**Georgian Mathematical Journal**
Volume 14 (2007), Number 4, 793–805

# STABILITY OF FINITE DIFFERENCE SCHEMES ON IRREGULAR MESHES FOR VON FOERSTER-TYPE 1-D EQUATIONS

PIOTR ZWIERKOWSKI

**Abstract.** We consider a generalized von Foerster equation in one dimensional spatial variable and construct finite difference schemes for the initial value problem. The stability of finite difference schemes on irregular meshes generated by characteristics is studied.

**2000 Mathematics Subject Classification:** Primary: 65M12. Secondary: 65M06, 92B99.
**Key words and phrases:** Von Foerster equation, finite differences, characteristics, stability.

## 1. Introduction

Suppose that $c\colon E \to \mathbb{R}$ and $\lambda\colon E \times \mathbb{R}_+^2 \to \mathbb{R}$, where $E = [0,a] \times \mathbb{R}_+$, $\mathbb{R}_+ = [0,+\infty)$, $a > 0$. Consider the initial value problem generalizing the classical von Foerster model of mathematical biology (see [1]–[3])

$$\partial_t u(t,x) + c\,(t,x)\,\partial_x u(t,x) = u(t,x)\lambda\big(t,x,u(t,x),z(t)\big), \tag{1}$$

where

$$z(t) = z[u(t,\cdot)] = \int\limits_0^\infty u(t,x)\,dx, \quad t \in [0,a], \tag{2}$$

with the initial condition

$$u(0,x) = v(x), \quad x \in \mathbb{R}_+, \tag{3}$$

where $v\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a given continuous and integrable function. The well-posedness of problem (1)–(3) demands the condition $c\,(t,0) \le 0$, $t \in [0,a]$, that is: the characteristics either go out of the set $E$ through the lateral boundary or meet the boundary and remain there.

There is rich literature concerning models describing the evolution of age-dependent populations [2], [4]–[6]. In these papers, nonlinear initial boundary value problems with non-local boundary conditions are examined. Such boundary conditions (renewal equations) describe a birth process, e.g., a number of newborn inhabitants at a given time.

In papers [1], [7] there are considered models with the right-hand side of characteristic equation dependent on $t$, $x$ and $z$, which describes the total number of inhabitants. Since the renewal equation is not considered, the function $c$ on the lateral boundary is assumed to be nonpositive.

ISSN 1072-947X / \$8.00 / 

Due to [7] the above models can be generalized in the following ways: (i) including many species and many space variables, (ii) taking into consideration past densities and past total sizes of species.

The main tools used to prove the existence and uniqueness of solutions are the characteristics method and fixed point theorems. We refer the reader interested in models of mathematical biology to [8]–[10].

In the known literature the authors consider methods of approximation of solutions to von Foerster-type equations, in particular various methods of discretization are presented. In [11], [12] finite difference schemes are examined for initial value problems on bounded domains with non-local boundary conditions. A convergence theorem concerning Euler schemes in unbounded domains is given in [13], assuming that all characteristics flow in one direction. The proof is based on the Lax-Richtmyer equivalence theorem, which states that convergence of a finite difference scheme is equivalent to its stability and consistency. In [14], the stability of Euler schemes is proved in unbounded domains without the assumption that the sign of the function $c$ is constant in the whole domain.

According to [15], problem (1)–(3) is transformed to the following system of ordinary differential equations:

$$\varphi'(t) = c(t, \varphi(t)), \quad \varphi(0) = x, \quad x \in \mathbb{R}_+, \tag{4}$$

$$\zeta'(t) = \zeta(t)\lambda(t, \varphi(t), \zeta(t), z(t)), \quad \zeta(0) = v(\varphi(0)). \tag{5}$$

By $y(\cdot; x)$ denote the unique solution of (4), by $u(\cdot; x)$ denote the unique solution of (5) along $y(\cdot; x)$ provided that system (4),(5) is uniquely solvable. A change of variables $\mu = y(t; x)$ in the formula $z(t) = \int_0^\infty u(t, \mu)\, d\mu$ yields

$$z(t) = \int_0^\infty u\left(t, y\left(t; x\right)\right) \frac{\partial}{\partial x} y(t; x)\, dx, \quad t \in [0, a].$$

The purpose of the paper is to present finite difference schemes for (1)–(3) using the method of characteristics, see [16]. In this approach a mesh is obtained by numerical integration of (4). The proposed method significantly differs from the generalized Euler method, since the mesh generated by characteristics is irregular. Moreover, there is no need to assume the Courant–Friedrich–Levy condition, cf. [13], [14], [17].

Note that the function $z$ is the non-local term defined on an unbounded domain. To apply the classical theory concerning numerical integration rules we truncate the set $E$ to some bounded domain. A general theory concerning finite difference schemes for first order ordinary differential equations is applied. Approximate solutions of (4) determine the mesh in the domain $E$. To obtain better approximation of the characteristic curves, a second order method is applied. We emphasize that using the second order method is essential for our difference scheme. Solutions of (5) along discrete characteristics are computed by the Euler method. The function $z$ is approximated by an irregular rectangle quadrature whose knots are determined by discrete characteristics.

Similarly to [13]–[15] we apply recurrent inequalities to prove our stability theorem. The presented theory is illustrated by numerical experiments in $\mathbb{R}^3$. Due to the biological interpretation of the considered problem, we investigate only its nonnegative solutions.

