Georgian Mathematical Journal
Volume 14 (2007), Number 4, 607–626

ON WIENER'S CRITERION FOR AN ELLIPTIC EQUATION WITH NONUNIFORM DEGENERATION

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Abstract. For some class of nonuniformly degenerated elliptic equations of second order, a necessary and sufficient condition for boundary points to be regular is found. This condition is an analogue of Wiener's criterion for the Laplace equation.

2000 Mathematics Subject Classification: 35B65, 35J67.
Key words and phrases: Wiener's criterion, boundary regularity conditions, elliptic equations, nonuniformly degenerated.

1. Introduction

Let E_n be an n-dimensional space of points $x = (x_1, x_2, \dots, x_n)$, $n \geq 2$, D be the bounded domain in E_n with boundary ∂D, $0 \in \partial D$. We will consider the equation

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) = 0 \tag{1.1}$$

in the domain D. The elements of the matrix $A = \|a_{ij}(x)\|$ given in E_n are symmetrical, real, measurable and satisfy the condition

$$\mu \sum_{j=1}^{n} \rho(x)^{\alpha_j} \eta_j^2 \leq \eta \cdot A\eta \leq \mu^{-1} \sum_{j=1}^{n} \rho(x)^{\alpha_j} \eta_j^2, \tag{1.2}$$

where η is an arbitrary vector in E_n, $\mu = const \leq 1$. Here and in what follows the point denotes a usual scalar product in E_n, $\rho(x) = \max_{1 \leq i \leq n} |x_i|^{1/\sigma_i}$. With regard to the constants we also make the following assumptions:

$$\sigma_i = \frac{\alpha_i + \beta}{2} > 0, \quad |\alpha_i| < \sum_{j=1}^{n} \sigma_j, \quad \beta \geq 0, \tag{1.3}$$

where $i = 1, 2, \dots, n$.

The aim of this paper is to prove a Wiener type criterion of the regularity of a boundary point (see [1] and [2]) in the case of the first boundary value problem for equation (1.1), which admits nonuniform power degeneration at the point 0. For equation (1.1) without degeneration ($\alpha_i = 0$, $i = 1, 2, \dots, n$), the criterion of regularity of boundary points was obtained in the fundamental work [3] (for another proof see [4]). The corresponding results for nondivergent equations with Dini continuous coefficients (or with a ellipticity function) can be found

ISSN 1072-947X / \$8.00 / © Heldermann Verlag *www.heldermann.de*

in [5], [6] ([7]). With regard to equation (1.1) we also want to mention the important work [8] in the case of uniform degeneration of $\omega(x)$ (which replaces all $\rho(x)^{\alpha_i}$, $i = 1, 2, \dots, n$, in condition (1.2)) satisfying the Muckenhoupt condition A_2.

Note that in the case of equation (1.1) the question of internal regularity satisfying condition (1.2) is considered in [9]–[11].

2. Definitions, the Notation and Auxiliary Statements

We denote by $\nabla u = (u_{x_1}, u_{x_2}, \dots, u_{x_n})$ the gradient of the function $u(x)$; $\rho^\alpha \nabla u$ is the vector $\big(\rho(x)^{\alpha_1} u_{x_1}, \rho(x)^{\alpha_2} u_{x_2}, \dots, \rho(x)^{\alpha_n} u_{x_n}\big)$. $\|\rho(x)^{\alpha/2}\nabla u\|_{2,D}$ stands for the sum $\sum_{j=1}^{n} \big\|\rho(x)^{\alpha_j/2}\nabla u_{x_j}\big\|_{2,D}$, where $\|\cdot\|_{2,D}$ is a usual Lebesgue norm in $L_2(D)$.

We denote by $\mathrm{Lip}(D)$ the space of Lipshitz continuous functions having a continuous extension in $\overline{D}$, by $\mathrm{Lip}_0(D)$ a subspace of functions $\mathrm{Lip}(D)$ vanishing on ∂D. $W^1_{2,\alpha}(D)$ is the closure of the space $\mathrm{Lip}(D)$ with respect to the norm

$$\|f\| = \|f\|_{2,D} + \|\rho^{\alpha/2}\nabla u\|_{2,D},$$

where $\|f\|_{2,D} = \|f\|_{L_2(D)}$. $\overset{\circ}{W}{}^1_{2,\alpha}(D)$ is a subspace of $W^1_{2,\alpha}(D)$, in which the set of all functions $\mathrm{Lip}_0(D)$ is dense.

We denote by Q^x_R a quasiball $\{y \in E_n : \rho(y - x) < R\}$.

Definition 2.1. We say that a function $u(x) \in W^1_{2,\alpha}(D)$ is not greater (smaller) than the constant M on a set $E \subset \overline{D}$ in the sense of $W^1_{2,\alpha}(D)$ if there exists a sequence of functions $u_j \in \mathrm{Lip}(D)$ such that 1) $u_j(x) \le M$ ($u_j(x) \ge M$) on E and 2) $\|u_j - u\| \to 0$ in the norm of the space $W^1_{2,\alpha}(D)$ as $j \to \infty$.

Definition 2.2. Any sequence $u_j(x) \in \mathrm{Lip}(D)$ satisfying condition 2) is called approximating for a function $u(x) \in W^1_{2,\alpha}(D)$.

Definition 2.3. A function $u(x) \in W^1_{2,\alpha}(D)$ is said to be equal to M on a set $E \subset \overline{D}$ in the sense of $W^1_{2,\alpha}(D)$ if simultaneously $u(x) \ge M$ and $u(x) \le M$ on E in the sense of $W^1_{2,\alpha}(D)$.

Note that a function $u(x) \in W^1_{2,\alpha}(D)$ is equal to M if and only if there exists an approximating sequence $u_j(x)$ equal to M on the set E.

Let $k \in R$, $u(x) \in W^1_{2,\alpha}(D)$. Denote by $\{u\}_k = \max(u(x), k)$, $\{u\}^k = \min(u(x), k)$ the sections of the function $u(x)$.

Lemma 2.1. *A section $\{u\}_k$ of a function $u(x) \in W^1_{2,\alpha}(D)$ belongs to this space and, if u_j approximates $u(x)$, then sections $\{u_j\}_k$ will approximate $\{u\}_k$, $\|\{u_j\}_k - \{u\}_k\| \to 0$ in the norm $W^1_{2,\alpha}(D)$ as $j \to \infty$.*

Proof. We extract from u_j an a.e. converging subsequence. Preserving for it the notation of the original sequence, we have

$$u_j \to u, \quad \nabla u_j \to \nabla u \quad \text{a.e.} \quad x \in D \quad \text{as} \quad j \to \infty. \tag{2.1}$$

It is obvious that $\{u_j\}_k \in \mathrm{Lip}(D)$, $\{u\}_k = k + (u-k)\chi_{u>k}$, $\{u_j\}_k = k + (u_j - k)\chi_{u_j>k}$, where χ_E denotes the characteristic function of the set E. Therefore

$$\{u_j\}_k - \{u\}_k = (u_j - u)\chi_{u_j>k} + (u-k)(\chi_{u_j>k} - \chi_{u>k}). \tag{2.2}$$

By the Minkovski inequality, from (2.2) we have

$$\|\{u_j\}_k - \{u\}_k\|_{2,D} \le \left\|(u_j - u)\chi_{u_j>k}\right\|_{2,D} + \left\|(u-k)(\chi_{u_j>k} - \chi_{u>k})\right\|_{2,D} \to 0$$

as $j \to \infty$. Indeed, the first summand does not exceed $\|u_j - u\|_{2,D}$ which tends to zero, while the second summand tends to zero by virtue of Lebesgue's majorant theorem, since $\chi_{u_j>k} \to \chi_{u>k}$ a.e. for almost all $x \in D$. Furthermore, (2.2) readily implies that

$$\nabla\{u_j\}_k - \nabla\{u\}_k = \nabla(u_j - u)\chi_{u_j>k} + \nabla u(\chi_{u_j>k} - \chi_{u>k}).$$

Then

$$\begin{aligned}\left\|\rho^{\alpha/2}\nabla(\{u_j\}_k - \{u\}_k)\right\|_{2,D} &\le \left\|\chi_{u_j>k}\cdot\rho^{\alpha/2}\nabla(u_j - u)\right\|_{2,D}\\ &\quad + \left\|(\chi_{u_j>k} - \chi_{u>k})\rho^{\alpha/2}\nabla u\right\|_{2,D} \to 0,\end{aligned}$$

also by virtue of Lebesgue's theorem and the fact that $u_j(x)$ approximates $u(x)$.

