*}{\varepsilon_1}\right] \int\limits_{Q_{2R}\setminus Q_R} \frac{u_{,i}u_{,i}}{|x|^2}dx.$$

Applying Hardy's inequality (1.4) to the second term on the right-hand side implies the estimate

$$\int\limits_{Q_{2R}} E_2(u)\cdot\Theta_R(x)dx \le \widetilde{K} \int\limits_{Q_{2R}\setminus Q_R} E_2(u)dx,$$

where the constant $\widetilde{K}$ does not depend on the radius R and on the function $u(x)$. Consequently, for any positive number $P > 0$ and any radius $R > P$, due to the last relation we obtain

$$\int\limits_{Q_P} E_2(u)dx \le \widetilde{K} \int\limits_{Q_{2R}\setminus Q_R} E_2(u)dx. \tag{2.13}$$

Since u is a generalized solution, condition (2.4) is automatically fulfilled and the right-hand side of (2.13) tends to zero as $R \longrightarrow \infty$. Note that the left-hand side of (2.13) is independent of the radius R. Thus we have obtained

$$\int\limits_{Q_P} E_2(u)\,dx \equiv \int\limits_{Q_P} u_{,ik}u_{,ik}dx \equiv 0$$

for any positive number P. □

Remark 2.2. When $n = 2$ (half-plane) one can easily check that, due to (2.4), condition (1.3) holds, and $j+n-p = 0$ ($j = 0$ and $p = 2$). Thus the generalized Hardy's inequality is not true. Hence in this paper the uniqueness theorem is proved only for the case where a space dimension is $n \ge 3$.

A similar reasoning leads to

Theorem 2.2 (Uniqueness of a solution of the Riquier BVP)**.** *Let the function $u(x)$ be a classical solution of the homogeneous equation* (2.1) (*i.e.*, $f(x) \equiv 0$) *with the boundary conditions* (2.3) *in the half-space* $\Re^n_+$ *and* $n > 2$. *If the additional condition* (2.4) *is fulfilled, then* $u(x) = \widehat{K}x_n$ *in* $\Re^n_+$ *with an arbitrary constant* $\widehat{K}$. *But if, in addition, condition* (2.5) *is fulfilled too, then the solution is unique and* $u(x) \equiv 0$.

Proof. Consider now a function $v(x) \equiv (u(x) - \widehat{K}x_n)\cdot\Theta_R(x)$, where $\Theta_R(x)$ is defined in (2.8). Substituting this function into the integral identity (1.2), as in the previous proof, we obtain

$$\int\limits_{Q_{2R}} E_2(u)\cdot\Theta_R(x)dx \le (\varepsilon+\varepsilon_1)\int\limits_{Q_{2R}\setminus Q_R} E_2(u)\cdot\Theta_R dx$$
$$+\frac{K_1}{\varepsilon}\int\limits_{Q_{2R}\setminus Q_R}\frac{(u-\widehat{K}x_n)_{,i}(u-\widehat{K}x_n)_{,i}}{|x|^2}dx+\frac{K_2}{\varepsilon_1}\int\limits_{Q_{2R}\setminus Q_R}\frac{(u-\widehat{K}x_n)^2}{|x|^4}dx.$$

Hence every second order derivative of the solution of the BVP (2.1)–(2.3) equals zero. Therefore, from the boundary condition (2.3) it follows that $u(x)=\widehat{K}x_n$. It is easy to verify that if the additional condition (2.5) is fulfilled, then $u(x)\equiv 0$. □

Remark 2.3. One can easily check that these theorems are true for any unbounded conical domain with vertex at the origin.

Now let as study the structure of solutions of the corresponding BVPs and the relationship between the function on the right-hand side of equation (2.1) and the character of the asymptotic behavior of these solutions.

Remark 2.4. It is well-known that in the half-space the Polyharmonic function $u(x)$ admits two different representations:

- I. Almazi's representation (see, e.g., [23] or [1])

$$u(x)=\sum_{j=1}^{m}|x|^{2(m-j)}v_j(x), \tag{2.14}$$

- II. The representation (see, e.g., [14] or [22])

$$u(x)=\sum_{j=1}^{m}x_n^{(m-j)}w_j(x), \tag{2.15}$$

where $v_j(x)$ and $w_j(x)$ are harmonic functions for any $j=\overline{1,m}$.

We have generalized one proposition from [23] proved by I. Vekua in the two-dimensional case.

