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SZEGÖ ASYMPTOTICS OF EXTREMAL POLYNOMIALS ON THE SEGMENT $[-1,+1]$: THE CASE OF A MEASURE WITH FINITE DISCRETE PART

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Abstract. The strong asymptotics of monic extremal polynomials with respect to the norm $L_p(\sigma)$ are studied. The measure σ is concentrated on the segment $[-1,1]$ plus a finite set of mass points in a region of the complex plane exterior to the segment $[-1,1]$.

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1. Introduction

Let σ be a positive Borel measure supported by a compact set E of the complex plane. Denote by $T_{n,2}(z)$ the monic polynomial $T_{n,2}(\sigma,z)=z^n+$ lower degree terms, orthogonal to the measure σ, i.e.,

$$(T_{n,2},z^k)_{L_p(\sigma)}:=\int_E T_{n,2}(\xi)\overline{\xi}^k d\sigma(\xi)=0, \quad k=0,1,\dots,n-1. \tag{1}$$

It is well known that these polynomials satisfy the extremal property

$$m_{n,2}(\sigma)=\min_{Q=z^n+\cdots}\|Q\|_{L_2(\sigma)},$$

where

$$m_{n,2}(\sigma):=(T_{n,2},T_{n,2})^{1/2}_{L_2(\sigma)}=\|T_{n,2}\|_{L_2(\sigma)}.$$

Thus the n-th orthogonal polynomial can be defined as a monic polynomial of degree n with a minimal norm in the Hilbert space $L_2(\sigma)$. From this point of view, we can define a large class of monic polynomials $T_{n,p}(\sigma,z)=z^n+$ lower degree terms called extremal polynomials that realize a minimal norm in $L_p(\sigma)$, i.e.,

$$\|T_{n,p}\|_{L_p(\sigma)}=\min_{Q=z^n+\cdots}\|Q\|_{L_p(\sigma)}=m_{n,p}(\sigma)$$

There is vast literature on orthogonal polynomials, but on extremal polynomials is insufficient. A special area of research in this subject has been the study of the asymptotic behavior of $T_{n,p}(z)$ when n tends to infinity. Beginning with the results obtained by Geronimus in 1952 [1], who considered the case where the support E of the measure is a rectifiable Jordan curve, in particular, Widom [8] investigated the case $p=\infty$. In 1987, Lubinsky and Saff proved the asymptotics of $m_{n,p}(\sigma)$ and $T_{n,p}$ outside the segment $[-1,1]$ under a general condition on the weight function [6]. Another result on zero distributions of

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extremal polynomials on the unit circle was presented by X. Li and K. Pan in [5]. In 1992, Kaliaguine [2] obtained the power asymptotics for extremal polynomials when E is a rectifiable Jordan curve plus a finite set of mass points. Recently, Khaldi presented an extension of Kaliaguine's results in [4], where he studied the case of a measure supported on a rectifiable Jordan curve plus an infinite set of mass points.

In the special case $p = 2$ of orthogonal polynomials, Kaliaguine established in [3] the power asymptotics for such a polynomial on an arc plus a finite discrete part, and recently Peherstorfer and Yuditskii [7] solved this problem for a measure supported on the segment $[-2, 2]$ plus a denumerable set of mass points which accumulate at the end points of the interval.

In this paper, we establish the strong asymptotics of the $L_p(\sigma)$ extremal polynomials $\{T_{n,p}(\sigma, z)\}$ associated with the measure σ which has a decomposition of the form $\sigma = \alpha + \gamma$, where α is a measure with $\text{supp}(\alpha) = [-1, 1]$, absolutely continuous with respect to the Lebesgue measure $d\theta$ on the segment E, i.e.,

$$d\alpha(x) = \rho(x)dx, \quad \rho \geq 0, \quad \int_{-1}^{+1} \rho(x)dx < +\infty,$$

satisfying some extra conditions, and γ is a measure supported on the finite set $\{z_k\}_{k=1}^{l} \subset \mathbb{C}\backslash[-1, +1]$, i.e.,

$$\gamma = \sum_{k=1}^{l} A_k \delta(z - z_k); \quad A_k > 0. \tag{2}$$

2. Preliminaries

2.1. **Hardy space and the Szegö function.** Let $E = [-1, 1]$, $\Omega = \{\mathbb{C}\backslash[-1, 1]\} \cup \{\infty\}$, $G = \{w \in \mathbb{C} : |w| > 1\} \cup \{\infty\}$. The conformal mapping $\Phi : \Omega \to G$ is defined by $\Phi(z) = z + \sqrt{z^2 - 1}$, its inverse $\Psi(w) = \frac{1}{2}\left(w + \frac{1}{w}\right)$, and the capacity $C(E) = \lim_{z \to \infty} \left(\frac{\Phi(z)}{z}\right) = \frac{1}{2}$.