## 2. Discretization of the Differential Problem

Since only a finite number of terms can be involved in practical computations, we truncate the domain $E$ and the initial set $E_0 = \mathbb{R}_+$ to some bounded domains determined by a respective characteristic. The irregular mesh in these domains is defined as follows. For a given number $N \in \mathbb{N}$ put $h = \frac{a}{N}$ and choose $N_h \in \mathbb{N}$ such that $hN_h \to \infty$ as $h \to 0$. Let $y^{(0,j)}$, $j = 0, 1, \dots, N_h$, be nodal points on the initial set $E_0$ such that the conditions: $(i)$ $y^{(0,0)} = 0$, $(ii)$ $C_0 h \le y^{(0,j+1)} - y^{(0,j)} \le C_1 h$, $j = 0, 1, \dots, N_h - 1$, with some positive constants $C_0$, $C_1$ are satisfied. Note that the initial set $E_0$ is truncated to the interval $\left[0, y^{(0,N_h)}\right]$. Denote $E_{0.h} = \left\{y^{(0,j)}, j = 0, 1, \dots, N_h\right\}$. If $C_0 = C_1$, then the mesh $E_{0.h}$ is regular. By $\left(t^{(i)}, y^{(i,j)}\right)$ denote the knots of the mesh, where $t^{(i)} = ih$, $i = 0, 1, \dots, N$. The number $y^{(i,j)}$ stands for the value of the discrete characteristic at the point $t = t^{(i)}$ which starts at the point $y^{(0,j)} \in E_{0.h}$. For $i = 0, \dots, N-1$, $j = 0, 1, \dots, N_h$ the mesh points $\left(t^{(i+1)}, y^{(i+1,j)}\right)$ are determined by the improved Euler formula (the Heun method)

$$y^{(i+1,j)} = y^{(i,j)} + \frac{h}{2}\Big[c\left(t^{(i)}, y^{(i,j)}\right) + c\left(t^{(i+1)}, y^{(i,j)} + hc\left(t^{(i)}, y^{(i,j)}\right)\right)\Big]. \tag{6}$$

For $i = 0, 1, \dots, N$ define the numbers $S_i$ in the following way: $S_0 = 0$, $S_i = \min\left\{j = 0, 1, \dots, N_h \colon y^{(i,j)} \ge 0\right\}$, $i = 1, \dots, N$. The motivation for the definition of the number $S_i$ is the following. Let $j = 0, 1, \dots, N_h$. Since the condition $c(t, 0) \le 0$, $t \in [0, a]$, is satisfied it is possible that $y^{(i,j)} < 0$ for some $i$, $i = 1, \dots, N$. Defining the number $S_i$ we exclude computation outside the set $E$.

Denote $E_h = \left\{\left(t^{(i)}, y^{(i,j)}\right) : i = 0, 1, \dots, N,\ j = S_i, \dots, N_h, \right\}$, $E_h^{(i)} = \left\{y^{(i,j)} : j = S_i, \dots, N_h\right\}$, $i = 0, 1, \dots, N$. For a function $u\colon E_h \to \mathbb{R}$ write $u^{(i,j)} = u\left(t^{(i)}, y^{(i,j)}\right)$. Solutions of (5) along the $j$-th characteristic, $j = 0, \dots, N_h$, on $E_h$ are approximated by the Euler formula

$$u^{(i+1,j)} = u^{(i,j)} + h\, u^{(i,j)} \lambda\left(t^{(i)}, y^{(i,j)}, u^{(i,j)}, z^{(i)}\right), \quad u^{(0,j)} = v\left(y^{(0,j)}\right) \tag{7}$$

for $i = 0, \dots, N-1$, where

$$z^{(i)} = \sum_{j=S_i}^{N_h - 1} u^{(i,j)}\left(y^{(i,j+1)} - y^{(i,j)}\right) \tag{8}$$

for $i = 0, \dots, N$. Formula (8) defines a discrete operator $Q_h$ which is a finite one-dimensional quadrature. Note that the number $N_h$ can be chosen in such a way that $S_i < N_h - 1$ for $i = 1, \dots, N$. This property holds if the function $c$ is bounded.

*Remark* 1. The condition $hN_h \to +\infty$ as $h \to 0$ is satisfied for $N_h$ given by the formula

$$N_h = \left[\frac{1}{h^k}\log\frac{1}{h^k}\right], \qquad k \geq 1,$$

where $[x]$ denotes the entire part of $x$. Notice that the greater $k$ is, the faster $N_h$ increases as $h \to 0$.

Denote respectively by $L^\infty(\mathbb{R}_+)$ and $L^1(\mathbb{R}_+)$ the classes of all essentially bounded measurable functions and Lebesgue integrable functions defined on $\mathbb{R}_+$. For any metric space $X$ we denote by $C(X, \mathbb{R})$ the class of all continuous functions $u\colon X \to \mathbb{R}$. Let us define the following class of integrable functions. Suppose that $f\colon \mathbb{R}_+ \to \mathbb{R}$. The function $f \in L^1_{\mathcal{M}}$ if and only if there exists a decreasing function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $g \in L^1(\mathbb{R}_+)$ and $|f(x)| \leq g(x)$ for a.e. $x \in \mathbb{R}_+$.

The following normed spaces are introduced. In the space $l^\infty$ of all bounded sequences $\psi = (\psi_j)_{j\in\mathbb{N}}$ we have the natural supremum norm

$$\|\psi\|_\infty = \sup_{j=0,1,\ldots} |\psi_j| \quad \text{for} \quad (\psi_j) \in l^\infty.$$

The space $l^1$, of all summable sequences $\psi = (\psi_j)_{j\in\mathbb{N}}$, is equipped with the norm

$$\|\psi\|_1 = h\sum_{j=0}^{+\infty} |\psi_j| \quad \text{for} \quad (\psi_j) \in l^1.$$

Since the difference problem is considered on the bounded mesh, we define counterparts of the norms $\|\cdot\|_\infty$, $\|\cdot\|_1$ for a finite sequence $(\psi_j) \in l^\infty$, $(\psi_j) \in l^1$ as follows. By $P_h$ denote a finite, irregular mesh on $\mathbb{R}_+$ whose knots satisfy the conditions: $(i)$ $x^{(j)} < x^{(j+1)}, j = 0, 1, \ldots, N_h - 1$, $(ii)$ $C_0 h \leq x^{(j+1)} - x^{(j)} \leq C_1 h$, $j = 0, 1, \ldots, N_h - 1$, with some positive constants $C_0$, $C_1$, a discretization parameter $h > 0$ and a positive number $N_h$. Consider the function $\phi\colon \mathbb{N} \to \mathbb{R}$ such that $(\phi_j) \in l^\infty \cap l^1$ and $\phi_j = 0$ for $x^{(j)} \notin P_h$, and define

$$\|\phi\|_{\infty.h} = \sup_{j=0,1,\ldots,N_h} |\phi_j|, \qquad \|\phi\|_{1.h} = h\sum_{j=0}^{N_h} |\phi_j|.$$

Note that the above definition is also valid for a regular mesh, i.e. $x^{(j)} = hj$, $j = 0, 1, \ldots, N_h$.

