

OSCILLATORY SOLUTIONS OF A CLASS OF NONLINEAR IMPULSIVE DIFFERENTIAL EQUATIONS OF FIRST ORDER WITH RETARDED ARGUMENT

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Abstract. Sufficient conditions are found for oscillation of all solutions to a class of nonlinear impulsive differential equations of first order with retarded argument and fixed moments of impulse effect.

1. Introduction

The impulsive differential equations with deviating argument are adequate mathematical models of various processes and phenomena studied in physics, biology, electrotechnics, etc. In spite of the great possibilities for application, the theory of these equations is developing rather slowly due to difficulties of technical and theoretical character arising in their study.

Let us note that in the recent years the theory of the impulsive differential equations has been separated as a selfcontaining branch in the general theory of differential equations [1], [2], [3].

However, the oscillation theory of the impulsive differential equations has not been yet elaborated.

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The first work devoted to the oscillatory theory of the impulsive differential equations is the work of K. Gopalsamy and B.G. Zhang [4]. The oscillatory properties are studied for the solutions of impulsive differential equations of the type

$$\begin{aligned}x'(t) + p(t)x(t-h) &= 0, & t \neq \tau_k, \\ \Delta x(\tau_k) &= b_k x(\tau_k).\end{aligned}$$

Therein sufficient conditions are found for oscillation of all solutions to the problem under consideration. Moreover, sufficient conditions for existence of a nonoscillatory solution are obtained when $p(t) = p = \text{const}$.

In the present work we obtain sufficient conditions for oscillation of all solutions to a class of nonlinear impulsive differential equations with retarded argument and fixed moments of impulse effect.

2. Preliminary notes

Let $\mathbb{N}_m = \{1, 2, \dots, m\}$, h_i be positive constants, $i \in \mathbb{N}_m$, $\bar{h} = \max\{h_i : i \in \mathbb{N}_m\}$, $h = \min\{h_i : i \in \mathbb{N}_m\}$, $\{\tau_k\}_{k=1}^\infty$ be a monotone increasing, unbounded sequence of positive numbers and let $\{b_k\}_{k=1}^\infty$ be a sequence of real numbers.

Consider the impulsive differential equation with retarded argument

$$\begin{aligned}x'(t) + a(t)x(t) + \sum_{i=1}^n p_i(t)f_i(x(t-h_1), \dots, x(t-h_m)) &= 0, & t \neq \tau_k \\ \Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k - 0) &= b_k x(\tau_k - 0),\end{aligned}\tag{1}$$

where $x(\tau_k - 0) = x(\tau_k)$, and the corresponding differential inequalities

$$\begin{aligned}x'(t) + a(t)x(t) + \sum_{i=1}^n p_i(t)f_i(x(t-h_1), \dots, x(t-h_m)) &\geq 0, & t \neq \tau_k \\ \Delta x(\tau_k) &= b_k x(\tau_k),\end{aligned}\tag{2}$$

$$\begin{aligned}x'(t) + a(t)x(t) + \sum_{i=1}^n p_i(t)f_i(x(t-h_1), \dots, x(t-h_m)) &\leq 0, & t \neq \tau_k \\ \Delta x(\tau_k) &= b_k x(\tau_k),\end{aligned}\tag{3}$$

with the initial condition

$$x(t) = \varphi(t), \quad t \in [-\bar{h}, 0],\tag{4}$$

where $\varphi \in C([-\bar{h}, 0], \mathbb{R})$.

Introduce the following conditions:

H1. The functions $a, p_1, \dots, p_n: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ ($\mathbb{R}_+ = (0, \infty)$) are piecewise continuous with points of discontinuity $\{\tau_k\}_{k=1}^\infty$ where they are continuous from the left.

H2. $f_i \in C(\mathbb{R}^m, \mathbb{R})$, $i \in \mathbb{N}_n$.