We have proved that

$$\|\{u_j\}_k - \{u\}_k\|_{2,D} \to 0 \quad \text{as} \quad j \to \infty$$

with respect to the norm $W^1_{2,\alpha}(D)$ for some subsequence u_j. If this relation is violated for some subsequence u_{j_n}, then, applying the above-given arguments to this subsequence, we come to a contradiction. □

Note that in case of Sobolev space $W^1_p(D)$ the analogous statements in [12, Ch. 2, §3] and also [3, Lemma 1.1] are proved using other approaches.

Definition 2.4. A function $u(x) \in W^1_{2,\alpha}(D)$ is called a solution of equation (1.1) in the domain D if

$$\int_D \nabla u \cdot A(x)\nabla\varphi\,dx = 0, \quad \forall\varphi \in \mathrm{Lip}_0(D). \tag{2.3}$$

It is not difficult to verify that if $u(x) \in W^1_{2,\alpha}(D)$ is a solution of equation (1.1), then identity (2.3) is fulfilled for any test function $\varphi \in \overset{\circ}{W}{}^1_{2,\alpha}(D)$.

Definition 2.5. A function $u \in W^1_{2,\alpha}(D)$ satisfying the inequality

$$\int_D \nabla u \cdot A(x)\nabla\varphi\,dx \underset{(\ge)}{\le} 0 \tag{2.4}$$

for any $\varphi \in \mathrm{Lip}_0(D)$, $\varphi \ge 0$, is called a subsolution (supersolution) of the operator L in D.

Definition 2.6. Let $H \subset E_n$ be a compact subset. We say that the capacity of the compactum H is the number

$$\operatorname{cap}_\alpha H = \inf \int\limits_{E_n} \rho^\alpha(x) \nabla u \cdot \nabla u \, dx, \tag{2.5}$$

where the lower bound is taken over all admissible functions $u \in \operatorname{Lip}_0(E_n)$, $u(x) \geq 1$ on H.

The next lemma is an analogue of the corresponding statement in [13].

Lemma 2.2 (Maximum Principle)**.** *Let $u(x) \in W^1_{2,\alpha}(D)$ be a solution of equation* (1.1) *in the domain D and $u(x) \leq M$ $(u(x) \geq M)$ on the boundary ∂D of the domain D in the sense of $W^1_{2,\alpha}(D)$. Then $u(x) \leq M$ $(u(x) \geq M)$ for almost all $x \in D$.*

Proof. Let $\varepsilon > 0$ be an arbitrary integer. By virtue of Lemma 2.1 the function $\varphi = \{u\}_{M+\varepsilon} - (M+\varepsilon)$ belongs to $\overset{\circ}{W}{}^1_{2,\alpha}(D)$. Substituting this test function into identity (2.3), we have

$$\int\limits_D \nabla\{u\}_{M+\varepsilon} \cdot A \nabla \{u\}_{M+\varepsilon} = 0,$$

from which, using the Sobolev–Gagliardo inequality and conditions (1.2), (1.3) we obtain

$$\begin{aligned} \|\{u\}_{M+\varepsilon} - (M+\varepsilon)\|_{\frac{n}{n-1},D} &\leq C \|\nabla \{u\}_{M+\varepsilon}\|_{1,D} \\ &\leq C \Big(\int\limits_D \sum_{j=1}^n \rho^{\alpha_j} |\nabla \{u\}_{M+\varepsilon}|^2 dx \Big)^{1/2} \Big(\int\limits_D \sum_{j=1}^n \rho^{-\alpha_j}(x)\, dx \Big)^{1/2} \\ &\leq C \int\limits_D \nabla\{u\}_{M+\varepsilon} \cdot A \nabla \{u\}_{M+\varepsilon} dx = 0, \quad C = C(n,\mu,\alpha,D). \end{aligned}$$

Then $\{u\}_{M+\varepsilon} \leq M + \varepsilon$ a.e. for almost all $x \in D$ and taking into account the arbitrariness of $\varepsilon > 0$, we find that $u(x) \leq M$ for almost all $x \in D$. □

3. Capacity and the Potential

Let $H \subset Q$ be a compact subset of the quasiball Q. Following [3, §4], let us consider the variational problem

$$D(v) = \int\limits_Q \nabla v \cdot A \nabla v \, dx \to \inf \tag{3.1}$$

in the class S of admissible functions, $v \in \operatorname{Lip}_0(Q)$, $v(x) \geq 1$ on H. The set $S \subset W^1_{2,\alpha}(Q)$ is convex, but not closed. Denote by $\overline{S}$ the closure of S. Since the functional $D(v)$ is continuous, the lower bound $D(v)$ on $\overline{S}$ coincides with the lower bound on S. The functional is strongly convex in $W^1_{2,\alpha}(Q)$: for any $u, v \in W^1_{2,\alpha}(Q)$ we have

$$D\left(\frac{u+v}{2}\right) - \frac{1}{2}\,D(u) - \frac{1}{2}\,D(v) = -\frac{1}{4}\,D(u-v) \leq -\frac{\mu}{4}\,\|u-v\|^2_{W^1_{2,\alpha}(Q)}.$$

Since the set $\overline{S}$ is convex and closed, there exists a function v_H such that the functional $D(v)$ attains its lower bound on $\overline{S}$. We will show that v_H is unique. Let $\inf D(f) = D(u) = D(v) = d$ simultaneously for two different functions $u, v \in W^1_{2,\alpha}(D)$. If $u = tv$, $t \in R$, then we have $d = D(v) = D(tv) = t^2 D(v) = t^2 d$, whence $t = 1$, i.e. $u = v$. If u, v are linearly independent, then the strict inequality

$$\left|\left|\int\limits_Q \nabla v \cdot A\nabla u\,dx\right| < \sqrt{D(u)\cdot D(v)}$$

is valid. Therefore for $\frac{u+v}{2} \in \overline{S}$ we have

$$D\left(\frac{u+v}{2}\right) = \frac{1}{4}\,D(u) + \frac{1}{4}\,D(v) + \frac{1}{2}\int\limits_Q \nabla u \cdot A\nabla v\,dx < d,$$

which contradicts the condition $D(u) = \inf$.

Definition 3.1. Let $H \subset Q$ be a compactum, $v_H \in \overset{\circ}{W}{}^1_{2,\alpha}(Q)$ be a unique solution of the variational problem

$$D(f) \to \inf, \quad f \in \overline{S}. \tag{3.2}$$

Then the function v_H is called the potential of the set H.

If $\{v_H\}^1$ is a section of the function v_H, then by virtue of Lemma 2.1 we have $\{v_H\}^1 \in \overline{S}$ and therefore $D(\{v_H\}^1) \leq D(v_H) = \inf$. Thus $v_H = 1$ on H in the sense of $\overset{\circ}{W}{}^1_{2,\alpha}(Q)$. For any $\varepsilon > 0$, $\psi \in \mathrm{Lip}_0(Q)$, $\psi \geq 0$ we have

$$D(v_H + \varepsilon\psi) \geq D(v_H) \quad \text{on } H, \tag{3.3}$$

whence, applying the well known variational technique, we obtain

$$\int\limits_Q \nabla v_H \cdot A\nabla\psi\,dx \geq 0, \tag{3.4}$$

i.e. v_H is a supersolution of the operator L in Q. If in (3.3) we take an arbitrary ε, for any $\psi \in \mathrm{Lip}_0(Q\backslash H)$, then for v_H we obtain the identity $\int\limits_Q \nabla v_H \cdot A\nabla\psi\,dx = 0$, i.e. v_H is a solution of equation (1.1) outside H. Finally, applying the maximum principle to the potential v_H, we obtain $0 \leq v_H \leq 1$. Thus we have proved the following statement.