Suppose that $R_h$ denotes an unbounded irregular mesh on $\mathbb{R}_+$ whose knots satisfy the conditions: $(i)$ $x^{(j)} < x^{(j+1)}, j = 0, 1, \ldots,$ $(ii)$ $C_0 h \leq x^{(j+1)} - x^{(j)} \leq C_1 h$, $j = 0, 1, \ldots,$ with some positive constants $C_0$, $C_1$ and a discretization parameter $h > 0$. Let $f_h$ denote the restriction of the function $f\colon \mathbb{R}_+ \to \mathbb{R}$ to the set $R_h$.

**Lemma 1.** *Suppose that* $f\colon \mathbb{R}_+ \to \mathbb{R}$ *and* $f \in L^1_{\mathcal{M}}$. *Then* $\|f_h\|_1 < \infty$.

*Proof.* By virtue of the definition of the class $L^1_{\mathcal{M}}$ there exists a nonnegative, Lebesgue integrable function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $|f(x)| \leq g(x)$ for a.e. $x \in \mathbb{R}_+$. Define a step function $w\colon \mathbb{R}_+ \to \mathbb{R}_+$ by the formula $w(x) = g(x^{(j)})$

for $x \in \left[x^{(j-1)}, x^{(j)}\right)$, $j = 1, \dots$ . Since the function $g$ is integrable and the inequality $0 \le w(x) \le g(x)$ holds for $x \in \mathbb{R}_+$, we obtain $\int\limits_{\mathbb{R}_+} w(x)\,dx < \infty$. The condition $(ii)$ of the definition of the set $R_h$ implies the estimate

$$C_0 h \sum_{j=1}^{p} g\left(x^{(j)}\right) \le \sum_{j=1}^{p} g\left(x^{(j)}\right)\left(x^{(j)} - x^{(j-1)}\right) = \int\limits_0^p w(x)\,dx.$$

Letting $p \to +\infty$, we obtain $\|g_h\|_1 < \infty$. The desired assertion follows from the inequality $|f(x)| \le g(x)$. □

*Remark* 2. The assertion of Lemma 1 remains true if $R_h$ is a regular mesh generated by a positive discretization parameter $h$, e.g., $R_h = \left\{x^{(j)} \colon x^{(j)} = hj, j = 0, 1, \dots\right\}$.

We adopt the main regularity assumptions on the given functions:

Assumption $[V]$. The initial function $v \colon \mathbb{R}_+ \to \mathbb{R}$ is nonnegative, bounded, continuous and $v \in L^1_{\mathcal{M}}$.

Assumption $[C]$. Suppose that $c \colon E \to \mathbb{R}$ is continuous, bounded and there is a constant $L_c > 0$ such that

$$|c(t,x) - c(t,\bar{x})| \le L_c\,|x - \bar{x}|$$

for $(t,x)$, $(t,\bar{x}) \in E$.

Assumption $[\Lambda]$. Suppose that $\lambda \colon E \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and there are constants $L_x$, $L_\lambda$, $L_z > 0$ such that

$$|\lambda(t,\, x,\, p,\, q) - \lambda(t,\, \bar{x},\, \bar{p},\, \bar{q})| \le L_x\,|x - \bar{x}| + L_\lambda\,|p - \bar{p}| + L_z\,|q - \bar{q}|$$

for $(t,x), (t,\bar{x}) \in E$, $p, q, \bar{p}, \bar{q} \in \mathbb{R}$.

Assumption $[\Lambda 0]$. Suppose that there is a constant $L > 0$ such that the function $\lambda \colon E \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the condition $\lambda(t,x,p,q) \le L$ for $(t,x,p,r) \in E \times \mathbb{R}^2$.

Assumption $[N]$. Suppose that $\lambda \colon E \times \mathbb{R}^2 \to \mathbb{R}$ and the discretization parameter $h$ satisfies the inequality

$$1 + h\lambda(t,x,p,q) \ge 0$$

for $(t,x,p,r) \in E \times \mathbb{R}^2_+$.

Note that if the function $\lambda$ is bounded, then Assumption [N] holds for a sufficiently small parameter $h$.

**Lemma 2.** *If Assumption* $[C]$ *is satisfied and* $h\Gamma < 1$, $\Gamma = L_c\left(1 + \frac{hL_c}{2}\right)$, *then for* $i = 0, 1, \dots, N$, $j = S_i, \dots, N_h - 1$ *the following estimate holds:*

$$y^{(i,j)} < y^{(i,j+1)}.$$

*Proof.* The proof is carried out by induction on $i$. Let $y^{(i,j)} \in E_h^{(i)}$, $i = 0, 1, \ldots, N$. From the condition $C_0 h \le y^{(0,j+1)} - y^{(0,j)} \le C_1 h$, $j = 0, 1, \ldots, N_h - 1$, it follows that $y^{(0,j+1)} - y^{(0,j)} > 0$, $j = 0, 1, \ldots, N_h - 1$. Suppose that the assertion holds for some $i$, $i = 0, 1, \ldots, N - 1$. Due to Assumption $[C]$ we have

$$y^{(i+1,j+1)} - y^{(i+1,j)} \ge \left(y^{(i,j+1)} - y^{(i,j)}\right)(1 - h\Gamma).$$

Since $y^{(i,j+1)} - y^{(i,j)} > 0$ and $1 - h\Gamma > 0$ we obtain the desired assertion. □

*Remark* 3. By virtue of Lemma 2, the operator $Q_h$ given by (8) is well-defined.

