Journal of Applied Analysis
Vol. 4, No. 1 (1998), pp. 63–74

OSCILLATORY PROPERTIES OF THE SOLUTIONS OF IMPULSIVE DIFFERENTIAL EQUATIONS WITH RETARDED ARGUMENT AND OSCILLATING COEFFICIENTS

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Received June 5, 1997 and, in revised form, November 4, 1997

Abstract. The impulsive equation with retarded argument

$$x'(t) + a(t)x(t) + p(t)x(t-\tau) = 0, \quad t \neq t_k,$$

$$\Delta x(t_k) + a_k x(t_k) + p_k x(t_k - \tau) = 0,$$

is considered, where the function $p(t)$ and the sequence $\{p_k\}$ are not of constant sign. Sufficient conditions are found for oscillation of all solutions to the equation under consideration.

1. Introduction

In the last twenty years significantly has increased the number of the publications devoted to the oscillatory behaviour of the solutions of functional-differential equations. A bigger part of the works on that topic published before 1977 is presented in [5]. In monographs [4] and [3] published in 1987 and 1991, respectively, the oscillation and asymptotic properties of

1991 *Mathematics Subject Classification.* 34A37.
Key words and phrases. Impulsive differential equations, oscillatory properties, retarded argument.

ISSN 1425-6908

the solutions of different classes of functional-differential equations were sistematically studied.

The first paper where the oscillatory properties are studied for the impulsive differential equations with retarded argument of the type

$$x'(t) + p(t)x(t-\tau) = 0, \quad t \neq t_k,$$
$$\Delta x(t_k) + a_k x(t_k) = 0$$

is that of Gopalsamy and Zhang [2]. The authors have found sufficient conditions for oscillation of all solutions assuming that $p(t)$ is a positive function. Moreover, sufficient conditions are obtained for the existence of non-oscillatory solution if $p(t) \equiv p = \text{const} > 0$.

In the present paper we find sufficient conditions for oscillation of the solutions of linear impulsive differential equations with retarded argument of the type

$$\begin{aligned} &x'(t) + a(t)x(t) + p(t)x(t-\tau) = 0, \quad t \neq t_k, \\ &\Delta x(t_k) + a_k x(t_k) + p_k x(t_k - \tau) = 0. \end{aligned} \tag{1}$$

We assume that the function $p(t)$ and the sequence p_k change their sign for $t \geq 0$, $k \in \mathbb{N}$.

2. Preliminary notes

First of all, we will consider the linear impulsive differential equation with retarded argument

$$\begin{aligned} &x'(t) + p(t)x(t-\tau) = 0, \quad t \neq t_k, \\ &\Delta x(t_k) + p_k x(t_k - \tau) = 0 \end{aligned} \tag{2}$$

and the corresponding linear impulsive inequalities

$$\begin{aligned} &x'(t) + p(t)x(t-\tau) \leq 0, \quad t \neq t_k, \\ &\Delta x(t_k) + p_k x(t_k - \tau) \leq 0 \end{aligned} \tag{3}$$

and

$$\begin{aligned} &x'(t) + p(t)x(t-\tau) \geq 0, \quad t \neq t_k, \\ &\Delta x(t_k) + p_k x(t_k - \tau) \geq 0, \end{aligned} \tag{4}$$

where $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^-) = x(t_k)$.

Introduce the following conditions:

H1. The constant τ is positive and the sequence $\{t_k\}_{k=1}^{\infty}$ is such that

$$0 = t_0 < t_1 < t_2 < \dots, \qquad \lim_{k \to +\infty} t_k = +\infty.$$

H2. The function $p\colon\ \mathbb{R}_+ \to \mathbb{R}$ is piecewise continuous in $\mathbb{R}_+ = [0, +\infty)$ with points of discontinuity $\{t_k\}$, where it is continuous from the left.

Let $J = [\alpha, \beta) \subset [\tau, +\infty)$.

Definition 1. The function $x = \varphi(t)$ is called *a solution* of the equation (2) in the interval J if:

1. $\varphi(t)$ is defined in $J_1 = [\alpha - \tau, \beta)$.
2. $\varphi(t)$ is absolutely continuous on each interval $J_1 \cap (t_{k-1}, t_k]$, $k \in \mathbb{N}$.
3. $\varphi'(t) + p(t)\varphi(t - \tau) = 0$ almost everywhere in $J \cap (t_{k-1}, t_k]$, $k \in \mathbb{N}$.
4. $\varphi(t_k^-) = \varphi(t_k)$, $\varphi(t_k^+) - \varphi(t_k) + p_k\varphi(t_k - \tau) = 0$ for $t_k \in J$.