H3. If $u_k \neq 0$ and $\operatorname{sgn} u_1 = \operatorname{sgn} u_2 = \dots = \operatorname{sgn} u_m$, then

$$u_k f_i(u_1, u_2, \dots, u_m) > 0 \quad k \in \mathbb{N}_m, \quad i \in \mathbb{N}_n.$$

H4. There exist constants $L_i > 0$ and $\alpha_{ij} \geq 0$ ($i \in \mathbb{N}_n$, $j \in \mathbb{N}_m$) such that

$$|f_i(u_1, u_2, \dots, u_m)| \geq L_i \prod_{j=1}^m |u_j|^{\alpha_{ij}}, \quad \sum_{j=1}^m \alpha_{ij} = 1, \quad i \in \mathbb{N}_n.$$

H5. There exist constants l_1 and l_2 such that

$$\lim_{k \rightarrow \infty} (\tau_k - kl_1) = l_2.$$

H6. There exists a constant $M > 0$ such that $0 \leq b_k < M$, $k \in \mathbb{N}$.

H7. $\tau_{k+1} - \tau_k \geq T > \bar{h}$ for each $k \in \mathbb{N}$.

Let us denote by $i[a, b]$ the number of the impulse moments in the interval $[a, b]$ ($0 < a < b < \infty$).

We construct the sequence

$$\{t_i\}_{i=1}^\infty = \{\tau_i\}_{i=1}^\infty \cup \{\tau_{is}\}_{i=1, s=1}^m,$$

where $\tau_{is} = \tau_i + h_s$ ($i \in \mathbb{N}$, $s \in \mathbb{N}_m$) and $t_i < t_{i+1}$, $i \in \mathbb{N}$.

Definition 1. By *solution* of the equation (1) with an initial function (4) we mean any function $x: [-\bar{h}, \infty) \rightarrow \mathbb{R}$ for which the following conditions are valid:

1. If $-\bar{h} \leq t \leq 0$, then $x(t) = \varphi(t)$.
2. If $0 \leq t \leq t_1 = \tau_1$, then x coincides with the solution of the problem

$$x'(t) + a(t)x(t) + \sum_{i=1}^n p_i(t)f_i(x(t-h_1), \dots, x(t-h_m)) = 0$$

with initial condition (4).

3. If $t_i < t \leq t_{i+1}$, $t_i \in \{\tau_i\}_{i=1}^\infty \setminus \{\tau_{is}\}_{i=1, s=1}^m$, then x coincides with the solution of the problem

$$x'(t) + a(t)x(t) + \sum_{i=1}^n p_i(t)f_i(x(t-h_1), \dots, x(t-h_m)) = 0$$

$$x(t_i + 0) = (1 + b_{k_i})x(t_i),$$

where the number k_i is determined from the equality $t_i = \tau_{k_i}$.

4. If $t_i < t \leq t_{i+1}$, $t_i \in \{\tau_{is}\}_{i=1, s=1}^{\infty, m} \setminus \{\tau_i\}_{i=1}^{\infty}$, then x coincides with the solution of the problem

$$x'(t) + a(t)x(t+0) + \sum_{i=1}^n p_i(t)f_i(x(t-h_1+0), \dots, x(t-h_m+0)) = 0$$

$$x(t_i+0) = x(t_i).$$

5. If $t_i < t \leq t_{i+1}$, $t_i \in \{\tau_i\}_{i=1}^{\infty} \cap \{\tau_{is}\}_{i=1, s=1}^{\infty, m}$, then x coincides with the solution of the problem

$$x'(t) + a(t)x(t+0) + \sum_{i=1}^n p_i(t)f_i(x(t-h_1+0), \dots, x(t-h_m+0)) = 0$$

$$x(t_i+0) = (1 + b_{k_i})x(t_i).$$

Remark 1. The definition of a solution to the problem (2), (4) ((3), (4)) is analogous to that of Definition 1.

Definition 2. The solution x of the inequality (2) is said to be *finally positive* if there exists a point $t_0 > 0$ such that x is defined for $t \geq t_0$ and $x(t) > 0$ for $t \geq t_0$.

In an analogous way the notion of *finally negative* solution of the inequality (3) can be introduced.

Definition 3. The nonzero solution x of the equation (1) is said to be *nonoscillatory* if there exists a point $t_0 > 0$ such that the function x is defined for $t \geq t_0$ and $x(t)$ does not change its sign for $t \geq t_0$. Otherwise, the solution x is called *oscillatory*.