**Lemma 3.** *Suppose that Assumptions* $[V]$, $[C]$, $[N]$ *are satisfied and* $h\Gamma < 1$. *Then the discrete function* $u \colon E_h \to \mathbb{R}$ *determined by* (7) *is nonnegative. Moreover, if Assumption* $[\Lambda 0]$ *is satisfied, then the nonnegative function* $u \colon E_h \to \mathbb{R}$ *is bounded.*

*Proof.* By induction, we show the nonnegativeness of the function $u$. Since the initial function $v$ is nonnegative, the assertion holds for $i = 0$. Suppose that the values of the function $u$ are nonnegative for $\left(t^{(i)}, x^{(j)}\right) \in E_h$, $i = 0, \ldots, N - 1$. Applying Assumption [N] to formula (7) we conclude that the assertion follows for $\left(t^{(i+1)}, x^{(j)}\right) \in E_h$.

Assumptions [N], [Λ0] applied to (7) yield the recurrence inequality $u^{(i+1,j)} \le (1 + hL)\, u^{(i,j)}$, which gives the recurrence relation

$$\left\|u^{(i+1,\cdot)}\right\|_{h.\infty} \le \left\|u^{(i,\cdot)}\right\|_{h.\infty}(1 + hL)$$

with the initial condition $\left\|u^{(0,\cdot)}\right\|_{h.\infty} \le \|v\|_\infty = \sup\left\{v(x) \colon x \in \mathbb{R}_+\right\}$. Hence we obtain the estimate $\left\|u^{(i,\cdot)}\right\|_{h.\infty} \le U_\infty$, where $U_\infty = \|v\|_\infty e^{aL}$. □

Since the initial function $v \colon \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $L^1_{\mathcal{M}}$, there exists a nonnegative, integrable and decreasing function $V \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $v(x) \le V(x)$ for a.e. $x \in \mathbb{R}_+$. By $V_h$ denote the restriction of the function $V$ to the set $E_{0.h}$. By virtue of Lemma 1 there exists a positive number $\bar{V}$, independent of $N_h$, such that $\|V_h\|_{h.1} < \bar{V}$.

**Lemma 4.** *Suppose that Assumptions* $[C]$, $[N]$, $[V]$, $[\Lambda 0]$ *are satisfied and* $h\Gamma < 1$. *Then there holds an estimate* $\left\|u^{(i,\cdot)}\right\|_{h.1} \le U_1$ *for* $i = 0, 1, \ldots, N$, *where* $U_1 = \bar{V} e^{aL}$.

*Proof.* Applying Assumptions [N], [Λ0] to (7), we obtain the inequality $u^{(i+1,j)} \le (1 + hL)\, u^{(i,j)}$, which yields the recurrence inequality

$$\left\|u^{(i+1,\cdot)}\right\|_{h.1} \le (1 + hL)\left\|u^{(i,\cdot)}\right\|_{h.1}$$

with the initial condition $\left\|u^{(0,\cdot)}\right\|_{h.1} \le \bar{V}$. Hence we obtain the desired estimate. □

**Lemma 5.** *Suppose that Assumption* $[C]$ *is satisfied,* $h\Gamma < 1$, $u \colon E_h \to \mathbb{R}_+$ *and the norms* $\left\|u^{(i,\cdot)}\right\|_{h.1}$ *are bounded. Then the following estimate is true:*

$$\sum_{j=S_i}^{N_h - 1} u^{(i,j)}\left(y^{(i,j+1)} - y^{(i,j)}\right) \le C_1 (1 + h\Gamma)^i \left\|u^{(i,\cdot)}\right\|_{1.h},$$

$i=0,1,\ldots,N,$ *provided that* $u^{(i,j)}=0$ *for* $j<S_i,\ \ i=1,\ldots,N.$

*Proof.* Let $y^{(i,j)}\in E^{(i)}$, $i=0,1,\ldots,N$. From Assumption $[C]$ it follows that

$$y^{(i+1,j+1)}-y^{(i+1,j)}\le \left(y^{(i,j+1)}-y^{(i,j)}\right)(1+h\Gamma) \tag{9}$$

with $C_0h\le y^{(0,j+1)}-y^{(0,j)}\le C_1h,\ \ j=0,1,\ldots,N_h-1.$ Consider the comparison difference equation for (9):

$$\alpha^{(i+1,j)}=\alpha^{(i,j)}\left(1+h\Gamma\right).$$

There is no loss of generality in assuming $\alpha^{(0,j)}=C_1h,\ \ j=0,1,\ldots,N_h$. Taking into consideration the initial condition, we have

$$\alpha^{(i,j)}=C_1h\left(1+h\Gamma\right)^i \quad \text{and} \quad y^{(i,j+1)}-y^{(i,j)}\le\alpha^{(i,j)}.$$

Hence, for $i=0,1,\ldots,N$, $j=S_i,\ldots,N_h-1$ the following inequality holds

$$y^{(i,j+1)}-y^{(i,j)}\le C_1h\left(1+h\Gamma\right)^i.$$

Multiplying the above inequality by $u^{(i,j)}$ and summing the terms over $j=S_i,\ldots,N_h-1,$ we obtain the desired assertion. □

In order to make the descriptions concise, denote

$$c^{(i,j)}=c\left(t^{(i)},y^{(i,j)}\right),\quad \lambda^{(i,j)}[u,z]=\lambda\left(t^{(i)},y^{(i,j)},u^{(i,j)},z^{(i)}\right)$$

for $i=0,1,\ldots,N$, $j=S_i,\ldots,N_h$.