In an analogous way, we define solutions of the equation (1) and inequalities (3) and (4).

Definition 2. The solution $x(t)$ of the equation (2) (or of the inequalities (3), (4)) is called *regular* if it is defined in some interval $[T_x, +\infty) \subset \mathbb{R}_+$ and

$$\sup\big\{|x(t)|\colon\ t \geq T\big\} > 0 \quad \text{for each} \quad t \geq T_x.$$

Definition 3. The solution $x(t)$ of the inequality (3) is said to be *finally positive* if there exists $T > 0$ such that $x(t) > 0$ for $t \geq T$.

Definition 4. The solution $x(t)$ of the inequality (4) is said to be *finally negative* if there exists $T > 0$ such that $x(t) < 0$ for $t \geq T$.

Definition 5. The solution $x(t)$ of the equation (2) is said to be *oscillatory* if it changes its sign in the interval $[T, +\infty)$, where T is an arbitrary number.

3. Main results

Theorem 1. *Let the conditions* H1 *and* H2 *be fulfilled and let there exist a sequence of non-intersected intervals $J_n = [\xi_n, \eta_n)$ with $\eta_n - \xi_n = 2\tau$, such that:*

1. *For each $n \in \mathbb{N}$, $t \in J_n$ and $t_k \in J_n$ one has*

$$p(t) \geq 0, \quad p_k \geq 0. \tag{5}$$

2. *There exists $\nu_1 \in \mathbb{N}$ such that for $n \geq \nu_1$*

$$\int\limits_{\eta_n - \tau}^{\eta_n} p(s)ds + \sum_{\eta_n - \tau \leq t_k < \eta_n} p_k \geq 1. \tag{6}$$

Then:

1. *The inequality* (3) *has no finally positive solution.*
2. *The inequality* (4) *has no finally negative solution.*
3. *Each regular solution of the equation* (2) *is oscillatory.*

Proof. First, we shall prove that the inequality (3) has no finally positive solution. Let us suppose the opposite. Then there exists a solution $x(t)$ of (3) such that we have $x(t) > 0$, $t \geq T$ for sufficiently large T. Then $x(t-\tau) > 0$ for $t \geq T + \tau$.

Since $\xi_n \to +\infty$ as $n \to +\infty$, there exists $\nu_0 \in \mathbb{N}$ such that $\xi_n > T + \tau$ for $n \geq \nu_0$. Then, it follows from (3) and (5) that $x'(t) \leq 0$, $\Delta x(t_k) \leq 0$ for $t,\ t_k \in J_n$, i.e., $x(t)$ is a nonincreasing function for $t \in J_n$, $n \geq \nu_0$.

Let $\nu = \max(\nu_0, \nu_1)$ and $n \geq \nu$.

We integrate (3) from $\eta_n - \tau$ to η_n and obtain

$$x(\eta_n) - x(\eta_n - \tau) + \int\limits_{\eta_n-\tau}^{\eta_n} p(s)x(s-\tau)ds + \sum_{\eta_n - \tau \leq t_k < \eta_n} p_k x(t_k - \tau) \leq 0.$$

Since $x(s-\tau) \leq x(\eta_n - \tau)$ for $s \in [\eta_n - \tau, \eta_n)$, then

$$x(\eta_n) + x(\eta_n - \tau)\left\{ \int\limits_{\eta_n-\tau}^{\eta_n} p(s)ds + \sum_{\eta_n - \tau \leq t_k < \eta_n} p_k - 1 \right\} \leq 0. \qquad (7)$$

It follows from (7) that for each $n \geq \nu$ the inequality

$$\int\limits_{\eta_n-\tau}^{\eta_n} p(s)ds + \sum_{\eta_n - \tau \leq t_k < \eta_n} p_k < 1$$

holds true, which contradicts (6).

In order to prove that (4) has no finally negative solution, it is sufficient to note that if $x(t)$ is a solution of (4), then $-x(t)$ is a solution of (3).