3. Main results

Lemma 1. *Let condition H5 hold. Then for each $\Delta a > 0$ there exists a constant $l = l(\Delta a)$, $l \in \mathbb{N}$, such that $i[a, a + \Delta a] \leq l$ for $a > 0$.*

Proof. In view of $\tau_k > 0$ ($k \in \mathbb{N}$), it follows that $l_1 > 0$. Let $\varepsilon_0 = l_1/2$. Then there exists $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$ the following inequality is fulfilled:

$$\begin{aligned} l_2 - \varepsilon_0 < \tau_k - kl_1 < l_2 + \varepsilon_0, \quad \text{i.e.,} \\ l_2 - \varepsilon_0 + kl_1 < \tau_k < l_2 + \varepsilon_0 + kl_1. \end{aligned}$$

Analogously we obtain

$$l_2 - \varepsilon_0 + (k+1)l_1 < \tau_{k+1} < l_2 + \varepsilon_0 + (k+1)l_1$$

whence we deduce

$$\tau_{k+1} - \tau_k > [l_2 - \varepsilon_0 + (k + 1)l_1] - [l_2 + \varepsilon_0 + kl_1] = l_1 - 2\varepsilon_0 > 0.$$

Consequently, if $a > \tau_{k_0}$ then there are at most $\Delta a / (l_1 - 2\varepsilon_0)$ impulse moments in the interval $[a, a + \Delta a]$.

Finally, in each interval of the form $[a, a + \Delta a]$, $a > 0$ we have at most

$$l = k_0 + \frac{\Delta a}{l_1 - 2\varepsilon_0}$$

impulse moments. □

Theorem 1. *Let the following conditions hold:*

1. *Conditions H1 — H6 are fulfilled.*

2. $\liminf_{t \rightarrow \infty} \int_{t-h_i}^t a(s) ds \geq k_i,$

where k_i are positive constants, $i \in \mathbb{N}_m$.

3. $\liminf_{t \rightarrow \infty} \int_{t-h}^t p_i(s) ds > \frac{(1 + M)^{2l}}{L_i e^{k_i+1}}, i \in \mathbb{N}_m.$

Then:

1. *The inequality (3) has no finally positive solutions.*
2. *The inequality (2) has no finally negative solutions.*
3. *All solutions of the equation (1) are oscillatory.*

Proof. Let x be a positive solution of the inequality (3) for $t \geq t_0 \geq 0$. It is clear that $x(t - \bar{h}_j) > 0$, $j \in \mathbb{N}_m$, and $f_i(x(t - h_1), \dots, x(t - h_m)) > 0$, $i \in \mathbb{N}_n$ for $t \geq t_0 + \bar{h}$.

Let $t > T \geq t_0 + \bar{h}$. Multiplying (3) by $\exp\left(\int_T^t a(s) ds\right)$, we deduce

$$\begin{aligned} & \left(x(t) \exp\left(\int_T^t a(s) ds\right) \right)' \\ & + \sum_{i=1}^n p_i(t) \exp\left(\int_T^t a(s) ds\right) f_i(x(t - h_1), \dots, x(t - h_m)) \leq 0. \end{aligned} \quad (5)$$

Set

$$z(t) = x(t) \exp\left(\int_T^t a(s) ds\right). \quad (6)$$

Then, from (5) and (6) we obtain

$$z'(t) + \sum_{i=1}^n p_i(t) \exp\left(\int_T^t a(s) ds\right) \times \\ f_i\left(z(t-h_1) \exp\left(-\int_T^{t-h_1} a(s) ds\right), \dots, z(t-h_m) \exp\left(-\int_T^{t-h_m} a(s) ds\right)\right) \leq 0 \quad (7)$$

$$\Delta z(\tau_k) = b_k z(\tau_k).$$

It follows from (5) that $z(t) > 0$ for $t \geq T$. Then $z(t-h_j) > 0$, $j \in \mathbb{N}_m$ and

$$f_i\left(z(t-h_1) \exp\left(-\int_T^{t-h_1} a(s) ds\right), \dots, z(t-h_m) \exp\left(-\int_T^{t-h_m} a(s) ds\right)\right) > 0,$$

for $i \in \mathbb{N}_n$ and $t \geq T$.

In view of the above inequalities and of (7), it follows that z is a nonincreasing function in the set $J = (T, \tau_s) \cup \left[\bigcup_{i=1}^{\infty} (\tau_i, \tau_{i+1})\right]$, where $\tau_{s-1} < T < \tau_s$.