Lemma 3.1. *Let $H \subset Q$ be a compact subset. Then there exists a unique solution of problem* (3.2) *with the properties: the function v_H delivers* inf *to the functional $D(f)$ and is a unique function in this sense; the function v_H is a* *supersolution of the operator L in Q; v_H is a solution of equation* (1.1) *outside H; v_H satisfies the inequality $0 \le v_H(x) \le 1$ a.e. in Q; $v_H = 1$ for $x \in \operatorname{int} H$.*

Now let us consider the operator

$$L_1 v = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\rho(x)^{\alpha_i}\,\frac{\partial v}{\partial x_i}\right). \tag{3.5}$$

All that has been said above refers to the functional

$$D_1(v) = \int_Q \rho(x)^{\alpha} \nabla v \cdot \nabla v \, dx, \tag{3.6}$$

too, i.e. the following statement is true.

Lemma 3.2. *For any compactum $H \subset Q$, the variational problem*

$$D_1(v) \to \inf, \quad v \in \overline{S},$$

has a unique solution v_H. The number

$$\overline{\operatorname{cap}}_\alpha H = D_1(v_H) = \inf \tag{3.7}$$

is called the capacity of the compactum H with respect to the quasiball Q (if Q coincides with E_n, then we have the capacity from Definition 2.6*). The function v_H is a solution of the equation $L_1 u = 0$ in $Q \backslash H$, a supersolution of the operator L_1 in Q, and also $0 \le v_H \le 1$, $v_H = 1$ on H in the sense of $W^1_{2,\alpha}(D)$.*

The next two lemmas are the obvious formulas for a classical solution of problem (1.1).

Lemma 3.3. *Let $H \subset E_n$ be a compact subset, $v \in \overset{\circ}{W}{}^1_{2,\alpha}(E_n)$ be its potential, and $v^{(k)}(x)$ approximate $v(x)$ as $k \to \infty$, $G = E_n \setminus H$. Then for almost all $t \in (0,1)$ we have the identity*

$$t \int_{\partial \Gamma_t} \nabla v^{(k)} \cdot AN \, d\sigma = \int_{E_n \backslash \Gamma_t} \nabla v^{(k)} \cdot A \nabla v^{(k)} \, dx + \delta_k, \tag{3.8}$$

where $\delta_k \to 0$ as $k \to \infty$, $N = (N_1, N_2, \dots, N_m)$ is the inward normal vector to the domain boundary

$$\Gamma_t = \left\{x \in E_n : v^{(k)}(x) > t\right\}$$

(*depending also on k*).

Proof. Let $\varphi = g_h(x) v^{(k)}(x)$, where $g_h(x) = \frac{t - v^{(k)}(x)}{h}$ for $x \in \Gamma_{t-h} \setminus \Gamma_t$, $g_h(x) = 1$ for $x \in E_n \setminus \Gamma_{t-h}$, $g_h(x) = 0$ for $x \in \Gamma_t$, $t \in (0,1]$, h is sufficiently small. It is obvious that the function $\varphi \in \overset{\circ}{W}{}^1_{2,\alpha}(G)$, and since from the identity

$$\int_G \nabla v \cdot A \nabla \varphi \, dx = 0$$

it follows that

$$\int_G \nabla v^{(k)} \cdot A\nabla\varphi\, dx = \delta_k \to 0 \quad \text{as} \quad k \to \infty, \tag{3.9}$$

after substituting the expression for φ into this identity we obtain

$$-\frac{1}{h}\int_{\Gamma_{t-h}\setminus\Gamma_t} v^{(k)}\nabla v^{(k)} \cdot A\nabla v^{(k)}\, dx + \int_{E_n\setminus\Gamma_t} g_h \nabla v^{(k)} \cdot A\nabla v^{(k)}\, dx = \delta_k. \tag{3.10}$$

Using Federer's formula [14, p. 40] and Lebesgue's theorem [15, §1, Ch. 1] for almost all $t \in (0,1)$, we obtain for the first summand in (3.10)

$$\frac{1}{h}\int_{\Gamma_{t-h}\setminus\Gamma_t} v^{(k)}\nabla v^{(k)} \cdot A\nabla v^{(k)}\, dx = \frac{1}{h}\int_{t-h}^{t} s\, ds \left(\int_{\partial\Gamma_s} N \cdot A\nabla v^{(k)}\, d\sigma_s\right)$$
$$\to t \cdot \int_{\partial\Gamma_t} \nabla v^{(k)} \cdot AN\, d\sigma_t \tag{3.11}$$

as $h \to 0$, since $N = \nabla v^{(k)}/|\nabla v^{(k)}|$ on the surface $\partial\Gamma_s$. By virtue of Lebesgue's majorant theorem we have

$$\int_{E_n\setminus\Gamma_t} g_h\nabla v^{(k)} \cdot A\nabla v^{(k)}\, dx \to \int_{E_n\setminus\Gamma_t} \nabla v^{(k)} \cdot A\nabla v^{(k)}\, dx \quad \text{as} \quad h \to 0. \tag{3.12}$$

Passing to the limit as $h \to 0$ and taking into account (3.11), (3.12) we obtain (3.8). □

Lemma 3.4. *Let $H \subset E_n$ be a compactum, $v(x)$, $v^{(k)}(x)$, Γ_t be the same as in Lemma* 3.3. *Then for a.e. for almost all $0 < s < t < 1$ the identity*

$$\int_{\partial\Gamma_s} \nabla v^{(k)} \cdot AN\, d\sigma_s = t \cdot \int_{\partial\Gamma_t} \nabla v^{(k)} \cdot AN\, d\sigma_t + \delta_k \tag{3.13}$$

is valid, where $\delta_k \to 0$ as $k \to \infty$, $d\sigma_s$ is an element of the $(n-1)$-dimensional Lebesgue measure on the surface $\partial\Gamma_s$ (depends on k).

The proof of this lemma is analogous to that of Lemma 3.3. In identity (3.9) we set the test function $\varphi = g_h(x)f_h(x)$, where $g_h(x)$ is the same as in Lemma 3.3, $f_h(x) = \frac{v^{(k)}(x)-s}{h}$ for $x \in \Gamma_s \setminus \Gamma_{s+h}$, $f_h(x) = 1$ for $x \in \Gamma_{s+h}$, $f_h(x) = 0$ for $x \in E_n \setminus \Gamma_s$. Then

$$-\frac{1}{h}\int_{\Gamma_s\setminus\Gamma_{s+h}} v^{(k)}\nabla v^{(k)} \cdot A\nabla v^{(k)}\, dx + \frac{1}{h}\int_{\Gamma_{t-h}\setminus\Gamma_t} v^{(k)}\nabla v^{(k)} \cdot A\nabla v^{(k)}\, dx = \delta_k \to 0$$

as $k \to \infty$. If $h \to 0$, then analogously to (3.11) we obtain (3.13).