## 3. Stability of the Scheme

To prove the stability of the finite difference scheme for problem (1)–(3) consider a perturbed scheme with respect to the truncations of the domain, perturbations of the right-hand sides and the initial condition. As in the previous section, we truncate the unbounded domain $E$ to some bounded domain determined by respective characteristic. The procedure presented in Section 2 will be applied to the difference schemes with perturbed right hand-sides. The knots are derived by the improved Euler formula (the Heun method) with perturbations:

$$\bar{y}^{(i+1,j)}=\bar{y}^{(i,j)}+\frac{h}{2}\left[\bar{c}^{(i,j)}+c\left(t^{(i+1)},\bar{y}^{(i,j)}+h\bar{c}^{(i,j)}\right)\right]+h^2\eta^{(i,j)}, \tag{10}$$

where $\bar{c}^{(i,j)}=c\left(t^{(i)},\bar{y}^{(i,j)}\right),\ \ i=0,1,\ldots,N,\ \ j=0,1,\ldots,N_h$. The initial condition is given by $\bar{y}^{(0,j)}=y^{(0,j)}$, $j=0,\ldots,N_h$. Denote $\bar{E}_h=\left\{\left(t^{(i)},\bar{y}^{(i,j)}\right):i=0,1,\ldots,N,\ \ j=S_i,\ldots,N_h\right\}$. The numbers $S_i$, $i=0,1,\ldots,N$, were defined in the previous section. Note that some points of the mesh $\bar{E}_h$ may be placed below the $X$-axis.

The value of any discrete function $\bar{u}\colon\bar{E}_h\to\mathbb{R}_+$ at the knot $\left(t^{(i)},\bar{y}^{(i,j)}\right)$ is denoted by $\bar{u}^{(i,j)}=\bar{u}\left(t^{(i)},\bar{y}^{(i,j)}\right)$. Consider the following scheme with the perturbed right hand-side

$$\bar{u}^{(i+1,j)}=\bar{u}^{(i,j)}+h\,\bar{u}^{(i,j)}\lambda\left(t^{(i)},\bar{y}^{(i,j)},\bar{u}^{(i,j)},\bar{z}^{(i)}\right)+h\xi^{(i,j)}, \tag{11}$$

and the perturbed initial condition $\bar{u}^{(0,j)} = v\left(\bar{y}^{(0,j)}\right) + \xi^{(0,j)}$ for $j = 0, 1, \ldots, N_h$. The perturbed function $\bar{z}$ is given by formula (8) with the discrete functions $\bar{y}$, $\bar{u}$ : $\bar{z}^{(i)} = (Q_h \bar{u})_i$, $i = 0, 1, \ldots, N$.

Assumption $[P]$. Suppose that the perturbations $\eta$ and $\xi$ satisfy the conditions:

$$(i) \qquad \sup_{i=1,\ldots,N} \left\|\eta^{(i,\cdot)}\right\|_{\infty.h} \le C_{\eta.h},$$

$$(ii) \qquad \sup_{i=1,\ldots,N} \left\|\xi^{(i,\cdot)}\right\|_{\infty.h} \le \bar{C}_{\xi.h}, \qquad \sup_{i=1,\ldots,N} \left\|\xi^{(i,\cdot)}\right\|_{1.h} \le \hat{C}_{\xi.h},$$

$$\left\|\xi^{(0,\cdot)}\right\|_{\infty.h} \le \bar{C}^0_{\xi.h}, \qquad \left\|\xi^{(0,\cdot)}\right\|_{1.h} \le \hat{C}^0_{\xi.h},$$

where $C_{\eta.h}$, $\bar{C}_{\xi.h}$, $\hat{C}_{\xi.h}$, $\bar{C}^0_{\xi.h}$, $\hat{C}^0_{\xi.h} \to 0$ as $h \to 0$.

We write an auxiliary estimate for $\left\|(\bar{y} - y)^{(i,\cdot)}\right\|_{\infty.h}$.

**Lemma 6.** *Suppose that* $\left(t^{(i)}, y^{(i,j)}\right) \in E_h$, $\left(t^{(i)}, \bar{y}^{(i,j)}\right) \in \bar{E}_h$. *If Assumptions* $[C]$ *and* $[P](i)$ *are satisfied, then*

$$\left\|(\bar{y} - y)^{(i,\cdot)}\right\|_{\infty.h} \le h C_{\eta.h} X,$$

$i = 0, 1, \ldots, N$, *where* $X = \frac{e^{a\Gamma} - 1}{\Gamma}$, $\Gamma = L_c\left(1 + \frac{hL_c}{2}\right)$.

*Proof.* Denote $\varepsilon^{(i,j)} = \bar{y}^{(i,j)} - y^{(i,j)}$. Subtracting (6) from (10), we obtain the explicit recurrence error equation

$$\begin{aligned}\varepsilon^{(i+1,j)} = \varepsilon^{(i,j)} &+ \frac{h}{2}\left[\bar{c}^{(i,j)} + c\left(t^{(i+1)}, \bar{y}^{(i,j)} + h\bar{c}^{(i,j)}\right)\right] + h^2 \eta^{(i,j)} \\ &- \frac{h}{2}\left[c^{(i,j)} + c\left(t^{(i+1)}, y^{(i,j)} + hc^{(i,j)}\right)\right],\end{aligned}$$

$i = 0, 1, \ldots, N-1$, $j = S_i, \ldots, N_h$. It follows from Assumption $[C]$ that

$$\left|\varepsilon^{(i+1,j)}\right| \le \left|\varepsilon^{(i,j)}\right| (1 + h\Gamma) + h^2 \left|\eta^{(i,j)}\right|.$$

Taking the maximum over $j = S_i, \ldots, N_h$, we have

$$\left\|\varepsilon^{(i+1,\cdot)}\right\|_{\infty.h} \le \left\|\varepsilon^{(i,\cdot)}\right\|_{\infty.h} (1 + h\Gamma) + h^2 \left\|\eta^{(i,\cdot)}\right\|_{\infty.h}. \tag{12}$$

Consider the comparison recurrence equation with respect to (12):

$$\theta^{(i+1)} = \theta^{(i)} (1 + h\Gamma) + h^2 \left\|\eta^{(i,\cdot)}\right\|_{\infty.h}.$$

Taking into consideration the initial condition $\theta^{(0)} = 0$ and the estimate given in Assumption $[P](i)$, we have

$$\left\|\varepsilon^{(i,\cdot)}\right\|_{\infty.h} \le \theta^{(i)} \quad \text{and} \quad \theta^{(i)} \le h C_{\eta.h} \frac{(1 + h\Gamma)^i - 1}{\Gamma},$$

$i = 0, 1, \ldots, N$. Since $(1 + h\Gamma)^i \le e^{ih\Gamma} \le e^{Nh\Gamma} = e^{a\Gamma}$, $i = 0, \ldots, N$, we have the estimate

$$\left\|\varepsilon^{(i,\cdot)}\right\|_{\infty.h} \le h C_{\eta.h} X, \qquad i = 0, 1, \ldots, N,$$

which completes the proof. □

Now, we present the estimate of the term $\left|\bar{z}^{(i)} - z^{(i)}\right|$.