We introduce the notation

$$w(t) = \frac{z(t-h)}{z(t)}, \quad t \geq T_2 = T + \bar{h}.$$

Fixing t ($t \geq T_2$) and renumber the points of impulsive effect so that

$$t-h < \tau_1 < \tau_2 < \dots < \tau_k < t, \quad k \leq l.$$

Then

$$z(t-h) \geq z(\tau_1) = \frac{z(\tau_1+0)}{1+b_1} \geq \dots \geq \frac{z(t)}{\prod_{i=1}^k (1+b_i)} \geq \frac{z(t)}{(1+M)^l}, \quad \text{i.e.,}$$

$$w(t) = \frac{z(t-h)}{z(t)} \geq \frac{1}{(1+M)^l}. \quad (8)$$

We shall prove that the function w is bounded from above for $t \geq T_3 = T_2 + h$. Let t^* be an arbitrary point, $t^* \geq T_3$.

From condition 3 it follows that there exist constants $N_i > 0$, $i \in \mathbb{N}_m$ such that

$$\int_{t-h}^t p_i(s) ds \geq N_i > \frac{(1+M)^{2l}}{L_i e^{k_i+1}}, \quad i \in \mathbb{N}_m$$

for sufficiently large t ($t \geq T_2$). Therefore there exists a point $t \geq T_2$ such that $t - h \leq t^* < t$ and, moreover,

$$\int_{t-h}^{t^*} p_i(s)ds \geq \frac{N_i}{2} \quad \text{and} \quad \int_{t^*}^t p_i(s)ds > \frac{N_i}{2}, \quad i \in \mathbb{N}_m.$$

Fixing i ($i \in \mathbb{N}_m$), (7) implies the inequality

$$z'(t) + p_i(t) \exp\left(\int_T^t a(s)ds\right) \times$$

$$f_i\left(z(t-h_1) \exp\left(-\int_T^{t-h_1} a(s)ds\right), \dots, z(t-h_m) \exp\left(-\int_T^{t-h_m} a(s)ds\right)\right) \leq 0 \quad (9)$$

$$\Delta z(\tau_k) = b_k z(\tau_k).$$

Integrating (9) from $t-h$ to t^* we obtain

$$z(t-h) - z(t^*) + \sum_{\tau_k \in [t-h, t^*]} b_k z(\tau_k)$$

$$\geq L_i \int_{t-h}^{t^*} p_i(s) \exp\left(\int_T^s a(u)du\right) \prod_{j=1}^m z^{\alpha_{ij}}(s-h_j) \prod_{j=1}^m \exp\left(-\alpha_{ij} \int_T^{s-h_j} a(u)du\right) ds. \quad (10)$$

Moreover,

$$\exp\left(\int_T^s a(u)du\right) \prod_{j=1}^m \exp\left(-\alpha_{ij} \int_T^{s-h_j} a(u)du\right)$$

$$= \prod_{j=1}^m \exp\left(\alpha_{ij} \int_{s-h_j}^s a(u)du\right) \geq \exp\left(\sum_{j=1}^m \alpha_{ij} \liminf_{s \rightarrow \infty} \int_{s-h_j}^s a(u)du\right) \quad (11)$$

$$\geq \exp\left(\sum_{j=1}^m \alpha_{ij} k_j\right) \geq e^k,$$

where $k = \max\{k_j : j \in \mathbb{N}_m\}$.

Since z is a nonincreasing function in the set $(T_2, \tau_k) \cup \left[\bigcup_{i=k}^\infty (\tau_i, \tau_{i+1})\right]$, it follows the inequality

$$z(s-h_i) \geq \frac{z(s-h)}{(1+M)^{i[s-h_i, s-h]}} \geq \frac{z(s-h)}{(1+M)^l}.$$

Then

$$\prod_{j=1}^m z^{\alpha_{ij}}(s-h_j) \geq \prod_{j=1}^m \frac{z^{\alpha_{ij}}(s-h)}{(1+M)^{\alpha_{ij}l}} = \frac{z(s-h)}{(1+M)^l}. \quad (12)$$

From (10), (11), and (12) we obtain that

$$\begin{aligned} z(t-h) - z(t^*) + \sum_{\tau_k \in [t-h, t^*]} b_k z(\tau_k) & \quad (13) \\ & \geq \frac{L_i e^k}{(1+M)^l} \int_{t-h}^{t^*} p_i(s) z(s-h) ds \geq \frac{L_i e^k N_i}{2(1+M)^l} \inf_{s \in [t-h, t^*]} z(s-h) \\ & \geq \frac{L_i N_i e^k}{2(1+M)^l} \frac{z(t^* - h)}{(1+M)^{i[t-2h, t^* - h]}}. \end{aligned}$$