Let ρ be an integrable non negative weight function on E satisfying the Szegő condition

$$\int_{-1}^{1} \frac{\operatorname{Log} \rho(x)}{\sqrt{1 - x^2}} dx > -\infty. \tag{3}$$

Then we can easily see that the weight function λ defined on the unit circle by

$$\lambda(e^{i\theta}) = \begin{cases} \rho(\xi) / \left|\Phi'_-(\xi)\right|, & \xi = \Psi(e^{i\theta}), \quad \pi < \theta < 2\pi, \\ \rho(\xi) / \left|\Phi'_+(\xi)\right|, & \xi = \Psi(e^{i\theta}), \quad 0 < \theta < \pi, \end{cases}$$

satisfies the usual Szegő condition

$$\int_{-\pi}^{\pi} Log(\lambda(e^{i\theta}))d\theta > -\infty.$$

Thus the Szegő function associated with the unit circle $T = \{t : |t| = 1\}$ and the weight function λ is defined by

$$D(w) = \exp\left\{-\frac{1}{2p\pi}\int_0^{2\pi} \operatorname{Log}\left(\rho(\cos\theta)\left|\sin\theta\right|\right)\frac{1+we^{-i\theta}}{1-we^{-i\theta}}d\theta\right\}, \quad |w| < 1, \tag{4}$$

and satisfies the following properties:

1) D is analytic on the open unit disk $U = \{w : |w| < 1\}$, $D(w) \neq 0, \forall w \in U$, and $D(0) > 0$.

2) D has boundary values almost everywhere on the unit circle T such that

$$\lambda(e^{i\theta}) = \rho(\cos\theta)\left|\sin\theta\right| = |D(t)|^{-p}$$

a.e. for $t = e^{i\theta} \in T$.

Definition 1. An analytic function f on Ω belongs to $H^p(\Omega,\rho)$ if and only if $f(\Psi(w))/D(1/w) \in H^p(G)$, where $H^p(G)$ is the usual Hardy space associated with the exterior G of the unit circle.

Any function $f \in H^p(\Omega,\rho)$ has boundary values f_+ and f_- on both sides of E, and f_+, $f_- \in L_p(\alpha)$.

In the Hardy space $H^p(\Omega,\rho)$ we will define

$$\|f\|^p_{H^p(\Omega,\rho)} = \oint_E |f(x)|^p\,\rho(x)dx = \lim_{R\to 1^+}\frac{1}{R}\int_{E_R}\frac{|f(z)|^p}{|D(z)|^p}\left|\Phi'(z)\right||dz|,$$

where $E_R = \{z \in \Omega : |\Phi(z)| = R\}$.

2.2. **Notation and the lemmas.** Let $1 \leq p < \infty$. We denote by $\mu(\rho)$, $\mu(\sigma)$ respectively the extremal values of the problems

$$\mu(\rho) = \inf\left\{\|\varphi\|^p_{H^p(\Omega,\rho)} : \varphi \in H^p(\Omega,\rho),\ \ \varphi(\infty) = 1\right\}, \tag{5}$$

$$\mu(\sigma) = \inf\Big\{\|\varphi\|^p_{H^p(\Omega,\rho)} : \varphi \in H^p(\Omega,\rho),\ \ \varphi(\infty) = 1,\ \ \varphi(z_k) = 0,\quad k = 1,2,\ldots,l\Big\}. \tag{6}$$

We denote by φ^* and ψ^* the extremal functions of problems (5) and (6).

Note that $\varphi^*(z) = D(1/\Phi(z))/D(0)$ is an extremal function of problem (5) and $\mu(\rho) = 2\pi/[D(0)]^p$ (see [2]).