Proposition 2.1. *If $\Lambda_{n|1}(x)$ is a fundamental solution of Laplace equation, then*

$$\Lambda_{n|2}(x)\equiv\frac{|x|^2}{2(4-n)}\Lambda_{n|1}(x) \tag{2.16}$$

is a fundamental solution of the biharmonic equation except the case with $n=4$.

Proof. By the theory of distributions (of generalized functions [16]),

$$\begin{aligned}\Delta\Delta(|x|^2\Lambda_{n|1}(x)) &= \Delta[2n\Lambda_{n|1}(x)+|x|^2\Delta\Lambda_{n|1}(x)+4x_k\Lambda_{n|1,k}(x)]\\ &= 4n\Delta\Lambda_{n|1}(x)+|x|^2\Delta\Delta\Lambda_{n|1}(x)+8x_k\Delta\Lambda_{n|1,k}(x)+8x_{k,j}\Lambda_{n|1,kj}(x).\end{aligned}$$

Recall that summation is assumed to be taken from 1 to n over the repeated indexes. Therefore for every "basic" function ($\varphi \in C_0^\infty(\Re^n)$), in view of the formulas (3.13) and (3.14) below, the following relation holds

$$\langle \Delta\Delta(|x|^2\Lambda_{n|1}(x))|\varphi(x)\rangle = 2(4-n)\langle\delta(x)|\varphi(x)\rangle,$$

which proves the proposition. □

By our notation, $G_{n|1}(x,y)$ is the classical Green's function for the Dirichlet problem for the Laplace equation in the half-space, i.e.,

$$G_{n|1}(x,y) = \frac{1}{\sigma_n(n-2)}\left[\frac{1}{|x-y|^{n-2}} - \frac{1}{|x-\overline{y}|^{n-2}}\right], \tag{2.17}$$

where σ_n is the surface area of the n-dimensional ball of radius 1, and $\overline{y} \equiv (y_1,\dots,y_{n-1},-y_n)$ for $y=(y_1,\dots,y_n)$.

Theorem 2.3. *The generalized unique solution of the BVP* (2.1), (2.2) *in the half-space* $\Re^n_+$ *has the form*

$$u(x) = \int\limits_{\Re^n_+} G_{n|2}(x,y)f(y)dy \tag{2.18}$$

with

$$G_{n|2}(x,y) = \begin{cases} \dfrac{1}{2(4-n)}\left[|x-y|^2G_{n|1}(x,y) + \dfrac{2x_ny_n}{\sigma_n|x-\overline{y}|^{n-2}}\right], & n\neq 4,\\ c_4\left[\ln\dfrac{1}{|x-y|} - \ln\dfrac{1}{|x-\overline{y}|}\right] - \dfrac{2c_4x_4y_4}{|x-\overline{y}|^2}, & n=4, \end{cases} \tag{2.19}$$

where c_4 *is a constant.*

Proof. Firstly, note that the Green's function (2.19) is a fundamental solution of the biharmonic equation (2.1). Due to the structure of this function, for any n the second term on the right-hand side of (2.19) is the first term of the representation (2.15), and hence this term is a biharmonic function; for $n \neq 4$ the first term on the right-hand side of (2.19) is the first term of Almanzi's representation (2.15) and so this term is a fundamental solution, i.e.,

$$\Delta^2 G_{n|2}(x,y) = \delta(x). \tag{2.20}$$

If $n=4$, then, as is easy to check by simple calculations, the second term on the right-hand side of (2.19) is a fundamental solution, i.e., identity (2.20) holds.

Secondly, the Green's function (2.19) satisfies the boundary conditions (2.2). In that case, the boundary of the domain is the hyperspace $x \in \Re^n_+ : x_n = 0$ and hence in both cases the second term of (2.19) is zero. Note that

$$|x-\overline{y}| = |x-y| \quad \text{when} \quad x_n = 0. \tag{2.21}$$

and therefore the first Dirichlet condition (2.2) holds. It is easy to check that if $n \neq 4$, then

$$\frac{\partial G_{n|2}(x,y)}{\partial x_n} = \frac{(x_n - y_n)}{4-n} G_{n|1}(x,y) + |x-y|^2 \frac{\partial G_{n|1}(x,y)}{\partial x_n} + \frac{2y_n}{\sigma_n |x-\overline{y}|^{(n-2)}} + \frac{2x_n y_n}{\sigma_n} \frac{\partial |x-\overline{y}|^{(2-n)}}{\partial x_n}. \tag{2.22}$$