Since $i[t-2h, t^* - h] \leq l$, then from (13) as well as from the fact that $z(t^*) > 0$, it follows that

$$z(t^* - h) \leq \frac{2(1+M)^{2l}}{L_i N_i e^k} \left[z(t-h) + \sum_{\tau_k \in [t-h, t^*]} b_k z(\tau_k) \right]. \quad (14)$$

Integrate (9) from t^* to t and analogously to (13) obtain that

$$\begin{aligned} z(t^*) - z(t) + \sum_{\tau_k \in [t^*, t]} b_k z(\tau_k) & \geq \frac{L_i N_i e^k}{2(1+M)^l} \inf_{s \in [t^*, t]} z(s-h) \quad (15) \\ & \geq \frac{L_i N_i e^k}{2(1+M)^{2l}} \frac{z(t-h)}{(1+M)^{i[t^* - h, t - h]}} \geq \frac{L_i N_i e^k}{2(1+M)^{2l}} z(t-h). \end{aligned}$$

Having in mind $z(t) \geq 0$, it follows from (15) that

$$z(t-h) \leq \frac{2(1+M)^{2l}}{L_i N_i e^k} \left[z(t^*) + \sum_{\tau_k \in [t^*, t]} b_k z(\tau_k) \right]. \quad (16)$$

From (14) and (16) we obtain the estimate

$$z(t^* - h) \leq A^2 z(t^*) + A^2 \sum_{\tau_k \in [t^*, t]} b_k z(\tau_k) + A \sum_{\tau_k \in [t-h, t]} b_k z(\tau_k), \quad (17)$$

where $A = 2(1+M)^{2l}/(L_i N_i e^k)$.

Moreover, the following inequalities hold true

$$\sum_{\tau_k \in [t^*, t]} z(\tau_k) \leq z(t^*) \sum_{s=1}^{i[t^*, t]} (1+M)^{s-1}, \quad (18)$$

$$\sum_{\tau_k \in [t-h, t^*]} z(\tau_k) \leq z(t-h) \sum_{s=1}^{i[t-h, t^*]} (1+M)^{s-1}, \quad (19)$$

$$z(t^* - h) \geq \frac{z(t-h)}{(1+M)^{i[t^*-h, t-h]}}. \quad (20)$$

It follows from (19) and (20) that

$$\sum_{\tau_k \in [t-h, t^*]} z(\tau_k) < z(t^* - h)(1+M)^{i[t^*-h, t-h]} \sum_{s=1}^{i[t-h, t^*]} (1+M)^{s-1}. \quad (21)$$

From (17), (18) and (21) we obtain that

$$z(t^* - h) \leq A^2 z(t^*) + A^2 M \sum_{\tau_k \in [t^*, t]} z(\tau_k) + MA \sum_{\tau_k \in [t-h, t^*]} z(\tau_k) \quad (22)$$

$$\begin{aligned} &\leq A^2 z(t^*) + A^2 M z(t^*) \frac{(1+M)^{i[t^*, t]} - 1}{M} \\ &\quad + AM z(t^* - h)(1+M)^{i[t^*-h, t-h]} \frac{(1+M)^{i[t-h, t^*]} - 1}{M} \\ &\leq A^2 z(t^*) + A^2 z(t^*) [(1+M)^l - 1] \\ &\quad + z(t^* - h) [A(1+M)^{2l} - A(1+M)^l]. \end{aligned}$$

From (22) it follows that

$$z(t^* - h) [1 - A(1+M)^{2l} + A(1+M)^l] \leq A^2 (1+M)^l z(t^*),$$

i.e., the function w is bounded from above at an arbitrary chosen point $t^* \geq T_3$. Therefore it is bounded for $t \geq T_3$.