4. The Harnack Inequality

Theorem 4.1. *Let $G_R = Q^0_R \setminus Q^0_{R/4}$, $u(x) \in W^1_{2,\alpha}(G_R)$ be a positive solution of equation (1.1) in G_R. Then there exists a constant $C = C(n, \mu, \alpha, \beta)$ such that*

$$\operatorname*{ess\,sup}_{\partial Q^0_{r/2}} u(x) \le C \operatorname*{ess\,inf}_{\partial Q^0_{r/2}} u(x). \tag{4.1}$$

Proof. By definition, for any $\varphi \in W^1_{2,\alpha}(G_R)$

$$\int\limits_{G_R} \nabla v \cdot A\nabla\varphi \, dx = 0. \tag{4.2}$$

Performing the change $x = R^\sigma\xi = (R^{\sigma_1}\xi_1, R^{\sigma_2}\xi_2, \dots, R^{\sigma_n}\xi_n)$, we obtain

$$\int\limits_{T} \nabla_\xi v \cdot B\nabla_\xi\psi \, d\xi = 0, \tag{4.3}$$

where $T = Q^0_1 \setminus Q^0_{1/4}$ is the image of the set G_R; $B = \|b_{ij}(\xi)\|$, $b_{ij} = R^{\beta-\sigma_i-\sigma_j} \times a_{ij}(x(\xi))$, $i, j = 1, 2, \dots, n$. $x(\xi) = R^\sigma\xi$, $\nabla_\xi v$ denotes the vector $(v_{\xi_1}, v_{\xi_2}, \dots, v_{\xi_n})$, and $\psi(\xi) = \varphi(x(\xi))$ is an arbitrary function from the space $\overset{\circ}{W}{}^1_2(T)$, $v(\xi) = u(x(\xi))$.

The matrix B is uniformly elliptic on T. Indeed,

$$\mu\sum_{j=1}^n \rho(x(\xi))^{\alpha_j} R^{\beta-2\sigma_j}\eta_j^2 \le \eta\cdot B\eta \le \mu^{-1}\sum_{j=1}^n \rho(x(\xi))^{\alpha_j} R^{\beta-2\sigma_j}\eta_j^2.$$

Since $x \in G_R$ for $\xi \in T$, we have

$$\frac{R}{4} < \rho(x) < R, \tag{4.4}$$

and therefore

$$|\eta|^2\,\frac{\mu}{4^{\alpha_0}} \le \eta\cdot B\eta \le \mu^{-1}4^{\alpha_0}|\eta|^2 \quad \forall \eta \in E_n, \tag{4.5}$$

where $\alpha_0 = \max\limits_{1\le j\le n}|\alpha_j|$. The image of the points $x \in \partial Q^0_{r/2}$ belongs to $T' = \partial Q^0_{1/2}$.

It is obvious that $T' \subset T$ and a usual distance $\operatorname{dist}(T', \partial T) > 0$ does not depend on R. The function $v(\xi)$ belongs to $W^1_2(T)$, since $u(x) \in W^1_{2,\alpha}(G_R)$. Indeed, from (4.4) we have

$$\int\limits_{G_R} \rho^\alpha \nabla u\cdot\nabla u\,dx = \sum_{j=1}^n R^{\sum\limits_{i=1}^n \sigma_i} \int\limits_T \rho(x(\xi))^{\alpha_j} R^{-2\sigma_j} u^2_{\xi_j}\,d\xi \sim R^{\sum\limits_{i=1}^n \sigma_i - \beta} \int\limits_T |\nabla_\xi u|^2\,d\xi.$$

Identity (4.3) implies that $v(\xi)$ is a positive solution of the equation

$$\sum_{i,j=1}^n \frac{\partial}{\partial\xi_i}\left(b_{ij}(\xi)\,\frac{\partial v}{\partial\xi_j}\right) = 0 \tag{4.6}$$

from $W_2^1(T)$ on T. In view of (4.5), equation (4.6) is elliptic on T and therefore the Harnack inequality

$$\operatorname*{ess\,sup}_{\xi \in T'} v(\xi) \leq C \operatorname*{ess\,inf}_{\xi \in T'} v(\xi) \tag{4.7}$$

is fulfilled for it (see [16]). Here the constant $C = C(n, \mu, \alpha, \beta)$. By the inverse transformation $\xi \to x$, from estimate (4.7) we derive (4.1). □

5. Lemma on the Increase of Positive Solutions

Lemma 5.1. *Let $D \subset Q_R^0$ be the domain having limiting points on the surface ∂Q_R^0 and a nonempty intersection with $Q_{R/4}^0$. Let $u \in W_{2,\alpha}^1(D)$ be a solution of equation* (1.1) *in D, $u(x) \leq 0$ on a part of the boundary Γ lying strictly inside Q_R^0 in the sense of $W_{2,\alpha}^1(D)$. Then there exists a constant $\eta_0 = \eta_0(n, \mu, \alpha, \beta)$ such that for any such function $u(x)$ there holds*

$$\operatorname*{ess\,sup}_{x \in D} u(x) \geq \Big(1 + \eta_0 R^{\beta - \sum_{j=1}^{n} \sigma_j} \operatorname{cap}_\alpha H_R\Big) \operatorname*{ess\,sup}_{x \in D \cap Q_{R/4}^0} u(x), \tag{5.1}$$

where the set $H_R = Q_{R/4}^0 \setminus D$ and $\operatorname{cap}_\alpha H_R$ is its capacity.

Proof. Denote $M = \sup_{x \in D} u(x)$, $G = Q_R^0 \setminus H_R$ and let $U_H \in \overset{\circ}{W}{}_{2,\alpha}^1(Q_R^0)$ be the potential of the set H_R. Let $\phi \in \operatorname{Lip}_0(Q_R^0)$ be the function equal to 1 on H_R. Assume that $z = M(1 - U_H)$ in Q_R^0. Then z is a solution of equation (1.1) in G, $z(x) \geq 0$, on H_R. Applying the maximum principle to the functions $z(x)$, $u(z)$ in the domain D, we have $z(x) \geq u(x)$ on $\Gamma \cup \partial Q_R^0$, since $z|_{\partial Q_R^0} = M \geq u(x)$ and, for $x \in \Gamma$ $u(x) = 0 \leq z(x)$. Then by virtue of Lemma 2.2 for almost all $x \in D$ we obtain $z(x) \geq u(x)$. Since $\sup_{x \in D \cap Q_{R/2}^0} u(x) \leq \sup_{x \in D \cap Q_{R/2}^0} z(x)$, we have

$$M \geq M \inf_{x \in D \cap Q_{R/2}^0} U_H + \sup_{x \in D \cap Q_{R/2}^0} u(x). \tag{5.2}$$

Let us estimate the expression $\inf_{x \in D \cap Q_{R/2}^0} U_H$ from below. To this end, we obtain the uniform estimate (from below) of the numbers $a^{(k)} = \sup_{x \in Q_{R/2}^0} U_H^{(k)}$ with sufficiently large k for the corresponding approximating sequence $U_H^{(k)}$. Denote the k-dependent set $\Gamma_t = \{x \in Q_R^0 : U_H^{(k)}(x) > t\}$. Then, by virtue of Lemma 3.4, for an arbitrarily small $\varepsilon > 0$ and $a_1 \in (a^{(k)}, a^{(k)} + \varepsilon)$ we have

$$\int_{\partial \Gamma_{1-\varepsilon}} \nabla U_H^{(k)} \cdot AN \, d\sigma = \int_{\partial \Gamma_{a_1}} \nabla U_H^{(k)} \cdot AN \, d\sigma + \delta_k, \tag{5.3}$$

where $\delta_k \to 0$ as $k \to \infty$. Note that $\Gamma_{a_1} \subset Q_{R/2}^0$, $N = \nabla U_H^{(k)} / |\nabla U_H^{(k)}|$ on $\partial \Gamma_{1-\varepsilon}$. By virtue of Lemma 3.3, the left-hand side of (5.3) is estimated by the expression

$$\int\limits_{\partial\Gamma_{1-\varepsilon}} \nabla U_H^{(k)} \cdot AN\, d\sigma \geq \frac{1}{1-\varepsilon} \int\limits_{Q_R^0\setminus\Gamma_{1-\varepsilon}} \nabla U_H^{(k)} \cdot A\nabla U_H^{(k)}\, dx + \delta_k, \tag{5.4}$$

which in its turn exceeds

$$\frac{\mu}{1-\varepsilon} \int\limits_{Q_R^0\setminus\Gamma_{1-\varepsilon}} \rho(x)^\alpha \nabla U_H^{(k)} \cdot \nabla U_H^{(k)}\, dx + \delta_k \geq \mu(1-\varepsilon)\overline{\mathrm{cap}}_\alpha \Gamma_{1-\varepsilon} + \delta_k.$$

The set $H_R \subset \Gamma_{1-\varepsilon}$ and therefore the right-hand side of inequality (5.5) is not smaller than $\mu(1-\varepsilon)^2 \operatorname{cap}_\alpha H_T$. Then (5.4) implies

$$\int\limits_{\partial\Gamma_{1-\varepsilon}} \nabla U_H^{(k)} \cdot AN\, d\sigma \geq \mu(1-\varepsilon)\overline{\mathrm{cap}}_\alpha H_R + \delta_k, \tag{5.5}$$

where $\overline{\mathrm{cap}}_\alpha H_R$ is a relative capacity H_R with respect to Q_R^0.