If $x_n = 0$, the first term on the right-hand side of (2.22) equals zero, since $G_{n|1}(x,y)$ is the Green's function of the Dirichlet BVP for the Laplace equation. It is evident that the fourth term on the right-hand side of (2.22) also equals zero. Due to (2.17) simple calculations give

$$|x-y|^2 \frac{\partial G_{n|1}(x,y)}{\partial x_n} = \frac{|x-y|^2}{(n-2)} \left[\frac{-(n-2)(x_n - y_n)}{|x-y|^n} - \frac{-(n-2)(x_n + y_n)}{|x-\overline{y}|^n} \right].$$

Hence from (2.21) we have

$$|x-y|^2 \frac{\partial G_{n|1}(x,y)}{\partial x_n} = -\frac{2y_n}{\sigma_n |x-\overline{y}|^{(n-2)}}$$

and thus the second and the third term on the right-hand side of (2.22) vanish. Therefore the second Dirichlet condition (2.2) is fulfilled. By simple calculations it is easy to check that for $n = 4$ the function $G_{4|2}(x,y)$ also satisfies the second boundary condition (2.2).

Thirdly, it is easy to see that for any $i, k = \overline{1,n}$ the partial derivative $\frac{\partial^2 u}{\partial x_i \partial x_k}$ of the solution (2.18) has the order of decay $|x|^{1-n}$ at infinity. Therefore for the solution (2.18) inequality (2.4) is fulfilled when $n > 2$. By uniqueness theorem 2.1 the solution (2.18) of the BVP (2.1), (2.2) is unique. □

A solution of the Riquier BVP is constructed similarly.

Theorem 2.4. *The function*

$$u(x) = \int\limits_{\Re^n_+} G^*_{n|2}(x,y) f(y) dy \tag{2.23}$$

with

$$G^*_{n|2}(x,y) = \begin{cases} \dfrac{|x-y|^2}{2(4-n)} G_{n|1}(x,y) + \dfrac{2x_n y_n}{\sigma_n (4-n)(2-n)|x-\overline{y}|^{n-2}}, & n \neq 2 \ \ or \ \ 4, \\ \dfrac{|x-y|^2}{4} G_{2|1}(x,y) - \dfrac{4x_2 y_2}{\pi} \ln\Big(\dfrac{1}{|x-y|}\Big), & n = 2, \\ c_4 \Big[\ln \dfrac{1}{|x-y|} - \ln \dfrac{1}{|x-\overline{y}|} \Big], & n = 4, \end{cases} \tag{2.24}$$

is a classical unique solution of the BVP (2.1), (2.3) *if $n > 4$ and conditions* (2.4) *and* (2.5) *hold in the half-space $\Re^n_+$; here c_4 is a constant defined in* (2.19).

In the cases $n = 3, 4$ this solution is unique to within an additional term cx_n, where c is an arbitrary constant.

Proof. The proof of this theorem proceeds similarly to that of the previous one. But note that in this case for any $i = \overline{1, n}$, the partial derivative $\frac{\partial u}{\partial x_i}$ of the solution (2.18) has the order of decay $|x|^{2-n}$ at infinity. Therefore condition (2.5) is fulfilled when $n \geq 5$ and it is only in this case that by Theorem 2.1 the solution (2.23) is unique. Also, by the second boundary condition in (2.3), we do not have a unique solution for $n = 3, 4$. □

Remark 2.5. Neither of these theorems includes the case $n = 2$. For the solutions (2.18) and (2.23) condition (2.4) does not take place and therefore these solutions are not unique solutions of the corresponding BVPs.

Now let us consider the asymptotic behavior of these solutions.

Theorem 2.5. *If $u(x)$ (2.18) is a solution of the BVP (2.1), (2.2), then*

$$u(x) = M_n \left[\int\limits_{\Re^n_+} y_n^2 f(y) dy \right] \cdot \frac{x_n^2}{|x|^n} + O(|x|^{1-n}), \tag{2.25}$$

where $M_n = \frac{2}{\sigma_n (4-n)}$ if $n \neq 4$ and $M_4 = 2c_4$ if $n = 4$, and c_4 is defined in (2.19).