It follows from assertions 1 and 2 of Theorem 1 that the equation (2) has neither finally positive nor finally negative solutions. Therefore, each regular solution of (2) is oscillatory. □

For the equation with constant coefficients and constant delay

$$\begin{aligned} &x'(t) + px(t-\tau) = 0, \quad t \neq t_k, \\ &\Delta x(t_k) + p_0 x(t_k - \tau) = 0, \end{aligned} \qquad (8)$$

we obtain the following

Corollary 1. *Let $p \geq 0$, $p_0 \geq 0$, $\tau > 0$ and*

$$p\tau + p_0 i[T_n - \tau, T_n) \geq 1 \qquad (9)$$

for infinitely many numbers T_n with $\lim_{n\to+\infty} T_n = +\infty$, where $i[a,b)$ denotes the number of the points t_k lying in the interval $[a,b)$.

Then each regular solution of the equation (8) is oscillatory.

Let us suppose in addition, that the equation (8) *is* τ*-periodic, i.e., there exists a* $m \in \mathbb{N}$ *such that* $t_{k+m} = t_k + \tau$, $k \in \mathbb{Z}$, *or equivalently,* $i[t-\tau, t) \equiv m$.

Then each regular solution of the equation (8) *is oscillatory if*

$$p\tau + p_0 m \geq 1. \tag{10}$$

Theorem 2. *Let the conditions* H1 *and* H2 *be fulfilled and suppose there exists a sequence of non-intersected intervals* $J_n = [\xi_n, \eta_n)$ *with* $\eta_n - \xi_n \geq 2\tau$ *such that:*

1. *For each* $n \in \mathbb{N}$, $t \in J_n$ *and* $t_k \in J_n$,

$$p(t) \geq 0, \quad p_k \geq 0. \tag{11}$$

2. *There exist a constant* $K > 0$ *and an integer* $\nu_1 > 0$ *such that for each* $n \geq \nu_1$ *and* $t \in [\xi_n + \tau, \eta_n)$ *we have*

$$A(t) \equiv \int\limits_{t-\tau}^{t} p(s)ds + \sum_{t-\tau \leq t_k < t} p_k \geq K > e^{-1}. \tag{12}$$

3. *There exist a constant* $\delta > 0$ *and an integer* $\nu_2 > 0$ *such that for each* $n \geq \nu_2$ *there exists* $t_n^* \in [\eta_n - \tau, \eta_n)$ *such that*

$$B_n(t_n^*) C_n(t_n^*) \geq \delta, \tag{13}$$

where

$$B_n(t_n^*) = \int\limits_{\eta_n - \tau}^{t_n^*} p(s)ds + \sum_{\eta_n - \tau \leq t_k < t_n^*} p_k,$$

$$C_n(t_n^*) = \int\limits_{t_n^*}^{\eta_n} p(s)ds + \sum_{t_n^* \leq t_k < \eta_n} p_k.$$

4. *There exists* $\nu_3 > 0$ *such that for each* $n \geq \nu_3$ *the inequality*

$$\eta_n - \xi_n > (m_0 + 1)\tau \tag{14}$$

holds true, where

$$m_0 = \min\big\{m \in \mathbb{N}\colon\ \delta(eK)^m > 1\big\}. \tag{15}$$

Then:

1. *The inequality* (3) *has no finally positive solution.*
2. *The inequality* (4) *has no finally negative solution.*
3. *Each regular solution of the equation* (2) *is oscillatory.*

Proof. 1. Let us suppose that the inequality (3) has a finally positive solution $x(t)$, i.e., there exists sufficiently large $T > 0$ such that $x(t) > 0$ for $t \geq T$ and $x(t-\tau) > 0$ for $t \geq T + \tau$.

Since $\xi_n \to +\infty$ as $n \to +\infty$, there exists $\nu_0 > 0$ such that $\xi_n > T + \tau$ for $n \geq \nu_0$. Therefore, it follows from (3) and (11) that $x(t)$ is nonincreasing function in J_n, $n \geq \nu_0$.

Let $\nu = \max(\nu_0, \nu_1, \nu_2, \nu_3)$. Then for each $n \geq \nu$ the solution $x(t)$ is a nonincreasing function in J_n and conditions (12), (13) and (14) are valid.