We divide the inequality (9) by $z(t) > 0$ for $t \geq T_2$, integrate the inequality obtained from $t - h$ to t and get

$$\begin{aligned} \ln \frac{z(t-h)}{z(t)} + \sum_{i=1}^k \ln \frac{z(\tau_i+0)}{z(\tau_i)} &\geq \frac{L_i e^{k_i}}{(1+M)^l} \int_{t-h}^t p_i(s) w(s) ds, \\ \ln [(1+M)^l w(t)] &\geq \frac{L_i e^{k_i}}{(1+M)^l} \int_{t-h}^t p_i(s) w(s) ds. \end{aligned} \quad (23)$$

Introduce the notation $w_0 = \liminf_{t \rightarrow \infty} w(t)$. It is clear that $0 < w_0 < \infty$ by virtue of the boundedness of w . Moreover, it follows from (8) that $(1+M)^l w_0 \geq 1$. By the fact that $\max_{x \geq 1} [\ln x - ax] = -\ln a - 1$ and from (23) we obtain

$$\begin{aligned} \max_{(1+M)^l w_0 \geq 1} \left\{ \ln [(1+M)^l w_0] - \frac{L_i e^{k_i} w_0}{(1+M)^l} \liminf_{t \rightarrow \infty} \int_{t-h}^t p_i(s) ds \right\} \\ = -\ln \left[\frac{L_i e^{k_i}}{(1+M)^{2l}} \liminf_{t \rightarrow \infty} \int_{t-h}^t p_i(s) ds \right] \leq -1. \end{aligned}$$

The last inequality yields

$$\liminf_{t \rightarrow \infty} \int_{t-h}^t p_i(s) ds \leq \frac{(1+M)^{2l}}{L_i e^{k_i+1}},$$

which contradicts condition 3 of Theorem 1.

In order to prove that (2) has no finally negative solution, it is sufficient to note that if $x(t)$ is a solution of (3), then $-x(t)$ solves (2).

Now, assertions 1 and 2 of Theorem 1 show that the equation (1) has neither finally positive, nor finally negative solutions. Therefore, all solutions of the equation (1) are oscillatory. \square

Corollary 1. *Let the following conditions hold:*

1. *Conditions 1 and 2 of Theorem 1 are fulfilled.*

$$2. \liminf_{t \rightarrow \infty} \int_{t-h}^t \sum_{i=1}^n L_i p_i(s) ds > \frac{(1+M)^{2l}}{e^{k+1}},$$

where $k = \min\{k_i : i \in \mathbb{N}_m\}$.

Then:

1. *The inequality (3) has no finally positive solutions.*
2. *The inequality (2) has no finally negative solutions.*
3. *All solutions of the equation (1) are oscillatory.*

The proof of Corollary 1 is analogous to that of Theorem 1.

Theorem 2. *Let the following conditions hold:*

1. *Conditions H1 — H4, H7 are met.*
2. $b_k > -1, k \in \mathbb{N}$.
3. $\limsup_{k \rightarrow \infty} \frac{1}{1 + b_k} \int_{\tau_k}^{\tau_k+h} p_i(s) \exp\left(\int_{s-h}^s a(u)du\right) ds > \frac{1}{L_i}, i \in \mathbb{N}_m$.

Then:

1. *The inequality (3) has no finally positive solutions.*
2. *The inequality (2) has no finally negative solutions.*
3. *All solutions of the equation (1) are oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of the equation (1). Without loss of generality we may assume that $x(t) > 0$ for $t \geq t_0 \geq 0$. It is clear that $x(t - h_i) > 0, i \in \mathbb{N}_n$ and

$$f_i(x(t - h_1), \dots, x(t - h_m)) > 0 \quad \text{for } i \in \mathbb{N}_n, \quad t \geq t_0 + \bar{h} = T_2.$$

Then from (1) it follows that x is a nonincreasing function on the set $[T_2, \tau_s) \cup \left[\bigcup_{i=s}^{\infty} (\tau_i, \tau_{i+1}) \right]$, where $\tau_{s-1} < T_2 < \tau_s$.

We set $z(t) = x(t) \exp\left(\int_T^t a(s)ds\right), t > T \geq T_2$, inserting it into (3) and analogously to the proof of Theorem 1 obtain (9). Integrating (9) from τ_k to $\tau_k + h$ ($k \geq s$), we arrive at the inequality

$$z(\tau_k + h) - z(\tau_k + 0) + L_i \int_{\tau_k}^{\tau_k+h} p_i(s) \exp\left(\int_{s-h}^s a(u)du\right) \prod_{j=1}^m z^{\alpha_{ij}}(s - h_j) ds \leq 0. \quad (24)$$

On the other hand, for each $s \in [\tau_k, \tau_k + h]$ there is no point of jump in the interval $[s - h_i, s - h]$. Therefore,

$$\prod_{j=1}^m [z(s - h_j)]^{\alpha_{ij}} \geq [z(s - h)]^{\sum_{j=1}^m \alpha_{ij}} = z(s - h) \quad (25)$$

since the function z is nonincreasing in the interval $[s - h_i, s - h], s \in [\tau_k, \tau_k + h]$.