Proof. Note that for the solution (2.19) the coefficients of the first terms of Taylor's series in the neighborhood of $y = 0$ are

$$G_{n|2}(x, 0) = \left. \frac{\partial G_{n|2}(x, y)}{\partial y_k} \right|_{y=0} = \left. \frac{\partial^2 G_{n|2}(x, y)}{\partial y_k \partial y_j} \right|_{y=0} = 0$$

for every $k = \overline{1, n}$ and $j = \overline{1, n-1}$. But

$$\left. \frac{\partial^2 G_{n|2}(x, y)}{\partial y_n^2} \right|_{y=0} = M_n \cdot \frac{x_n^2}{|x|^n}.$$ □

Corollary 2.1.

- I. Half-plane. *If $n = 2$, then the BVP (2.1), (2.2) has a solution of the form (2.18), but this function does not always tend to zero as $x \longrightarrow \infty$:*

$$u(x_1, x_2) \longrightarrow 0 \quad \textit{when} \quad x_1 \longrightarrow \infty \quad \textit{and} \quad x_2 = \text{const},$$
$$u(x_1, x_2) \longrightarrow \text{const}, \quad \textit{when} \quad x_2 \longrightarrow \infty \quad \textit{and} \quad x_1 = \text{const}.$$

- II. Half-space. *If $n > 2$, then the BVP (2.1), (2.2) has a solution of the form (2.18), this function always tends to zero as $x \longrightarrow \infty$, but the order of its decay is greater than that of fundamental solutions at least by 1:*

$$u(x_1, \ldots, x_n) = O(|x|^{-n}) \quad \textit{as} \quad |x| \longrightarrow \infty \quad \textit{and} \quad x_n = \text{const},$$
$$u(x_1, \ldots, x_n) = O(|x|^{2-n}) \quad \textit{as} \quad x_n \longrightarrow \infty.$$

Theorem 2.6. *If* $u(x)$ *in* (2.23) *is a solution of the BVP* (2.1), (2.3), *then*

$$u(x) = M_n^* \left[\int\limits_{\Re_+^n} y_n f(y) dy \right] \cdot \frac{x_n}{|x|^n} + O(|x|^{2-n}), \tag{2.26}$$

where $M_n^* = \frac{1}{\sigma_n(2-n)}$ *if* $n \neq 2, 4$ *and* $M_4^* = 2c_4$ *if* $n = 4$, *and* c_4 *is defined in* (2.19).

Proof. Note that for the solution (2.23) the coefficients of the first terms of Taylor's series in the neighborhood of $y = 0$ are

$$G_{n|2}^*(x, 0) = \frac{\partial G_{n|2}^*(x, y)}{\partial y_k}\bigg|_{y=0} = 0$$

for every $k = \overline{1, n-1}$. But

$$\frac{\partial G_{n|2}^*(x, y)}{\partial y_n}\bigg|_{y=0} = M_n^* \cdot \frac{x_n}{|x|^{n-2}}. \qquad \square$$

Corollary 2.2. *The structure of the solution* (2.23) *is asymmetric, but then the behavior of solutions coincides with that of fundamental solutions.*

3. On Some Properties of Solutions of Dirichlet Problems for Polyharmonic Equation in Polyhedral Angles

In the domain $\Re_l^n$ the Dirichlet problem

$$\frac{\partial^j u(x)}{\partial \overrightarrow{\nu}^j}\bigg|_{\partial \Re_l^n} = 0, \quad j = 0, \dots, m-1, \tag{3.1}$$

is considered for equation (1.1), where $\overrightarrow{\nu}$ is a unit outer normal to $\partial \Re_l^n$.

Remark 3.1. It is easy to check that a classical (smooth) solution of problem (1.1)–(3.1) is also a generalized solution.

Recall that due to Definition 1.1 every solution of BVP (3.1) has a finite "energy integral".

$$\int\limits_{\Re_l^n} E_m(u) dx < \infty. \tag{3.2}$$

Theorem 3.1 (Uniqueness of a Generalized solution of the Dirichlet BVP)**.** *Let the function* $u(x)$ *be a generalized solution of the homogeneous equation* (1.1) (*i.e.,* $f(x) \equiv 0$) *with the boundary conditions* (3.1). *Then* $u(x) \equiv 0$ *in* $\Re_l^n$.