Lemma 1. *The extremal functions φ^* and ψ^* are related by $\psi^* = B(z).\varphi^*$ and $\mu(\sigma) = \left(\prod_{k=1}^{l}|\Phi(z_k)|\right)^p \mu(\rho)$, where*

$$B(z) = \prod_{k=1}^{l}\frac{\Phi(z)-\Phi(z_k)}{\Phi(z)\overline{\Phi(z_k)}-1}\,\frac{|\Phi(z_k)|^2}{\Phi(z_k)} \tag{7}$$

is the Blaschke product.

Proof. Proceeding in the same way as in the case of a curve in [2, p. 231], we obtain the following result. □

Lemma 2.

$$\limsup_{n\to\infty} 2^n m_{n,p}(\sigma) \le \left[\mu\left(\rho(2^{lp}\left|\omega_l\right|^p)\right)\right]^{1/p} \le [\mu(\sigma)]^{1/p}, \tag{8}$$

where $\omega_l(z) = \prod_{k=1}^{l}(z-z_k)$.

Proof. The proof is the same as in [2, pp. 233–234]. □

3. Main results

Definition 2. A positive measure α supported on $[-1,1]$ is said to belong to a class $\mathbf{A}$ if it is absolutely continuous with respect to the Lebesgue measure as well as if for each $r<\infty$, $(\alpha')^{-1} \in L_r[-1,1]$.

First we state the Lubinsky and Saff's result [6].

Theorem 1 ([6])**.** *Let $\alpha \in \mathbf{A}$ be a positive measure. We can associate with the measure α, the function D, and the extremal values $m_{n,p}(\alpha)$ and $\mu(\rho)$ given by (4), (1), and (5). Then the monic extremal polynomials $T_{n,p}(\alpha,z)$ have the following asymptotic behavior ($n\to\infty$):*
(1) $\lim 2^n m_{n,p}(\alpha) = [\mu(\rho)]^{1/p}$;
(2) $\lim T_{n,p}(\alpha,z) = \{\Phi(z)/2\}^n \dfrac{D\left(1/\Phi(z)\right)}{D(0)}[1+\chi_n(z)]$
where $\chi_n(z)\to 0$ uniformly on the compact subsets of Ω.

Now we give the main result of this paper.

Theorem 2. *Let a measure $\sigma = \alpha+\gamma$ be such that $\alpha\in\mathbf{A}$ and $\gamma = \sum_{k=1}^{l} A_k\,\delta(z-z_k)$. Associate with the measure σ the functions D, B and the extremal values $m_{n,p}(\sigma)$ and $\mu(\sigma)$ given by (4), (7), (1), and (6), respectively. Then the monic extremal polynomials $T_{n,p}(\sigma,z)$ have the following asymptotic behavior as $n\to\infty$:*
(1) $\lim 2^n m_{n,p}(\sigma) = [\mu(\sigma)]^{1/p}$.
(2) $T_{n,p}(\sigma,z) = \{\Phi(z)/2\}^n B(z)\dfrac{D\left(1/\Phi(z)\right)}{D(0)}[1+\chi_n(z)]$,
where $\chi_n(z)\to 0$ uniformly on compact subsets of Ω.

Proof. Putting $t_k = \dfrac{1}{\Phi(z_k)}$, $\xi = \Psi(t)$ where $t=e^{i\theta}$, we have $B(\xi) = b(\bar{t})$ with $b(t) = \prod_{k=1}^{l} \dfrac{t-t_k}{t\overline{t_k}-1}\dfrac{\overline{t_k}}{|t_k|^2}$, $k=1,\dots,l$.

Now we consider the integral

$$I_n = \int_0^{\pi}\left|\frac{2^n t^n T_{n,p}(\Psi(t))}{D(t)} - \left(\frac{b(t)}{D(0)} + \frac{t^{2n} b(\bar{t}) D(\bar{t})}{D(0)D(t)}\right)\right|^2 \frac{d\theta}{2\pi}$$

and transform it in a standard way as the following sum

$$I_n = \int_0^\pi \left|\frac{2^n t^n T_{n,p}(\Psi(t))}{D(t)}\right|^2 \frac{d\theta}{2\pi} + \int_0^\pi \left|\frac{b(t)}{D(0)} + \frac{t^{2n} b(\bar t) D(\bar t)}{D(0)D(t)}\right|^2 \frac{d\theta}{2\pi}$$

$$- 2\mathcal{R}e \int_0^\pi \frac{2^n t^n T_{n,p}(\Psi(t))}{D(t)} \overline{\left(\frac{b(t)}{D(0)} + \frac{t^{2n} b(\bar t) D(\bar t)}{D(0)D(t)}\right)} \frac{d\theta}{2\pi}. \tag{9}$$