*Remark* 4. By the definition of the functions $\bar{z}^{(i)}$, $z^{(i)}$ we have

$$\bar{z}^{(i)} - z^{(i)} = \sum_{j=S_i}^{N_h-1} \left( \bar{u}^{(i,j)} \left( \bar{y}^{(i,j+1)} - \bar{y}^{(i,j)} \right) - u^{(i,j)} \left( y^{(i,j+1)} - y^{(i,j)} \right) \right).$$

Adding a term

$$\sum_{j=S_i}^{N_h-1} \left( \bar{u}^{(i,j)} \left( y^{(i,j+1)} - y^{(i,j)} \right) - \bar{u}^{(i,j)} \left( y^{(i,j+1)} - y^{(i,j)} \right) \right),$$

which is equal to zero, to the right hand-side of the above equality we have

$$\begin{aligned} \left|\bar{z}^{(i)} - z^{(i)}\right| \le & \sum_{j=S_i}^{N_h-1} \bar{u}^{(i,j)} \left( \left|\bar{y}^{(i,j+1)} - y^{(i,j+1)}\right| + \left|\bar{y}^{(i,j)} - y^{(i,j)}\right| \right) \\ & + \sum_{j=S_i}^{N_h-1} \left|\bar{u}^{(i,j)} - u^{(i,j)}\right| \left( y^{(i,j+1)} - y^{(i,j)} \right). \end{aligned}$$

Applying Lemma 6 to the terms $\left|\bar{y}^{(i,j+1)} - y^{(i,j+1)}\right|$, $\left|\bar{y}^{(i,j)} - y^{(i,j)}\right|$ and Lemma 5 to $\left|\bar{u}^{(i,j)} - u^{(i,j)}\right| \left( y^{(i,j+1)} - y^{(i,j)} \right)$, we obtain the estimate

$$\left|\bar{z}^{(i)} - z^{(i)}\right| \le 2X \left\|\bar{u}^{(i,\cdot)}\right\|_{1.h} C_{\eta.h} + C_1 e^{a\Gamma} \left\|\bar{u}^{(i,\cdot)} - u^{(i,\cdot)}\right\|_{1.h}, \tag{13}$$

where $X = \frac{e^{a\Gamma}-1}{\Gamma}$, $\Gamma = L_c\left(1 + \frac{hL_c}{2}\right)$. Note that using the second order method to approximate solutions of (4) is essential to obtain estimate (13).

Now, we prove a stability theorem for the proposed difference scheme.

**Theorem 1.** *Suppose that* $\left(t^{(i)}, y^{(i,j)}\right) \in E_h$, $\left(t^{(i)}, \bar{y}^{(i,j)}\right) \in \bar{E}_h$, *Assumptions* [$C$], [$P$] *are satisfied, the discrete functions* $u\colon E_h \to \mathbb{R}$, $\bar{u}\colon \bar{E}_h \to \mathbb{R}_+$ *are bounded and for* $i = 1, \dots, N$ *the norms* $\left\|u^{(i,\cdot)}\right\|_{1.h}$, $\left\|\bar{u}^{(i,\cdot)}\right\|_{1.h}$ *are bounded. If*

- (*i*) $u$ *is a nonnegative solution of problem* (7),
- (*ii*) $\bar{u}$ *is a solution of problem* (11) *with perturbations satisfying Assumption* [$P$](*ii*),
- (*iii*) *the function* $\lambda \in C(E \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ *satisfies Assumptions* [$\Lambda$], [$\Lambda 0$],
- (*iv*) *the function* $\lambda \in C(E \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ *and the discretization parameter* $h$ *satisfy Assumption* [$N$],

*then the sequences* $\left\{\left\|\bar{u}^{(i,\cdot)} - u^{(i,\cdot)}\right\|_{\infty.h}\right\}_i$, $\left\{\left\|\bar{u}^{(i,\cdot)} - u^{(i,\cdot)}\right\|_{1.h}\right\}_i$ *converge uniformly to* 0 *as* $h \to 0$.

*Proof.* The proof is based on the recurrence inequalities similarly as in the proofs of Lemmas 5 and 6. Recall that $\varepsilon^{(i,j)} = \bar{y}^{(i,j)} - y^{(i,j)}$. By $\omega^{(i,j)} = \bar{u}^{(i,j)} - u^{(i,j)}$ denote the error of the scheme. Subtracting (7) from (11), we get the explicit recurrence error equation

$$\omega^{(i+1,j)} = \omega^{(i,j)} + h\left( \bar{u}^{(i,j)} \lambda^{(i,j)}[\bar{u}, \bar{z}] - u^{(i,j)} \lambda^{(i,j)}[u, z] \right) + h\xi^{(i,j)}.$$

Applying Assumption [Λ] and the nonnegativeness of the function $u$, we obtain the inequality

$$\left|\omega^{(i+1,j)}\right| \le \left(1 + hL + hL_\lambda u^{(i,j)}\right)\left|\omega^{(i,j)}\right| + hL_x u^{(i,j)}\left|\varepsilon^{(i,j)}\right| + hL_z u^{(i,j)}\left|\bar{z}^{(i)} - z^{(i)}\right| + h\left|\xi^{(i,j)}\right|. \tag{14}$$