Proof. It is easy to check that for the function Θ_R defined in (2.8) and for any multiindex α we have

$$|D^\alpha \Theta_R(x)|^2 \leq \frac{K_\alpha \Theta(|x|)}{|x|^{2|\alpha|}}, \quad \text{where} \quad R < |x| < 2R. \tag{3.3}$$

Here the constants K_α depend only on order α of a partial differential and R is an arbitrary positive number. Note that $v(x) \equiv u(x)\cdot\Theta_R(x) \in H_m(Q_{2R}, \partial Q_{2R})$. Substituting this function into the integral identity (1.2), we obtain

$$\int\limits_{Q_{2R}} E_m(u, u\Theta_R)dx = \int\limits_{Q_{2R}} \left[\sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha u D^\alpha u\right]\Theta_R dx + \int\limits_{Q_{2R}\setminus Q_R} \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha u\left[\sum_{\beta+\iota=\alpha \iota\neq 0} \frac{(|\beta|+|\iota|)!}{\beta!\iota!} D^\beta u D^\iota \Theta_R\right]dx = 0.$$

Therefore

$$\int\limits_{Q_{2R}} E_m(u)\Theta_R dx = \left|\int\limits_{Q_{2R}\setminus Q_R} \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha u[\sum_{\beta+\iota=\alpha \iota\neq 0} \frac{(|\beta|+|\iota|)!}{\beta!\iota!} D^\beta u D^\iota \Theta_R]dx\right|.$$

(2.11) and (3.3) imply that

$$\int\limits_{Q_{2R}} E_m(u)\Theta_R(x)dx \leq \sum_{|\alpha|=m} \frac{m!}{\alpha!}\varepsilon_\alpha \int\limits_{Q_{2R}\setminus Q_R} E_m(u)\Theta_R(x)dx + \sum_{|\alpha|=m} \frac{m!}{\alpha!}\cdot\frac{K_\alpha}{\varepsilon_\alpha} \int\limits_{Q_{2R}\setminus Q_R} \left[\sum_{|\beta|=0}^{m-1} E_{|\beta|}(u)\cdot\frac{1}{R^{2(m-|\beta|)}}\right]dx,$$

where the constants K_α are defined in (3.3) and ε_α are arbitrary positive numbers from (2.11).

Every domain $\Re_l^n$ is "conical" with the vertex at the origin. So, we apply repeatedly the Hardy's inequality (1.4) to the second term on right-hand side of the latter inequality until the order of the partial differential achieves the order m. Note that the generalized Hardy's inequality holds if $n \geq 2, p = 2$ and $j \neq 0$. After an appropriate choice of ε_α we have

$$\int\limits_{Q_{2R}} E_m(u)\Theta_R(x)dx \leq \widetilde{K} \int\limits_{Q_{2R}\setminus Q_R} E_m(u)dx.$$

The constant $\widetilde{K}$ does not depend on the radius R and the function $u(x)$. For an arbitrary real number P there exist a radius $R > P$ and $Q_P \subset Q_R$. Therefore

$$\int\limits_{Q_P} E_m(u)dx \leq \widetilde{K} \int\limits_{Q_{2R}\setminus Q_R} E_m(u)dx. \tag{3.4}$$

Note that by the condition (3.2), the right-hand side of (3.4) tends to zero as $R \to 0$. On the other hand, the left-hand side of (3.4) does not depend on R. Hence for any positive number P the "energy integral" in the domain Q_P equals zero, i.e.,

$$\int\limits_{Q_P} E_m(u)dx \equiv \int\limits_{Q_P} \left[\sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha u D^\alpha u\right]dx \equiv 0.$$

Therefore partial differentials of any order of solutions equal zero and due to the structure of domains and the boundary conditions (3.1) we deduce that $u(x) \equiv 0$ in $\Re_l^n$. □

To construct the Green's function for the corresponding problems let us consider

Proposition 3.1. *If* $\Lambda_{n|m-1}(x)$ *is a fundamental solution of polyharmonic equation*

$$\Delta^{m-1}\Lambda_{n|m-1}(x) = \delta(x), \tag{3.5}$$

then the function

$$\Lambda_{n|m}(x) = \frac{|x|^2}{2(m-1)(2m-n)}\Lambda_{n|m-1}(x) \tag{3.6}$$

is a fundamental solution of the polyharmonic equation (1.1) *except the case where the order of the equation coincides with the space dimension* (*i.e.,* $n = 2m$). *In this particular case, the fundamental solution is constructed separately.*

Proof. For $m = 2$ this proposition holds, see (2.16). For any sufficiently smooth functions $V(x)$ and $U(x)$

$$\Delta(V(x)U(x)) = U(x)\Delta V(x) + V(x)\Delta U(x) + 2V_{,k}(x)U_{,k}(x). \tag{3.7}$$