By virtue of Lemma 3.3, for the right-hand part of (5.3) we obtain

$$\int\limits_{\partial\Gamma_{a_1}} \nabla U_H^{(k)} \cdot AN\, d\sigma \leq \frac{1}{a_1} \int\limits_{Q_R^0\setminus\Gamma_{a_1}} \nabla U_H^{(k)} \cdot A\nabla U_H^{(k)}\, dx + \delta_k. \tag{5.6}$$

Let $\varphi_{a_1} \in \overset{\circ}{W}{}^1_{2,\alpha}(Q_R^0)$ be the potential of the set $\overline{\Gamma}_{a_1}$ generated by functional (3.6). Then for this potential we have

$$\int\limits_{Q_R^0} \rho^\alpha \nabla\varphi_{a_1} \cdot \nabla\varphi_{a_1} dx = \overline{\mathrm{cap}}_\alpha \overline{\Gamma}_{a_1} \tag{5.7}$$

(see (3.7)). Choosing the test function $\varphi = U_H^{(k)} - a_1\varphi_{a_1}$ from equation (1.1), we obtain (since $\varphi \in \overset{\circ}{W}{}^1_{2,\alpha}(Q_R^0 \setminus \overline{\Gamma}_{a_1})$)

$$\begin{aligned} \int\limits_{Q_R^0\setminus\Gamma_{a_1}} \nabla U_H^{(k)} \cdot A\nabla U_H^{(k)}\, dx &= a_1 \int\limits_{Q_R^0\setminus\Gamma_{a_1}} \nabla U_H^{(k)} \cdot A\nabla\varphi_{a_1}\, dx + \delta_k \\ &\leq a_1 \Bigg(\int\limits_{Q_R^0} \nabla U_H^{(k)} \cdot A\nabla U_H^{(k)}\, dx \Bigg)^{1/2} \Bigg(\int\limits_{Q_R^0} \nabla\varphi_{a_1} \cdot A\nabla\varphi_{a_1}\, dx \Bigg)^{1/2} + \delta_k. \end{aligned}$$

Therefore

$$\begin{aligned} \int\limits_{Q_R^0\setminus\Gamma_{a_1}} \nabla U_H^{(k)} \cdot A\nabla U_H^{(k)}\, dx &\leq \mu^{-1} a_1^2 \int\limits_{Q_R^0\setminus\Gamma_{a_1}} \nabla\varphi_{a_1} \cdot \nabla\varphi_{a_1}\, dx + \delta_k \\ &= \mu^{-1} a_1^2 \operatorname{cap}_\alpha \overline{\Gamma}_{a_1} + \delta_k \leq \mu^{-1} a_1^2 \operatorname{cap}_\alpha Q_{R/2}^0 + \delta_k, \end{aligned} \tag{5.8}$$

since $\Gamma_{a_1} \subset \Gamma_{a^{(k)}} \subset Q_{R/2}^0$ and (5.7) is fulfilled.

From (5.3), (5.5) , (5.6), (5.8) it follows that

$$(1-\varepsilon)\mu^2\,\overline{\mathrm{cap}}_\alpha H_R \le a_1 \overline{\mathrm{cap}}_\alpha Q^0_{R/2} + \frac{\delta_k}{a_1} + \delta_k\,. \tag{5.9}$$

We replace here $\frac{\delta_k}{a_1}$ by δ_k, since $a_1 \in (a^{(k)}, a^{(k)}+\varepsilon)$ and, for sufficiently large k's, $\alpha^{(k)}$ is close to $\inf\limits_{x\in Q^0_{R/2}} U_H$. Hence, taking into account the arbitrariness of ε, we have

$$a^{(k)} \ge \mu^2 \left(\frac{\overline{\mathrm{cap}}_\alpha H_R}{\overline{\mathrm{cap}}_\alpha Q^0_{R/2}} \right) + \delta_k\,. \tag{5.10}$$

It is obvious that

$$\overline{\mathrm{cap}}_\alpha H_R \ge \mathrm{cap}_\alpha\, H_R. \tag{5.11}$$

Let $\eta(x) = \prod\limits_{j=1}^{n} \eta_0\big(\frac{x_j}{R^{\sigma_j}}\big)$, $\eta_0(t)$ be an infinite times differentiable function on $(-\infty,\infty)$, with a compact support, $0 \le \eta(t) \le 1$, $|\eta'(t)| \le C_0$; $C_0 = C_0(n)$, $\eta(t) = 1$ for $t \in \big[0, \frac{1}{2}\big]$, $\eta(t) = 0$ for $|t| \ge 1$. Then

$$\overline{\mathrm{cap}}_\alpha Q^0_{R/2} \le \int\limits_{Q^0_R} \rho^\alpha \nabla\eta \cdot \nabla\eta\, dx. \tag{5.12}$$

Therefore

$$\mathrm{cap}_\alpha\, Q^0_{R/2} \le C_0 \sum_{j=1}^{n} R^{-2\sigma_j} \int\limits_{Q^0_R} \rho(x)^{\alpha_j}\, dx \le C_0 C_1 R^{\sum\limits_{i=1}^{n}\sigma_i - \beta}, \tag{5.13}$$

$C_1 = C_1(n,\alpha)$, since for condition (1.3) we have

$$\int\limits_{Q^0_R} \rho(x)^{\alpha_j}\, dx \le C_1 R^{\sum\limits_{i=1}^{n}\sigma_i + \alpha_j}. \tag{5.14}$$

By (5.11), (5.13) and (5.10) we obtain

$$a^{(k)} \ge C R^{\beta - \sum\limits_{i=1}^{n}\sigma_i}\, \mathrm{cap}_\alpha\, H_R + \delta_k, \tag{5.15}$$

where $C = C(n,\mu,\alpha,\beta)$, $\delta_k \to 0$ as $k \to \infty$. If in estimate (5.15) we assume that $k \to \infty$ and apply Theorem 4.1, then we have

$$\inf_{x\in Q^0_{R/1}} U_H \ge \eta_0 R^{\beta - \sum\limits_{i=1}^{n}\sigma_i}\, \mathrm{cap}_\alpha\, H_R, \tag{5.16}$$

where $\eta_0 = \eta_0(n,\mu,\alpha,\beta)$. Due to (5.16), from (5.2) we obtain (5.1) (keeping in mind that $\sup\limits_{Q^0_{R/2}} u \ge \sup\limits_{Q^0_{R/4}} u$ and using the maximum principle). □

6. Regularity of a Boundary Point

Let D be a bounded domain and f a continuous function on ∂D. We will consider the Dirichlet problem

$$Lu = 0 \quad \text{in} \quad D, \quad u|_{\partial D} = f. \tag{6.1}$$

Assume that $\phi \in W^1_{2,\alpha}(D)$. Using the classical theory of a Hilbert space (or the variational method of §3), we easily prove the existence and uniqueness of a solution of the problem

$$Lu = 0 \quad \text{in} \quad D, \quad u - \phi \in \overset{\circ}{W}{}^1_{2,\alpha}(D). \tag{6.2}$$