It follows from (3) that

$$\begin{aligned} & x'(t) + p(t)x(t) \leq 0, \quad t \neq t_k, \\ & \Delta x(t_k) + p_k x(t_k) \leq 0 \end{aligned} \tag{16}$$

for $t,\ t_k \in [\xi_n + \tau, \eta_n)$, $n \geq \nu$.

Since $1 - p_k \leq e^{-p_k}$, then (16) implies

$$\begin{aligned} & x'(t) \leq -p(t)x(t), \quad t \neq t_k, \\ & x(t_k^+) \leq e^{-p_k} x(t_k) \end{aligned} \tag{17}$$

for $t,\ t_k \in [\xi_n + \tau, \eta_n)$, $n \geq \nu$.

By means of the Theorem of impulsive differential inequalities ([1], Theorem 2.3), it follows from (17) that

$$x(t) \leq x(t-\tau) \exp\left[-\int_{t-\tau}^{t} p(s)ds - \sum_{t-\tau \leq t_k < t} p_k \right]$$

for $t \in [\xi_n + \tau, \eta_n)$, $n \geq \nu$. From the above inequality, and from (12) it follows

$$\frac{x(t-\tau)}{x(t)} \geq e^K \geq eK, \quad t \in [\xi_n + \tau, \eta_n).$$

Repeating the above procedure we arrive at

$$\frac{x(t-\tau)}{x(t)} \geq (eK)^{m_0} \tag{18}$$

for $t \in [\xi_n + m_0\tau, \eta_n)$, $n \geq \nu$.

Since $\xi_n + m_0\tau < \eta_n - \tau$, then (18) holds true for each $n \geq \nu$ and $t = t_n^* \in [\eta_n - \tau, \eta_n)$, i.e.,

$$\frac{x(t_n^* - \tau)}{x(t_n^*)} \geq (eK)^{m_0}. \tag{19}$$

On the other hand, integration of (3) from $\eta_n - \tau$ to t_n^* implies the inequality

$$x(t_n^*) - x(\eta_n - \tau) + \int_{\eta_n-\tau}^{t_n^*} p(s)x(s-\tau)ds + \sum_{\eta_n-\tau\le t_k<t_n^*} p_k x(t_k-\tau) \le 0,$$

which implies

$$x(t_n^*-\tau)B_n(t_n^*) \le x(\eta_n-\tau). \tag{20}$$

Analogously, from the inequality

$$x(\eta_n) - x(t_n^*) + \int_{t_n^*}^{\eta_n} p(s)x(s-\tau)ds + \sum_{t_n^*\le t_k<\eta_n} p_k x(t_k-\tau) \le 0$$

we obtain

$$x(\eta_n-\tau)C_n(t_n^*) \le x(t_n^*). \tag{21}$$

Then, it follows from (13), (20) and (21) that

$$\frac{x(t_n^*-\tau)}{x(t_n^*)} \le \frac{1}{\delta}. \tag{22}$$

Finally, (19) and (22) give

$$\frac{1}{\delta} \ge (eK)^{m_0},$$

which contradicts (15).

The proof of the assertions 2 and 3 is carried out as in Theorem 1. □

Remark 1. The condition (12) is fulfilled if we suppose that

$$\liminf_{t\to+\infty} \left\{ \int_{t-\tau}^{t} p(s)ds + \sum_{t-\tau\le t_k<t} p_k \right\} > e^{-1}$$

for $t \in \cup_{n=1}^{\infty}[\xi_n+\tau, \eta_n)$.

Remark 2. In the case when the equation (2) is an equation without impulse effect ($p_k = 0$, $k \in \mathbb{N}$), the condition (12) has the form

$$A(t) \equiv \int_{t-\tau}^{t} p(s)ds \ge K > e^{-1}, \quad t \in [\xi_n+\tau, \eta_n),\ n \ge \nu_1$$

and it implies that condition (13) is fulfilled also with $\delta = K^2/4$.

In the case when $p_k \neq 0$, we cannot derive (13) as consequence of (12). In fact, if $p(t) \equiv 0$, $t_k = k\tau$, then condition (12) has the form

$$p_k \geq K > e^{-1}$$

and in this case $B_n(t)C_n(t) \equiv 0$. Therefore, the condition (13) is not satisfied.