Substituting the function (3.6) multiplied by c_{nm}^{-1}, where

$$c_{nm} = [2(m-1)(2m-n)]^{-1}, \quad n > 1, \quad n \neq 2m, \tag{3.8}$$

into equation (1.1) and using induction with respect to the order m of the Laplace operator, we have

$$\begin{aligned}\Delta^m c_{nm}^{-1}\Lambda_{n|m}(x) &\equiv \Delta^{m-1}[\Delta|x|^2\Lambda_{n|m-1}(x)]\\ &= \Delta^{m-1}[2n\Lambda_{n|m-1}(x) + |x|^2\Delta\Lambda_{n|m-1}(x) + 4x_{,k}\Lambda_{n|m-1,k}]\\ &= 2n\delta(x) + \Delta^{m-1}[|x^2|\Delta\Lambda_{n|m-1}(x) + 4x_k\Lambda_{n|m-1,k}(x)].\end{aligned}$$

Next,

$$\begin{aligned}\Delta^m c_{nm}^{-1}\Lambda_{n|m}(x) = 2n\delta(x) + \Delta^{m-2}[&2n\Delta\Lambda_{n|m-1}(x) + |x|^2\Delta^2\Lambda_{n|m-1}(x)\\ &+ 8x_k\Delta\Lambda_{n|m-1,k}(x) + 8\epsilon_{jk}\Lambda_{n|m-1,jk}(x)],\end{aligned}$$

where ϵ_{kj} is the Kronecker symbol and hence

$$\begin{aligned}\Delta^m c_{nm}^{-1}\Lambda_{n|m}(x) = 2\cdot 2n\delta(x) + \Delta^{m-2}(&|x|^2\Delta^2\Lambda_{n|m-1}(x)\\ &+ 2\cdot 4x_k\Delta\Lambda_{n|m-1,k}(x)) + 8\delta(x).\end{aligned} \tag{3.9}$$

In the general case, similarly,

$$\begin{aligned}\Delta^m c_{nm}^{-1}\Lambda_{n|m}(x) &= \iota\cdot 2n\delta(x)\\ &+ \Delta^{m-\iota}(|x|^2\Delta^\iota\Lambda_{n|m-1}(x) + \iota\cdot 4x_k\Delta^{\iota-k}\Lambda_{n|m-1,k}(x)) + 8\left[\sum_{j=0}^{\iota-1} j\right]\delta(x)\end{aligned} \tag{3.10}$$

for each number $\iota = \overline{1, m}$ and if $\iota = m$, then

$$\Delta^m c_{nm}^{-1} \Lambda_{n|m}(x) = m \cdot 2n\delta(x) + |x|^2 \Delta\delta(x) + m \cdot 4x_k \delta_{,k} + 8\left[\sum_{j=0}^{m-1} j\right]\delta(x). \quad (3.11)$$

According to the theory of generalized functions, for every "basic" function $\phi \in C_0^\infty(\Re_l^n)$ we have

$$\langle |x|^2 \cdot \Delta\delta(x)|\phi(x)\rangle = 2n\langle \delta(x)|\phi(x)\rangle \quad (3.12)$$

and

$$\left\langle x_k \cdot \frac{\partial \delta(x)}{\partial x_k} \mid \phi(x) \right\rangle = -n\langle \delta(x)|\phi(x)\rangle. \quad (3.13)$$

By (3.9)–(3.10) and (3.12)–(3.13) we rewrite relation (3.11) as

$$\Delta^m c_{nm}^{-1} \Lambda_{n|m}(x) = [2nm + 2n - 4mn + 4m(m-1)]\delta(x). \quad (3.14)$$

By induction with respect to m, we deduce that $\Lambda_{n|m}(x)$ is a fundamental solution of polyharmonic equations. □

Due to the identity

$$\Delta[-\ln|x|] = \frac{2-n}{|x|^2}$$

we have

Proposition 3.2. *In the case where $n = 2m$,*

$$\Lambda_{n|\frac{n}{2}}(x) = -c_n \ln|x| \quad (3.15)$$

is a fundamental solution of equation (1.1)*, where the constant c_n depends only on the space dimension.*

Corollary 3.1. *The fundamental solution $\Lambda_{n|m}(x)$ can be represented by the fundamental solution of Laplace equation as*

$$\Lambda_{n|m}(x) = \frac{|x|^{2(m-1)}}{2^{(m-1)}(m-1)!} \frac{1}{\prod_{k=1}^{m}(2k-n)} \Lambda_{n|1}(x) \quad (3.16)$$

when:

a) *the space dimension is an arbitrary odd number;*

b) *the space dimension is an even number, but it is greater than the order of the equation* (*i.e.*, $n > 2m$).