Let us now proceed to constructing a solution of problem (6.1). There exists a sequence of smooth, for example, Lipschitz functions f_k approximating f uniformly on ∂D. There also exists an extension ϕ_k of functions f_k from ∂D to all $\overline{D}$ with the preservation of its Lipschitz property (see [15, p. 206]). Let u_k be a solution of problem (6.2) for $\phi = \phi_k$. Then u_k is a fundamental sequence in the uniform metric. Indeed, according to the maximum principle, we have

$$\operatorname*{ess\,sup}_{\overline{D}} |u_k - u_m| \le \sup_{\partial D} |u_k - u_m| = \sup_{\partial D} |f_k - f_m| \to 0 \quad \text{as} \quad k, m \to \infty.$$

Therefore the sequence $u_k \to u_f$ uniformly in $\overline{D}$ to some function $u_f(x)$. The function u_f is a solution of equation (1.1) in any strictly internal domain $D' \subset D$. Indeed, let $\xi \in \mathrm{Lip}_0(D)$ be equal to 1 in D' and to 0 outside D. Then we have $\int\limits_D \nabla u_k \cdot A\nabla(u_k\xi^2)\,dx = 0$, whence it follows that $\int\limits_D \nabla u_k \cdot A\nabla u_k \xi^2\,dx \le 4\int\limits_D \nabla\xi \cdot A\nabla\xi\,dx$. Therefore

$$\int\limits_{D'} \rho^\alpha \nabla u_k \cdot \nabla u_k\,dx \le \mu^{-1} \int\limits_{D'} \nabla u_k \cdot A\nabla u_k\,dx \le C(D'),$$

where $C(D')$ does not depend on k. Then $u_{k_n} \to v$ weakly in $W^1_{2,\alpha}(D')$ for some subsequence u_{k_n}. Since, in addition, $u_k \to u_f$ uniformly, we have $v = u_f$. Thus the function u_f, which does not depend on the approximation technique, is a solution of equation (1.1) in D'. The function $u_f(x)$ is called a generalized solution of problem (6.1).

Definition 6.1. A point $x_0 \in \partial D$ is called regular if a generalized solution $u_f(x)$ of problem (6.1) is continuous at x_0 for any continuous function f on ∂D.

The behavior of the solution of the Dirichlet problem for equation (1.1) near a nonregular boundary point depends on the domain geometry in an immediate proximity of this point. Since for the boundary points $x_0 \ne 0$ equation (1.1) has no degeneration, the regularity criterion for these points coincides with Wiener's criterion for the Laplace equation (see [3]). Hence we show interest in boundary point 0. The main result of this work is

Theorem 6.1. *Let $D \subset E_n$ be a bounded domain, $0 \in \partial D$. Then for boundary point 0 to be regular for equation (1.1) it is necessary and sufficient that*

$$\int_0 R^{\beta - \sum_{i=1}^{n} \sigma_i} \operatorname{cap}_\alpha H_R \frac{dR}{R} = \infty, \tag{6.3}$$

or, which is the same,

$$\sum_{m=1}^{n} \gamma(4^{-m}) = \infty, \tag{6.4}$$

where the set $H_R = Q_R^0 \setminus D$ and $\operatorname{cap}_\alpha H_R$ is its capacity, $\gamma(t) = t^{\beta - \sum_{i=1}^{n} \sigma_i} \operatorname{cap}_\alpha H_t$.

Proof. Sufficiency. Because of the continuity of the function f, for any $\varepsilon > 0$, there exists $0 < \delta_1 < 1$ such that $|f(x) - f(0)| < \frac{\varepsilon}{2}$ for $|x| < \delta_1$, $x \in D$. We have $f_k \in \operatorname{Lip}(\partial D)$ for which $|f_k - f| < \frac{\varepsilon}{2}$ when $x \in \partial D$. Let u_k be a solution of the problem

$$Lu_k = 0 \quad \text{in} \quad D, \quad u_k - \phi_k \in \overset{\circ}{W}{}^1_{2,\alpha}(D), \tag{6.5}$$

where $\phi_k \in \operatorname{Lip}(\overline{D})$ is an extension of f_k with preservation of its Lipshitz property to all $\overline{D}$ (for the existence of such an extension see [15, Ch. VI, p. 206]). The function $z = u_k - f(x_0) - \varepsilon$ is also a solution of the equation $Lz = 0$ in D for any $x \in \{|x| < \delta_1\} \cap D$. The function $z(x)$ is continuous in D. Denote $D' = \{x \in D : z(x) > 0\}$, $\sigma_0 = \min_{1 \le k \le n} \{\sigma_k\}$. Let $m_0 \in N$, $4^{-m_0 \sigma_0} < \delta_1$. Then $Q^0_{4^{-m_0}} \subset \{|x| < \delta_1\}$ and applying Lemma 5.1 to $Q^0_{4^{-m+1}}$ and $Q^0_{4^{-m}}$ and assuming that $m = m_0 + 1, m_0 + 2, \dots, l$, we have

$$M_{m-1} \ge (1 + \eta_0 \gamma_m) M_m, \tag{6.6}$$

where $\gamma_m = \gamma(4^{-m})$, $M_m = \sup_{Q^0_{4^{-m}}} z$.

Applying repeatedly estimate (6.6) we obtain

$$M_{m_0} \ge \prod_{j=m_0}^{l} (1 + \eta_0 \gamma_j) M_l \ge M_l \exp\left(\sum_{j=m_0}^{l} \ln(1 + \eta_0 \gamma_j) \right)$$
$$\ge M_l \exp\left(\eta_0 C_0 \sum_{j=m_0}^{l} \gamma_j \right),$$

where $C_0 = C_0(n)$. Hence

$$M_l \le M_{m_0} \exp\left(- \eta_0 C_0 \sum_{j=m_0}^{l} \gamma_j \right). \tag{6.7}$$

An analogous estimate holds for the function $z_1 = f(x_0) + \varepsilon - u_k$, too:

$$\mu_l \le \mu_{m_0} \exp\left(- \eta_0 C_0 \sum_{j=m_0}^{l} \gamma_j \right), \tag{6.8}$$

where $\mu_m = \sup\limits_{x\in D\cap Q^0_{4-m}} z_1(x)$. From (6.7) and (6.8) we obtain the estimate for $|u_k(x) - f(0)|$. Since $u_k \to u_f$ uniformly in $\overline{D}$, we have

$$\sup_{D\cap Q^0_{4-l}} |u_f(x) - f(0)| \le \varepsilon + 2 \max_{\partial D\cap Q^0_{4-m}} |f(x)| \exp\Big(- \eta_0 C_0 \sum_{j=m_0}^{l} \gamma_j \Big). \tag{6.9}$$

Choosing $l = l(\varepsilon) \in N$ so that the second summand is smaller than ε for $l > l(\varepsilon)$, we obtain $|u_k(x)-f(0)| < 2\varepsilon$ for $x \in Q^0_\delta \cap D$ ($\delta = \min(4^{-l(\varepsilon)}, \delta_1^{1/\sigma_0})$) and therefore for $|x| < \delta^{\sigma^0}$ ($\sigma^0 = \max\limits_{1\le i\le n} \sigma_i$), too. Applying an analogous reasoning to regular boundary point 0, the continuity modulus of the solution $u_f(x)$ is estimated through the continuity modulus of the function f and the sum of series (6.4).

Let $|f(x) - f(0)| < \theta(|x|)$, where $\theta(x) \downarrow 0$ is the continuity modulus. Then

$$|u_f(x) - f(0)| \le 2 \max_{\partial D} |f| \exp\Big(- C_0 \int_{\rho(x)}^{\delta_1^{1/\sigma^0}} \tau^{\beta - \sum_{i=1}^{n} \sigma_i} \operatorname{cap}_\alpha H_\tau \frac{d\tau}{\tau} \Big) + \theta(\delta)$$
$$\text{for } \rho(x) \le \delta_1^{1/\sigma^0}. \tag{6.10}$$

Such estimates for equation (1.1) were obtained for the first time in [17].