Remark 3. Condition 4 of Theorem 2 is fulfilled if we suppose that

$$\lim_{n\to+\infty}(\eta_n - \xi_n) = +\infty.$$

Remark 4. If $p > 0$ in the equation (8) with constant coefficients, then condition (13) is fulfilled since

$$B_n(t_n^*)C_n(t_n^*) \geq \frac{p^2\tau^2}{4}.$$

Corollary 2. *Let $p > 0$, $p_0 \geq 0$, $\tau > 0$ and*

$$p\tau + p_0 \liminf_{t\to+\infty} i[t-\tau, t) > e^{-1}. \tag{23}$$

Then each regular solution of the equation (8) *is oscillatory.*

Moreover, if the equation (8) *is a τ-periodic and $i[t-\tau,t) \equiv m \in \mathbb{N}$, then each regular solution of the equation* (8) *is oscillatory if*

$$p\tau + p_0 m > e^{-1}. \tag{24}$$

Consider now the equation (1) together with the corresponding inequalities

$$\begin{aligned} &x'(t) + a(t)x(t) + p(t)x(t-\tau) \leq 0, \quad t \neq t_k,\\ &\Delta x(t_k) + a_k x(t_k) + p_k x(t_k - \tau) \leq 0 \end{aligned} \tag{25}$$

and

$$\begin{aligned} &x'(t) + a(t)x(t) + p(t)x(t-\tau) \geq 0, \quad t \neq t_k,\\ &\Delta x(t_k) + a_k x(t_k) + p_k x(t_k - \tau) \geq 0. \end{aligned} \tag{26}$$

We introduce the following conditions:

H3. The function $a\colon \mathbb{R}_+ \to \mathbb{R}$ is piecewise continuous in $\mathbb{R}_+$ with points of discontinuity $\{t_k\}$, where it is continuous from the left.

H4. $1 - a_k > 0$, $k \in \mathbb{N}$.

We set into the equation (1) (or, into the inequalities (25), (26))

$$x(t) \equiv \varphi(t)z(t) \equiv e^{-\int_0^t a(s)ds} \prod_{0<t_k<t}(1-a_k)z(t). \tag{27}$$

Making use of the relations

$$x'(t) = -a(t)\varphi(t)z(t) + \varphi(t)z'(t), \quad t \neq t_k,$$
$$\Delta x(t_k) = -a_k\varphi(t_k)z(t_k) + \varphi(t_k^+)\Delta z(t_k),$$

we obtain

$$\begin{gathered} z'(t) + p(t)\exp\left[\int\limits_{t-\tau}^{t} a(u)du\right] \prod_{t-\tau \le t_j < t} (1-a_j)^{-1} z(t-\tau) = 0, \\ t \neq t_k, \\ \Delta z(t_k) + p_k \exp\left[\int\limits_{t_k-\tau}^{t_k} a(u)du\right] \prod_{t_k-\tau \le t_j \le t_k} (1-a_j)^{-1} z(t_k-\tau) = 0. \end{gathered} \tag{28}$$

Since $1 - a_k > 0$, then $\varphi(t) > 0$ and the oscillatory properties of the equations (1) and (28) coincide. Moreover, the directions of the inequalities in (25) and (26) are preserved after the change of the variable (27). Therefore, applying Theorems 1 and 2 to the equation (28) we can derive oscillatory results for the equation (1).

The following theorems hold true.

Theorem 3. *Let the conditions* H1 — H4 *be fulfilled and let there exist a sequence of non-intersected intervals* $J_n = [\xi_n, \eta_n)$ *with* $\eta_n - \xi_n = 2\tau$, *such that:*

1. *For each* $n \in \mathbb{N}$, $t \in J_n$ *and* $t_k \in J_n$ *one has*

$$p(t) \ge 0, \quad p_k \ge 0. \tag{29}$$

2. *There exists* $\nu_1 \in \mathbb{N}$ *such that for* $n \ge \nu_1$

$$\begin{gathered} \int\limits_{\eta_n-\tau}^{\eta_n} p(s)\exp\left[\int\limits_{s-\tau}^{s} a(u)du\right] \prod_{s-\tau \le t_j < s} (1-a_j)^{-1} ds + \\ + \sum_{\eta_n-\tau \le t_k < \eta_n} p_k \exp\left[\int\limits_{t_k-\tau}^{t_k} a(u)du\right] \prod_{t_k-\tau \le t_j \le t_k} (1-a_j)^{-1} \ge 1. \end{gathered} \tag{30}$$