Theorem 3.2. *A generalized solution of the BVP* (3.1) *for equation* (2.1) *in the half-space $\Re_+^n$ is represented as*

$$u(x) = \int_{\Re_+^n} G_{n|m}(x,y) f(y) dy, \quad (3.17)$$

where the Green's function has the form

$$G_{n|m}(x,y) = c_{nm}|x-y|^2 G_{n|m-1}(x,y) - \frac{\kappa_{nm} x_n^{m-1} y_n^{m-1}}{|x-\overline{y}|^{n-2}}; \quad (3.18)$$

here k and c_{nm} are constants, the latter being defined by (3.8) *and Proposition* 3.2 *for $n = 2m$. This solution satisfies condition* (3.2) *and therefore it is unique.*

Proof. By Almanzi's representation (2.13) the first term on the right-hand side of (3.18) is a fundamental solution. Also, by the representation (2.15) the second term of (3.18) is the polyharmonic function

$$w_1(x) = \frac{\kappa_{nm} y_n^{m-1}}{|x - \overline{y}|^{n-2}}.$$

Note that $\frac{\partial}{\partial \overrightarrow{\nu}} \equiv -\frac{\partial}{\partial x_n}$. So, for every $j = \overline{0, m-2}$, the partial derivatives of order $j < m - 1$ of both terms on the right-hand side of (3.18) equal zero on the boundary. The constant κ_{nm} is chosen so that a partial derivative of the function (3.18) of order $m-1$ vanishes for $x_n = 0$. Therefore the solution (3.17) satisfies the boundary conditions (3.1). By the construction of the Green's function (3.18), it is easy to check that the condition (3.2) takes place when the space dimension is greater than $2m$, the order of equation (1.1), and therefore this solution is unique. □

To construct the Green's function and to simplify the calculation of higher order derivatives, let as use the following special notation: if $y \equiv (y_1, \ldots, y_n)$ is a point in $\Re_l^n$ and $\varsigma \equiv (\varsigma_1, \ldots, \varsigma_n)$ is a multiindex, where ς_k equals 1 or 0 for every $k = \overline{1, n}$, then

$$y^\varsigma \equiv ((-1)^{\varsigma_1} y_1, \ldots, (-1)^{\varsigma_n} y_n).$$

Using this notation, we have

$$|x - y^\varsigma| \equiv \left[\sum_{k=1}^{n} (x - (-1)^{\varsigma_k} y_k)^2 \right]^{\frac{1}{2}}$$

and it is easy to check that

$$\frac{\partial |x - y^\varsigma|^{-k}}{\partial x_j} = \frac{-k[x_j - (-1)^{\varsigma_j} y_j]}{|x - y^\varsigma|^{k+2}}, \tag{3.19}$$

and

$$\frac{\partial |x - y^\varsigma|^{-k}}{\partial y_j} = \frac{(-1)^{\varsigma_j} k[x_j - (-1)^{\varsigma_j} y_j]}{|x - y^\varsigma|^{k+2}}. \tag{3.20}$$

Theorem 3.3. *If $u(x)$ defined by* (3.17) *is a solution of the BVP* (1.1), (3.1), *then*

$$u(x) = c_{nm} \left[\int\limits_{\Re_+^n} y_n^m f(y) dy \right] \cdot \frac{x_n^m}{|x|^n} + O(|x|^{m-n+1}). \tag{3.21}$$

Proof. Note that the coefficients of the first terms of the Taylor's series in a neighborhood of $y = 0$ of the solution (3.17) are

$$G_{n|m}(x, 0) = \left. \frac{\partial G_{n|m}(x, y)}{\partial y_k} \right|_{y=0} = D^\alpha G_{n|m}(x, y)|_{y=0} = 0$$

for every $k=\overline{1,n}$ and $|\alpha|\le m$, where $\alpha_n\ne m$. But

$$\frac{\partial^m G_{n|m}(x,y)}{\partial y_n^m}\bigg|_{y=0} = M_{n|m}\cdot\frac{x_n^m}{|x|^n}. \qquad \square$$

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