To prove the necessity we need the following

Proposition 6.1. *Let $H \subset Q^0_R \setminus Q^0_{R/4}$ be the closed set, and $v_H \in \overset{\circ}{W}{}^1_{2,\alpha}(E_n)$ be the potential of the set H. Then*

$$\sup_{x\in Q^0_{R/8}} v_H \le C R^{\beta - \sum_{j=1}^{n} \sigma_j} \operatorname{cap}_\alpha H, \quad C = C(n, \mu, \beta, \alpha). \tag{6.11}$$

To show estimate (6.11) for sufficiently large $k \in N$, we obtain the uniform estimate for the approximating sequence $v_H^{(k)}(x)$.

Let $t \in (0,1)$, $\Gamma_t = \{x \in E_n : v_H^{(k)}(x) > t\}$, $b^{(k)} = \inf\limits_{x\in Q^0_{R/8}} v_H^{(k)}(x)$. Then $Q^0_{R/8} \subset \Gamma_{b^{(k)}}$. Using Lemmas 3.3 and 3.4, for an arbitrary small $\varepsilon > 0$ and $b_1 \in (b^{(k)} - \varepsilon, b^{(k)})$ we obtain

$$\int_{E_n\setminus\overline{\Gamma}_{b_1}} \nabla v_H^{(k)} \cdot A \nabla v_H^{(k)}\, dx \le b_1 \int_{\partial\Gamma_{b_1}} \nabla v_H \cdot AN\, d\sigma + \delta_k$$
$$\le \delta_k + b_1 \int_{\partial\Gamma_{1-\varepsilon}} \nabla v_H^{(k)} \cdot AN\, dx \le \frac{b_1}{1-\varepsilon} \int_{E_n\setminus\Gamma_{1-\varepsilon}} \nabla v_H^{(k)} \cdot A\nabla v_H^{(k)}\, dx + \delta_k, \tag{6.12}$$

where $\delta_k \to 0$ as $k \to \infty$. If $Q^0_{R/8} \subset \Gamma_{b_1}$, then we have

$$\int\limits_{E_n\setminus\overline{\Gamma}_{b_1}} \nabla v_H^{(k)} \cdot A\nabla v_H^{(k)}\,dx \geq \mu \int\limits_{E_n\setminus\Gamma_{b_1}} \rho^\alpha \nabla v_H^{(k)} \cdot \nabla v_H^{(k)}\,dx$$
$$\geq \mu b_1^2 \operatorname{cap}_\alpha \overline{\Gamma}_{b_1} \geq \mu b_1^2 \operatorname{cap}_\alpha Q^0_{R/8}. \tag{6.13}$$

Let us estimate $\operatorname{cap}_\alpha Q^0_{R/8}$ from below. Clearly, $\operatorname{cap}_\alpha Q^0_{R/8} \geq \operatorname{cap}_\alpha(Q^0_{R/8} \setminus Q^0_{R/16})$. Denote $K = Q^0_{R/8} \setminus Q^0_{R/16}$. Then $\operatorname{cap}_\alpha K = \inf \int\limits_{E_n} \rho^\alpha(x)\nabla\varphi\cdot\nabla\varphi\,dx$, where the lower bound is taken over all $\varphi \in \operatorname{Lip}_0(E_n)$ equal to 1 on K. It is obvious that

$$\operatorname{cap}_\alpha K \sim \inf_{\{\varphi\}} \sum_{j=1}^n R^{\alpha_j} \int\limits_{E_n} \varphi^2_{x_j}\,dx. \tag{6.14}$$

If we make the change $x \to \xi$ by the formula $x = R^\sigma \xi = (R^{\sigma_1}\xi_1, R^{\sigma_2}\xi_2, \dots, R^{\sigma_n}\xi_n)$ in integral (6.14), then K transforms to $P = Q^0_{1/8} \setminus Q^0_{1/16}$ and we have

$$\sum_{j=1}^n R^{\alpha_j} \int\limits_{E_n} \varphi^2_{x_j}\,dx = R^{\sum_{i=1}^n \sigma_i - \beta} \int\limits_{E_n} |\nabla\psi|^2\,d\xi, \tag{6.15}$$

where $\psi(\xi) = \varphi(x(\xi))$. Minimizing the right-hand side of (6.15) over all admissible functions ψ, we obtain

$$\sum_{j=1}^n R^{\alpha_j} \int\limits_{E_n} \varphi^2_{x_j}\,dx \geq C_1 R^{\sum_{i=1}^n \sigma_i - \beta}. \tag{6.16}$$

Here the value $C_1 = C_1(n, \mu, \alpha, \beta)$ is equal to the Wiener capacity of the compactum P which exceeds some constant $C(n, \mu, \alpha, \beta)$. From (6.14), (6.15) and (6.16) we find

$$\operatorname{cap}_\alpha Q^0_{R/8} \geq C R^{\sum_{j=1}^n \sigma_j - \beta}, \quad C = C(n, \mu, \alpha, \beta). \tag{6.17}$$

Taking into account (6.13) and (6.17), we obtain

$$\int\limits_{E_n\setminus\Gamma_{b_1}} \nabla v_H^{(k)} \cdot A\nabla v_H^{(k)}\,dx \geq C b_1^2 R^{\sum_{j=1}^n \sigma_j - \beta}. \tag{6.18}$$

Since $H \subset \Gamma_{1-\varepsilon}$, we have

$$\int\limits_{E_n\setminus\Gamma_{1-\varepsilon}} \nabla v_H^{(k)} \cdot A\nabla v_H^{(k)}\,dx \leq \int\limits_{E_n\setminus H} \nabla v_H^{(k)} \cdot A\nabla v_H^{(k)}\,dx \leq \mu^{-1} \operatorname{cap}_\alpha H. \tag{6.19}$$

From estimates (6.18), (6.19) and (6.12) we obtain the estimate for b_1, whence by virtue of the arbitrariness of ε we have

$$b^{(k)} \leq C R^{\beta - \sum_{j=1}^n \sigma_j} \operatorname{cap}_\alpha H. \tag{6.20}$$

Passing to the limit as $k \to \infty$, from (6.20) we obtain the estimate for $\inf\limits_{x \in Q^0_{R/8}} v_H(x)$ and thus, applying Theorem 4.1, we have

$$\sup_{x \in Q^0_{R/8}} v_H(x) \le C R^{\beta - \sum\limits_{j=1}^{n} \sigma_j} \operatorname{cap}_\alpha H, \tag{6.21}$$

$C = C(n, \mu, \alpha, \beta)$. Estimate (6.11) is proved.

Necessity. Assume that condition (6.4) is not fulfilled. Let $U(x)$ be the potential of the set $Q^0_{4^{-m_0}} \setminus D$, the integer $m_0 \in N$ will be chosen later. We will show that for any $\delta > 0$ there exists a point $x' \in Q^0_\delta$, $x' \ne 0$, $x' \in D$, for which $U(x') < \frac{1}{2}$. Since series (6.4) is convergent, there is an integer m_1 for which $4^{-m_1} < \delta$,

$$\gamma_{m_1 - 1} \le \frac{\mu^2}{8}\,. \tag{6.22}$$

Let us denote by $U_0(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(E_n)$ the potential of the set $G_{m_1-1} = Q^0_{4^{-m_1+1}} \setminus D$ and show that

$$\inf_{x \in Q^0_{4^{-m_1-1}}} U_0 < \frac{1}{4}\,.$$

We again pass over to the approximating sequence $U_0^{(k)}(x)$. For this,we denote $a^{(k)} = \inf\limits_{x \in Q^0_{4^{-m_1}}} U_0^{(k)}(x)$, $\Gamma_t = \{x \in E_n : U_0^{(k)}(x) > t\}$, $t \in (0,1)$. Then, taking into account that $Q^0_{4^{-m}} \subset \Gamma_{a^{(k)}}$ and using Lemmas 3.3, 3.4, for arbitrarily small $\varepsilon > 0$ and $a_1 \in (a^{(k)} - \varepsilon, a^{(k)})$ we have

$$\begin{aligned}
\mu C_1 4^{-mq} a_1^2 &\le \mu a_1^2 \operatorname{cap}_\alpha \overline{\Gamma}_{a_1} \le \mu \int\limits_{E_n \setminus \Gamma_{a_1}} \rho^\alpha \nabla U_0^{(k)} \cdot \nabla U_0^{(k)}\, dx \\
&\le \int\limits_{E_n \setminus \Gamma_{a_1}} \nabla U_0^{(k)} \cdot A \nabla U_0^{(k)}\, dx = a_1 \int\limits_{\partial \Gamma_{a_1}} \nabla U_0^{(k)} \cdot AN\, d\sigma + \delta_k \\
&= a_1 \int\limits_{\partial \Gamma_{1-\varepsilon}} \nabla U_0^{(k)} \cdot AN\, d\sigma = \frac{a_1}{1-\varepsilon} \int\limits_{E_n \setminus \Gamma_{a_1}} \nabla U_0^{(k)} \cdot A \nabla U_0^{(k)}\, dx + \delta_k. \qquad (6.23)
\end{aligned}$$