From (24) and (25) there follow the inequality

$$z(\tau_k + h) - z(\tau_k + 0) + L_i \int_{\tau_k}^{\tau_k+h} p_i(s) \exp\left(\int_{s-h}^s a(u)du\right) z(s - h) ds \leq 0,$$

whence, having in mind $z(\tau_k + h) > 0$, we arrive at

$$L_i z(\tau_k) \int_{\tau_k}^{\tau_k+h} p_i(s) \exp\left(\int_{s-h}^s a(u) du\right) ds \leq (1 + b_k) z(\tau_k),$$

$$\frac{1}{1 + b_k} \int_{\tau_k}^{\tau_k+h} p_i(s) \exp\left(\int_{s-h}^s a(u) du\right) ds \leq \frac{1}{L_i}.$$

The last inequality contradicts condition 3 of Theorem 2. □

Corollary 2. *Let the following conditions hold:*

1. Conditions H1 — H4, H7 are fulfilled.
2. $b_k > -1$, $k \in \mathbb{N}$.
3. $\limsup_{k \rightarrow \infty} \frac{1}{1 + b_k} \int_{\tau_k}^{\tau_k+h} \left(\sum_{i=1}^m L_i p_i(s) \right) \exp\left(\int_{s-h}^s a(u) du\right) ds > 1$.

Then:

1. The inequality (3) has no finally positive solutions.
2. The inequality (2) has no finally negative solutions.
3. All solutions of the equation (1) are oscillatory.

The proof of Corollary 2 is analogous to the proof of Theorem 2.

Corollary 3. *Let the following conditions hold:*

1. Conditions H1 — H4, H6, H7 are fulfilled.
2. $\limsup_{k \rightarrow \infty} \int_{\tau_k}^{\tau_k+h} p_i(s) \exp\left(\int_{s-h}^s a(u) du\right) ds > \frac{1 + M}{L_i}$, $i \in \mathbb{N}_m$.

Then:

1. The inequality (3) has no finally positive solutions.
2. The inequality (2) has no finally negative solutions.
3. All solutions of the equation (1) are oscillatory.

Corollary 4. *Let the following conditions hold:*

1. Conditions H1 — H4, H7 are fulfilled.
2. $b_k > -1$, $k \in \mathbb{N}$.
3. $\liminf_{k \rightarrow \infty} \int_{\tau_k-h}^{\tau_k} a(u) du \geq s > 0$, $s = \text{const}$.
4. $\limsup_{k \rightarrow \infty} \frac{1}{1 + b_k} \int_{\tau_k}^{\tau_k+h} p_i(s) ds > \frac{1}{L_i e^s}$, $i \in \mathbb{N}_m$.

Then the assertions of Corollary 3 are valid.

Corollary 5. *Let the following conditions hold:*

1. *Conditions 1, 2 and 3 of Corollary 4 are fulfilled.*

$$2. \limsup_{k \rightarrow \infty} \frac{1}{1 + b_k} \int_{\tau_k}^{\tau_k+h} \left(\sum_{i=1}^m L_i p_i(s) \right) ds > e^{-s}.$$

Then the assertions of Corollary 3 hold true.

Theorem 3. *Let the following conditions hold:*

1. *Conditions H1 — H6 are fulfilled.*

$$2. \limsup_{k \rightarrow \infty} \int_{\tau_k}^{\tau_k+h} p_i(s) \exp\left(\int_{s-h}^s a(u)du\right) ds > \frac{(1 + M)^{2l}}{L_i}, \quad i \in \mathbb{N}_m.$$

Then:

1. *The inequality (3) has no finally positive solutions.*

2. *The inequality (2) has no finally negative solutions.*

3. *All solutions of the equation (1) are oscillatory.*

Proof. Let $x(t)$ be a positive solution of the inequality (3) for $t \geq t_0 \geq 0$. Then $x(t - h_i) > 0$ and $f_i(x(t - h_1), \dots, x(t - h_m)) > 0$ for $t \geq t_0 + \bar{h}$, $i \in \mathbb{N}_m$.