Denote $\bar{U}_1 = \sup_{i=0,\dots,N}\left\|\bar{u}^{(i,\cdot)}\right\|_{1.h}$. By virtue of Lemma 6 and Remark 4, we obtain the recurrence inequality

$$\left\|\omega^{(i+1,\cdot)}\right\|_{\infty.h} \le (1 + hL + hL_\lambda U_\infty)\left\|\omega^{(i,\cdot)}\right\|_{\infty.h} + hL_z U_\infty C_1 e^{a\Gamma}\left\|\omega^{(i,\cdot)}\right\|_{1.h} + hC_{\eta.h}U_\infty X\left(hL_x + 2L_z\bar{U}_1\right) + h\left\|\xi^{(i,\cdot)}\right\|_{\infty.h}, \tag{15}$$

where the number $U_\infty$ is given in the proof of Lemma 3. Multiplying both sides of (14) by $h$, summing all terms over $j = S_i, \dots, N_h - 1$, applying Lemma 6 and Remark 4, we obtain

$$\left\|\omega^{(i+1,\cdot)}\right\|_{1.h} \le \left(1 + hL + hL_\lambda U_\infty + hL_z U_1 C_1 e^{a\Gamma}\right)\left\|\omega^{(i,\cdot)}\right\|_{1.h} + h^2 C_{\eta.h}U_1 X\left(hL_x + 2L_z\bar{U}_1\right) + h\left\|\xi^{(i,\cdot)}\right\|_{1.h}, \tag{16}$$

where the number $U_1$ is given in Lemma 4.

Just as in the proof of Lemma 6 consider the comparison recurrence equations with respect to (15) and (16):

$$\bar{\theta}^{(i+1)} = \left(1 + h\bar{L}\right)\bar{\theta}^{(i)} + hL_z U_\infty C_1 e^{a\Gamma}\hat{\theta}^{(i)} + h\bar{\Pi}_h + h\left\|\xi^{(i,\cdot)}\right\|_{\infty.h},$$
$$\hat{\theta}^{(i+1)} = \left(1 + h\hat{L}\right)\hat{\theta}^{(i)} + h\hat{\Pi}_h + h\left\|\xi^{(i,\cdot)}\right\|_{1.h},$$

where $\bar{L} = L + L_\lambda U_\infty$, $\bar{\Pi}_h = C_{\eta.h}U_\infty X\left(hL_x + 2L_z\bar{U}_1\right)$, $\hat{L} = L + L_\lambda U_\infty + L_z U_1 C_1 e^{a\Gamma}$, $\hat{\Pi}_h = hC_{\eta.h}U_1 X\left(hL_x + 2L_z\bar{U}_1\right)$. Taking into consideration the initial conditions $\left\|\omega^{(0,\cdot)}\right\|_{\infty.h} \le \bar{\theta}^{(0)} = \bar{C}^0_{\xi.h}$, $\left\|\omega^{(0,\cdot)}\right\|_{1.h} \le \hat{\theta}^{(0)} = \hat{C}^0_{\xi.h}$, we obtain the estimates $\left\|\omega^{(i,\cdot)}\right\|_{\infty.h} \le \bar{\theta}^{(i)}$ and $\left\|\omega^{(i,\cdot)}\right\|_{1.h} \le \hat{\theta}^{(i)}$. Hence the solutions of (16), (15) satisfy the inequalities

$$\begin{aligned}\left\|\omega^{(i,\cdot)}\right\|_{1.h} &\le \hat{\theta}^{(i)} \le e^{a\hat{L}}\hat{C}^0_{\xi.h} + \frac{e^{a\hat{L}} - 1}{\hat{L}}\left(\hat{\Pi}_h + \hat{C}_{\xi.h}\right) =: \hat{F}_h,\\ \left\|\omega^{(i,\cdot)}\right\|_{\infty.h} &\le \bar{\theta}^{(i)} \le e^{a\bar{L}}\bar{C}^0_{\xi.h} + \frac{e^{a\bar{L}} - 1}{\bar{L}}\left(L_z U_\infty e^{a\Gamma}\hat{F}_h + \bar{\Pi}_h + \bar{C}_{\xi.h}\right),\end{aligned} \tag{17}$$

respectively. The proof is complete. □

*Remark* 5. It follows from Theorem 1 that $\sup_{i=0,\dots,N}\left|\bar{z}^{(i)} - z^{(i)}\right| \to 0$ as $h \to 0$.

Note that the right hand-sides of (17) do not depend on $N_h$.

## 4. Numerical Experiments

The idea of truncations introduced in the paper is employed in numerical experiments. For a fixed discretization parameter $h > 0$ we truncate the domain $E$ to some bounded domains by discrete characteristics and observe that the errors behave stably as the length of the initial set increases. To simplify the presentation of numerical experiments only the length of the initial interval $[0, Y]$, $Y > 0$, is given. The mesh points on the initial set $[0, Y]$ are uniformly spaced, i.e. $y^{(0,j)} = jh$, $j = 0, \ldots, N_h$. We take $a = 1$ and $h = 0.01$, $h = 0.002$. With the prescribed functions $u : [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+$, $v(x) = u(0, x)$, $x \in [0, Y]$, $c\colon [0, 1] \times \mathbb{R}_+ \to \mathbb{R}$ we determine the respective right-hand side of the differential equation, i.e. the function $\lambda\colon [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$. The errors are given by the formulas

$$\Delta u = \max_{\substack{i=1,\ldots,N \\ j=0,\ldots,N_h}} \left| \tilde{u}^{(i,j)} - u\left(t^{(i)}, y^{(i,j)}\right) \right|, \quad \Delta z = \max_{i=1\ldots,N} \left| \tilde{z}^{(i)} - z(t^{(i)}) \right|,$$

where the discrete functions $\tilde{u}$, $\tilde{z}$ approximate the functions $u$, $z$, respectively. The values of the errors are listed in the tables for various lengths of the initial interval $[0, Y]$. In our examples the function $\lambda$ depends on the approximated values of the unknown functions $u$ and $z$, represented by $p$ and $q$. The approximation of the integer term $q$ is computed on a sufficiently large interval.