Let $\psi \in \overset{\circ}{W}{}^1_{2,\alpha}(E_n)$ be the potential of the set $\Gamma_{1-\varepsilon}$ generated by functional (3.6). Then $\varphi = u_0^{(k)} - (1-\varepsilon)\psi \in \overset{\circ}{W}{}^1_{2,\alpha}(E_n \setminus \Gamma_{1-\varepsilon})$ is the test function for (1.1) in $E_n \setminus \Gamma_{1-\varepsilon}$ and therefore

$$\int\limits_{E_n \setminus \Gamma_{1-\varepsilon}} \nabla U_0^{(k)} \cdot A \nabla \varphi\, dx = \delta_k \to 0$$

or

$$\int\limits_{E_n \setminus \Gamma_{1-\varepsilon}} \nabla U_0^{(k)} \cdot A \nabla U_0^{(k)}\, dx = (1-\varepsilon) \int\limits_{E_n \setminus \Gamma_{1-\varepsilon}} \nabla U_0^{(k)} \cdot A \nabla \psi\, dx + \delta_k,$$

whence we find

$$\int\limits_{E_n\setminus\Gamma_{1-\varepsilon}} \nabla u_0^{(k)} \cdot A\nabla u_0^{(k)}\,dx \le (1-\varepsilon)^2 \int\limits_{E_n\setminus\Gamma_{1-\varepsilon}} \nabla\psi\cdot A\nabla\psi\,dx + \delta_k$$

$$\le \delta_k + \mu^{-1}(1-\varepsilon)^2 \int\limits_{E_n\setminus\Gamma_{1-\varepsilon}} \rho^\alpha(x)\nabla\psi\cdot\nabla\psi\,dx$$

$$= \mu^{-1}(1-\varepsilon)^2 \operatorname{cap}_\alpha \overline{\Gamma}_{1-\varepsilon} + \delta_k. \tag{6.24}$$

Now, since $\varepsilon > 0$ is arbitrary, we conclude that the right-hand part of (6.24) does not exceed

$$\frac{3}{2}\,\mu^{-1}\operatorname{cap}_\alpha G_{m_1-1}. \tag{6.25}$$

Therefore (6.23) implies that

$$a^{(k)} \le \frac{3}{2}\,\mu^{-2}4^{mq}\operatorname{cap}_\alpha G_{m_1-1} + \delta_k, \quad q = \sum_{i=1}^n \sigma_i - \beta.$$

Making $k\to\infty$, from the latter inequality we obtain

$$\inf_{x\in Q^0_{4-m_1}} U_0(x) < \frac{3}{16}\,.$$

Then there exists a point $x' \in Q^0_\delta$ such that $U_0(x') < \frac{1}{4}$. We put

$$U = \sum_{m=m_0}^{m_1-2} U_m + U_0,$$

where $U_m \in \overset{\circ}{W}{}^1_{2,\alpha}(E_n)$ is the potential of the set $K_m = (\overline{Q}{}^0_{4-m} \setminus Q^0_{4-m-1}) \setminus D$. Now, applying estimate (6.9) of Proposition 6.1 to this potential and assuming $m = m_0+1, m_0+2, \dots, m_1-2$, we obtain

$$U_m(x') \le \sup_{x\in Q^0_{4-m-2}} U_m(x) \le C4^{mq}\operatorname{cap}_\alpha K_m, \quad C = C(n,\mu,\alpha,\beta).$$

Therefore $U(x') \le \frac{1}{4} + C\sum\limits_{m=m_0}^{m_1-2}\gamma_m$.

Let us now choose m_0 so large that the latter sum be $< \frac{1}{4}$. Then $U(x') < \frac{1}{2}$ and the necessity is proved. □

The approach to the construction of the potential U is analogous to that presented in [18, p. 43], while the proof of the sufficiency begins as described in [19, p. 50].

In the case of nondegenerated elliptic equations of second order we know the sufficient conditions for a boundary point which are of obvious geometric character, say, when the boundary points may have contact with the cone or funnel vertex lying outside the domain [18, pp. 44, 45]. In Proposition 6.2 below we give an analogous sufficient condition for the regularity of the boundary point at which the equation degenerates.

Condition K. Let $g \subset Q^0_{h_0}$ be some set of positive Wiener capacity ($\operatorname{cap}_\alpha g \neq 0$, $\alpha = 0$), $0 \in \partial g$, where h_0 is some integer.

Assume that for every $t \in (0, h_0)$, the transformation

$$y = (h_0/t)^\sigma x$$

brings the set $Q^0_t \setminus D$ to g. Then we say that the domain D satisfies the condition K.

Proposition 6.2. *Let $D \subset E_n$, $0 \in \partial D$. If the domain D satisfies the condition K. then the point 0 is regular.*

Proof. It is sufficient to apply Theorem 6.1, i.e. to show the divergence of the integral in condition (6.3).

It is not difficult to show the estimate

$$\operatorname{cap}_\alpha(Q^0_t \setminus D) \geq C_1 t^{\sum\limits_{k=1}^n \sigma_k - \beta}, \tag{6.26}$$

where C_1 is the Wiener capacity of the set g.

Indeed, let $u(x)$ be a test function for $Q^0_t \setminus D$, i.e. $v \in C^\infty_0(E_n)$, $v(x) \geq 1$ on $Q^0_t \setminus D$. Then according to the condition K the change of the variables $x = (t/h_0)^\sigma y$ transforms the set $Q^0_t \setminus D$ to g. Therefore

$$\int\limits_{E_n} \rho^\alpha(x) \nabla v \cdot A \nabla v \, dx = t^{\sum\limits_{k=1}^n \sigma_k - \beta} \int\limits_{E_n} (\nabla_y \widetilde{v})^2 \, dy, \tag{6.27}$$

where $\widetilde{v}(y) = v((t/h_0)^\sigma y)$. It is not difficult to verify that the function $\widetilde{v}$ in (6.27) is the test function for g, i.e. $\widetilde{v}|_g \geq 1$, $\widetilde{v} \in C^\infty_0(E_n)$, whence follows estimate (6.26). Using (6.26) in integral (6.3), we obtain its divergence.

The proposition is proved. □

Let the set $K_1 \equiv K(h_0) \subset Q^0_{h_0}$, $0 \in \partial K_1$, $\operatorname{cap}_\alpha K_1 \neq 0$, $t \in (0, h_0)$, be such that the transformation

$$x = (t/h_0)^\sigma y$$

brings this set to the part lying in the ball Q^0_t. We call such a set K_1 a metric cone with vertex at 0.

Proposition 6.3. *If $0 \in \partial D$ and the point 0 can be contacted outside the domain by the vertex of some metric cone K_1, then the point is regular.*

Acknowledgement

The authors express their gratitude to the referee for his helpful comments and remarks that contributed to the elimination of some inaccuracies in the original version of the paper.

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(Received 13.04.2006; revised 24.11.2006)

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