Then applying the Hölder inequality to the first term of (9) we get

$$\int_0^\pi \left|\frac{2^n t^n T_{n,p}(\Psi(t))}{D(t)}\right|^2 \frac{d\theta}{2\pi} \le \left(\int_0^\pi \left|\frac{2^n T_{n,p}(\Psi(t))}{D(t)}\right|^p \frac{d\theta}{2\pi}\right)^{2/p} \left(\int_0^\pi \frac{d\theta}{2\pi}\right)^{1-2/p}$$

$$\le \left(\int_0^\pi |2^n T_{n,p}(\Psi(t))|^p \, |D(t)|^{-p} \frac{d\theta}{2\pi}\right)^{2/p}$$

$$= \left(\frac{1}{2\pi}\right)^{2/p} \left(\int_{-1}^{+1} |2^n T_{n,p}(x)|^p \rho(x) dx\right)^{2/p}$$

$$\le \left[\frac{2^n m_{n,p}(\sigma)}{(2\pi)^{1/p}}\right]^2. \tag{10}$$

The second term of(9) we transform as the following sum

$$\int_0^\pi \left|\frac{b(t)}{D(0)} + \frac{t^{2n} b(\bar t) D(\bar t)}{D(0)D(t)}\right|^2 \frac{d\theta}{2\pi} = \int_0^\pi \left|\frac{b(t)}{D(0)}\right|^2 \frac{d\theta}{2\pi}$$

$$+ \int_0^\pi \left|\frac{b(\bar t)}{D(0)}\right|^2 \frac{d\theta}{2\pi} + 2\mathcal{R}e \int_0^\pi \frac{t^{-2n} b(t)}{D(0)} \overline{\left(\frac{b(\bar t) D(\bar t)}{D(0)D(t)}\right)} \frac{d\theta}{2\pi}.$$

When n tends to ∞, the last term approaches 0. For the first and the second term we have the estimate

$$\int_0^\pi \left|\frac{b(t)}{D(0)}\right|^2 \frac{d\theta}{2\pi} + \int_0^\pi \left|\frac{b(\bar t)}{D(0)}\right|^2 \frac{d\theta}{2\pi} = \int_0^{2\pi} \left|\frac{b(t)}{D(0)}\right|^2 \frac{d\theta}{2\pi}$$

$$= \frac{\prod_{k=1}^{l} \frac{1}{|t_k|^2}}{(D(0))^2} = \left[\frac{\mu(\sigma)}{2\pi}\right]^{2/p}. \tag{11}$$

To estimate the last integral of (9), we transform it as follows:

$$J_n = \int_0^\pi \frac{2^n t^n T_{n,p}(\Psi(t))}{D(t)} \overline{\left(\frac{t^{2n} b(t)}{D(0)} + \frac{b(\bar t) D(\bar t)}{D(0)D(t)}\right)} \frac{d\theta}{2\pi}$$

$$= \int_0^{\pi} \frac{2^n t^n T_{n,p}(\Psi(t))}{D(t)} \overline{\left(\frac{t^{2n} b(t)}{D(0)}\right)} \frac{d\theta}{2\pi} + \int_0^{\pi} \frac{2^n t^n T_{n,p}(\Psi(t))}{D(t)} \overline{\left(\frac{b(\bar{t}) D(\bar{t})}{D(0) D(t)}\right)} \frac{d\theta}{2\pi}$$

$$= \int_T \frac{2^n t^n T_{n,p}(\Psi(t))}{D(t) b(t)} \frac{|b(t)|^2}{D(0)} \frac{dt}{2\pi i t}. \tag{12}$$