Then:

1. *The inequality* (25) *has no finally positive solution.*
2. *The inequality* (26) *has no finally negative solution.*
3. *Each regular solution of the equation* (1) *is oscillatory.*

Theorem 4. *Let the conditions* H1 — H4 *be fulfilled and let there exist a sequence of non-intersected intervals* $J_n = [\xi_n, \eta_n)$ *with* $\eta_n - \xi_n \geq 2\tau$, *such that:*

1. *For each* $n \in \mathbb{N}$, $t \in J_n$ *and* $t_k \in J_n$:

$$p(t) \geq 0, \quad p_k \geq 0. \tag{31}$$

2. *There exist a constant* $K > 0$ *and an integer* $\nu_1 > 0$ *such that for each* $n \geq \nu_1$ *and* $t \in [\xi_n + \tau, \eta_n)$ *we have*

$$\tilde{A}(t) \equiv \int_{t-\tau}^{t} p(s) \exp\left[\int_{s-\tau}^{s} a(u)du\right] \prod_{s-\tau \leq t_j < s} (1-a_j)^{-1} ds +$$

$$+ \sum_{t-\tau \leq t_k < t} p_k \exp\left[\int_{t_k-\tau}^{t_k} a(u)du\right] \prod_{t_k-\tau \leq t_j \leq t_k} (1-a_j)^{-1} \geq K > e^{-1}. \tag{32}$$

3. *There exist a constant* $\delta > 0$ *and an integer* $\nu_2 > 0$ *such that for each* $n \geq \nu_2$ *there exists* $t_n^* \in [\eta_n - \tau, \eta_n)$ *such that*

$$\tilde{B}_n(t_n^*)\tilde{C}_n(t_n^*) \geq \delta, \tag{33}$$

where

$$\tilde{B}_n(t_n^*) = \int_{\eta_n-\tau}^{t_n^*} p(s) \exp\left[\int_{s-\tau}^{s} a(u)du\right] \prod_{s-\tau \leq t_j < s} (1-a_j)^{-1} ds +$$

$$+ \sum_{\eta_n-\tau \leq t_k < t_n^*} p_k \exp\left[\int_{t_k-\tau}^{t_k} a(u)du\right] \prod_{t_k-\tau \leq t_j < t_k} (1-a_j)^{-1},$$

$$\tilde{C}_n(t_n^*) = \int_{t_n^*}^{\eta_n} p(s) \exp\left[\int_{s-\tau}^{s} a(u)du\right] \prod_{s-\tau \leq t_j < s} (1-a_j)^{-1} ds +$$

$$+ \sum_{t_n^* \leq t_k < \eta_n} p_k \exp\left[\int_{t_k-\tau}^{t_k} a(u)du\right] \prod_{t_k-\tau \leq t_j \leq t_k} (1-a_j)^{-1}.$$

4. *There exists* $\nu_3 > 0$ *such that* $\eta_n - \xi_n > (m_0+1)\tau$ *for each* $n \geq \nu_3$, *where* $m_0 = \min\{m \in \mathbb{N}: \ \delta(eK)^m > 1\}$.

Then:

1. *The inequality* (25) *has no finally positive solution.*
2. *The inequality* (26) *has no finally negative solution.*
3. *Each regular solution of the equation* (1) *is oscillatory.*

For the equation

$$x'(t) + ax(t) + px(t-\tau) = 0, \quad t \neq t_k, \\ \Delta x(t_k) + a_0 x(t_k) + p_0 x(t_k - \tau) = 0, \tag{34}$$

the following assertion is valid:

Corollary 3. *Suppose that*

$$p > 0, \quad p_0 \geq 0, \quad \tau > 0, \quad a_0 < 1, \quad i[t-\tau, t) \equiv m \in \mathbb{N} \tag{35}$$

and

$$p\tau e^{a\tau}(1-a_0)^{-m} + p_0 m e^{a\tau}(1-a_0)^{-m-1} > e^{-1}. \tag{36}$$

Then each regular solution of the equation (34) *is oscillatory.*

Acknowledgement. The present investigation was supported by the Bulgarian Ministry of Education and Science under Grant MM–702.

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