**Example 1.** Let

$$c(t, x) = \sin^3(x) \sin\left(\frac{\pi}{4}(t + 1)(1 + \sinh(2) - \cosh(2))\right) \frac{1}{x + 1}$$

and

$$\lambda(t, x, p, q) = \frac{1}{t + 1} + \frac{\sin(x)}{x + 1} \left(\sin(2x) - \frac{2xp}{t + 1}\right) \sin(q), \quad v(x) = \frac{\sin^2(x)}{1 + x^2}.$$

The solution of (1)–(3) with the above function is

$$u(t, x) = \frac{(t + 1)\sin^2(x)}{1 + x^2}.$$

Moreover $z(t) = \frac{\pi}{4}(t + 1)(1 + \sinh(2) - \cosh(2))$. The errors of computations are given it the table.

| $Y$ | $h$ | $\Delta u$ | $\Delta z$ | $h$ | $\Delta u$ | $\Delta z$ |
|---|---|---|---|---|---|---|
| 10 | 0.01 | 7.0E-3 | 0.1 | 0.002 | 6.12E-3 | 0.1 |
| 100 | | 1.68E-3 | 10.8E-3 | | 7.45E-4 | 1.01E-2 |
| 1000 | | 1.3E-3 | 2.82E-3 | | 4.45E-4 | 4.34E-3 |

**Example 2.** Consider problem (1)–(3) with

$$c(t, x) = \sin(4.9875(t + 1)) \sin(x) \sin\left((t + 1)\sin^2(x) \exp(-0.1x)\right) / (1 + x),$$

$$\lambda(t, x, p, q) = \frac{1}{t + 1} + \sin(p) \sin(q) \left(2\cos(x) - 0.1 \sin(x)\right) / (1 + x)$$

and $v(x) = \sin^2(x) \exp(-0.1x)$. The function

$$u(t, x) = (t + 1)\sin^2(x) \exp(-0.1x)$$

is a solution of the above problem. Moreover, $z(t) = 4.9875(t + 1)$. The errors of computations are presented in the table.

| $Y$ | $h$ | $\Delta u$ | $\Delta z$ | $h$ | $\Delta u$ | $\Delta z$ |
|---|---|---|---|---|---|---|
| 10 | 0.01 | 0.4 | 3.9 | 0.002 | 0.41 | 3.89 |
| 50 | | 1.17E-2 | 7.04E-2 | | 8.07E-3 | 6.83E-2 |
| 100 | | 4.82E-3 | 3.1E-3 | | 1.08E-3 | 2.03E-3 |
| 500 | | 4.8E-3 | 2.9E-3 | | 1.08E-3 | 2.03E-3 |

In both numerical experiments we observed that for given $h$, as the values of $Y$ increase, the values of the errors $\Delta u$ and $\Delta z$ decrease. Moreover, for greater values of $Y$, which are not listed in the tables, both of the errors are equal to the errors given in the last row of the tables.

## References


1. A. L. Dawidowicz and K. Łoskot, Existence and uniqueness of solution of some integro-differential equation. *Ann. Polon. Math.* **47**(1986), No. 1, 79–87.
2. M. E. Gurtin and R. C. McCamy, Non-linear age-dependent population dynamics. *Arch. Rational Mech. Anal.* **54**(1974), 281–300.
3. A. M. Nakhushev, Equations of mathematical biology. (Russian) *Vysshaya Shkola, Moscow,* 1995.
4. M. Czyżewska-Ważewska, A. Lasota, and M. C. Mackey, Minimizing therapeutically induced anemia. *J. Math. Biol.* **13**(1981/82), No. 2, 149–158.
5. M. El-Doma, Analysis of nonlinear integro-differential equations arising in age-dependent epidemic models. *Nonlinear Anal.* **11**(1987), No. 8, 913–937.
6. V. Skakauskas, Large time behavior in a density-dependent population dynamics problem with age structure and child care. *Mathematical modelling of population dynamics,* 243–258, *Banach Center Publ.,* 63, *Polish Acad. Sci., Warsaw,* 2004.
7. H. Leszczyński and P. Zwierkowski, Existence of solutions to generalized von Foerster equations with functional dependence. *Ann. Polon. Math.* **83**(2004), No. 3, 201–210.
8. F. Brauer and C. Castillo-Chávez, Mathematical models in population biology and epidemiology. *Texts in Applied Mathematics,* 40. *Springer-Verlag, New York,* 2001.
9. H. von Foerster, Some remarks on changing populations. In: *The Kinetics of Cellural Proliferation, Grune and Stratton, New York,* 1959.
10. A. J. Lotka, Elements of mathematical biology. (formerly published under the title Elements of Physical Biology). *Dover Publications, Inc., New York, N. Y.*, 1958.
11. L. M. Abia and J. C. López-Marcos, On the numerical integration of non-local terms for age-structured population models. *Deterministic models with applications in population dynamics and other fields of biology* (*Sofia*, 1997). *Math. Biosci.* **157**(1999), No. 1-2, 147–167.
12. A. S. Ackleh and R. R. Ferdinand, A nonlinear phytoplankton aggregation model with light shading, *SIAM J. Appl. Math.* **60**(1999), No. 1, 316–336.
13. H. Leszczyński and P. Zwierkowski, Stability of finite difference schemes for certain problems in biology. *Appl. Math. (Warsaw)* **31**(2004), No. 1, 13–30.
14. P. Zwierkowski, Stability of forward-backward finite difference schemes for certain problems in biology. *Comment. Math. Prace Mat.* **45**(2005), No. 2, 191–203.
15. Z. Kamont, Hyperbolic functional differential inequalities and applications. *Mathematics and its Applications,* 486. *Kluwer Academic Publishers, Dordrecht,* 1999.

16. Z. Kamont and J. Newlin-Łukowicz, Runge-Kutta bicharacteristic methods for first order partial functional differential equations. *Neural Parallel Sci. Comput.* **11**(2003), No. 4, 457–474.
17. K. W. Morton and D. F. Mayers, Numerical solution of partial differential equations. An introduction. *Cambridge University Press, Cambridge,* 1994.

(Received 7.10.2005; revised 29.11.2006)

Author's address:

Faculty of Mathematics and Computer Science
University of Warmia and Mazury
Żołnierska Street 14, 10-561 Olsztyn
Poland
E-mail: zwierkow@matman.uwm.edu.pl