By using the residue Theorem we set

$$J_n = \frac{\prod_{k=1}^{l} \frac{1}{|t_k|^2}}{(D(0))^2} + 2^n \sum_{k=1}^{l} \frac{t_k^{n-1} T_{n,p}(z_k)}{D(t_k) b'(t_k)}, \tag{13}$$

the last term of (13) can be estimated as

$$\left| \sum_{k=1}^{l} \frac{t_k^{n-1} T_{n,p}(z_k)}{D(t_k) b'(t_k)} \right|$$

$$\leq \left[\sum_{k=1}^{l} \left(\left| \frac{1}{D(t_k) b'(t_k)} \right| \frac{|t_k^{n-1}|}{A_k^{1/p}} \right)^q \right]^{1/q} \left[\sum_{k=1}^{l} |T_{n,p}(z_k)|^p A_k \right]^{1/p}, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{14}$$

Taking into account (14) and Lemma 2, we obtain

$$J_n \leq \left[\frac{\mu(\sigma)}{2\pi} \right]^{2/p} + \delta_n, \tag{15}$$

where $\delta_n \to 0$ as $n \to \infty$.

Substituting (10), (11) and (15) into (9) we obtain

$$0 \leq I_n \leq \left[\frac{2^n m_{n,p}(\sigma)}{(2\pi)^{1/p}} \right]^2 - \left[\frac{\mu(\sigma)}{2\pi} \right]^{2/p} - 2\delta_n. \tag{16}$$

Finally, using the previous estimate we get

$$\liminf_{n \to \infty} 2^n m_{n,p}(\sigma) \geq [\mu(\sigma)]^{1/p}.$$

This with Lemma 2 prove the first statement of Theorem 2.

Now, to prove (2) of Theorem 2, first we estimate the integral

$$\left| \int_T \left[\frac{2^n t^n T_{n,p}(\Psi(t))}{D(t)} - \left(\frac{b(t)}{D(0)} + \frac{t^{2n} b(\bar{t}) D(\bar{t})}{D(0) D(t)} \right) \right] \frac{1}{1 - w\bar{t}} \frac{dt}{2\pi i t} \right|^2$$

$$\leq \frac{1}{1 - |w|} \int_T \left| \frac{2^n t^n T_{n,p}(\Psi(t))}{D(t)} - \left(\frac{b(t)}{D(0)} + \frac{t^{2n} b(\bar{t}) D(\bar{t})}{D(0) D(t)} \right) \right|^2 \frac{d\theta}{2\pi}$$

$$= \frac{1}{1 - |w|} I_n. \tag{17}$$

As an immediate consequence of (16) and (1) of Theorem 2 we get

$$\lim_{n\to\infty} I_n = 0. \tag{18}$$

So,

$$\int_T \left[\frac{2^n t^n T_{n,p}(\Psi(t))}{D(t)} - \left(\frac{b(t)}{D(0)} + \frac{t^{2n} b(\bar{t}) D(\bar{t})}{D(0)D(t)}\right)\right] \frac{1}{1-w\bar{t}} \frac{dt}{2\pi i t} = o(1). \tag{19}$$

On the other hand, we have

$$\int_T \left[\frac{2^n t^n T_{n,p}(\Psi(t))}{D(t)} - \left(\frac{b(t)}{D(0)} + \frac{t^{2n} b(\bar{t}) D(\bar{t})}{D(0)D(t)}\right)\right] \frac{1}{1-w\bar{t}} \frac{dt}{2\pi i t}$$
$$= \int_T \chi_n(\Psi(t)) \frac{1}{1-w\bar{t}} \frac{dt}{2\pi i t} - \int_T \frac{t^{2n} b(\bar{t}) D(\bar{t})}{D(0)D(t)} \frac{1}{1-w\bar{t}} \frac{dt}{2\pi i t}. \tag{20}$$

Applying the Cauchy formula to the first term in (20), we can see that

$$\int_T \chi_n(\Psi(t)) \frac{1}{1-w\bar{t}} \frac{dt}{2\pi i t} = \chi_n(z), \quad z = \Psi(w) \in \Omega. \tag{21}$$

Since the last term in (20) approaches 0 as $n \to \infty$, from (19), (20) and (21) we obtain the second statement of Theorem 2. □

Remark. In this work, we have studied the strong asymptotics of L_p extremal polynomials under the Szegö condition for $p \geq 2$ and the case of a measure with finite discrete part. The cases where $0 < p < 2$ and the measure is concentrated on an infinite part, still remain open problems.

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