Setting $z(t) = x(t) \exp\left(\int_T^t a(s)ds\right)$, substitute it into (3) and obtain (9).

Let us fix k ($k \in \mathbb{N}$) in such a way that in the interval $[\tau_k, \tau_k + h]$ there are m_k points of jump, i.e.,

$$\tau_k < \tau_{k+1} < \tau_{k+2} < \dots < \tau_{k+m_k} < \tau_k + h < \tau_{k+m_k+1},$$

where $m_k \leq l$. Integrate (9) from τ_k to $\tau_k + h$ and deduce

$$z(\tau_k + h) - z(\tau_k + 0) - \sum_{s=k+1}^{k+m_k-1} b_s z(\tau_s) + \tag{26}$$

$$+ L_i \int_{\tau_k}^{\tau_k+h} p_i(s) \exp\left(\int_{s-h}^s a(u)du\right) \prod_{j=1}^m [z(s - h_j)]^{\alpha_{ij}} ds.$$

Let for each $s \in [\tau_k, \tau_k + h]$ the interval $[s - h_j, s - h]$ contain $l_j^{(k)}$ points of jump, $l_j^{(k)} \in \mathbb{N}$, $l_j^{(k)} \leq l$.

Then

$$z(s - h_j) \geq \frac{z(s - h)}{(1 + M)^{l_j^{(k)}}} \geq \frac{z(s - h)}{(1 + M)^l}$$

and

$$\prod_{j=1}^m [z(s-h)]^{\alpha_{ij}} \geq \prod_{j=1}^m \frac{[z(s-h)]^{\alpha_{ij}}}{(1+M)^{\alpha_{ij}l}} = \frac{z(s-h)}{(1+M)^l}. \tag{27}$$

From (26) and (27) we obtain that

$$\begin{aligned} & \frac{L_i}{(1+M)^l} \int_{\tau_k}^{\tau_k+h} p_i(s) \exp\left(\int_{s-h}^s a(u)du\right) z(s-h) ds \\ & \leq (1+b_k)z(\tau_k) + \sum_{s=k+1}^{k+m_k-1} b_s z(\tau_s), \\ & \frac{L_i z(\tau_k)}{(1+M)^l} \int_{\tau_k}^{\tau_k+h} p_i(s) \exp\left(\int_{s-h}^s a(u)du\right) ds \\ & \leq (1+M)z(\tau_k) + M \sum_{s=k+1}^{k+m_k-1} z(\tau_s). \end{aligned} \tag{28}$$

On the other hand,

$$\begin{aligned} z(\tau_{k+1}) & \leq z(\tau_k + 0) = (1+b_k)z(\tau_k) \leq (1+M)z(\tau_k), \\ z(\tau_{k+2}) & \leq z(\tau_{k+1} + 0) \leq \dots \leq (1+M)^2 z(\tau_k), \\ & \dots\dots\dots \\ z(\tau_{k+m_k-1}) & \leq \dots \leq (1+M)^{m_k-1} z(\tau_k) \leq (1+M)^{l-1} z(\tau_k). \end{aligned}$$

Then

$$\sum_{s=k+1}^{k+m_k-1} z(\tau_s) < z(\tau_k) \sum_{i=1}^{l-1} (1+M)^i = z(\tau_k) \frac{(1+M)[(1+M)^{l-1}-1]}{M}. \tag{29}$$

It follows from (28) and (29) that

$$\frac{L_i}{(1+M)^l} z(\tau_k) \int_{\tau_k}^{\tau_k+h} p_i(s) \exp\left(\int_{s-h}^s a(u)du\right) ds \leq z(\tau_k)(1+M)^l.$$

The last inequality contradicts condition 2 of Theorem 3. □

Corollary 6. *Let the following conditions hold:*

1. *Conditions H1 — H6 are fulfilled.*

$$2. \limsup_{k \rightarrow \infty} \int_{\tau_k}^{\tau_k+h} \left(\sum_{i=1}^n L_i p_i(s) \right) \exp \left(\int_{s-h}^s a(u) du \right) ds > (1+M)^{2l}.$$

Then:

1. The inequality (3) has no finally positive solutions.
2. The inequality (2) has no finally negative solutions.
3. All solutions of the equation (1) are oscillatory.

The proof of Corollary 6 is analogous to the proof of Theorem 3.

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