Journal of Applied Analysis
Vol. 5, No. 1 (1999), pp. 59–70

ON MINIMAX INEQUALITY AND GENERALIZED QUASI–VARIATIONAL INEQUALITY IN H-SPACES

X. WU and Z. ZHANG

Received July 31, 1997 and, in revised form, June 22, 1998

Abstract. In this paper, we give a new minimax theorem and two new existence theorems of solutions for generalized quasi–variational inequalities in H-spaces. Our results improve and develop some famous results.

1. Preliminaries

In this paper, our subject is to establish a Sion type minimax theorem and two existence theorems of solutions for generalized quasi–variational inequalities. Our results improve and develop some famous results. In order to establish our main results, we first give some concepts and notations.

To begin with we explain the notion of a H-space and some related notions.

Let X be a topological space and $\mathcal{F}(X)$ the family of all nonempty finite subsets of X. Let Γ_A be a family of nonempty contractible subsets of X indexed by $A \in \mathcal{F}(X)$ such that $\Gamma_A \subset \Gamma_{A'}$ whenever $A \subset A'$. The pair $(X, \{\Gamma_A\})$ is called a H-space. Given a H-space $(X, \{\Gamma_A\})$, a nonempty

1991 *Mathematics Subject Classification.* 47H04, 47H10, 49J35, 52A99, 58E35.

Key words and phrases. Closed transfer complete, H-space, local intersection property, H-convex set, W-lower (upper) semicontinuous function.

This Work Is Supported by the Natural Science Foundations of China and Yunnan.

ISSN 1425-6908

subset D of X is called H-convex (resp. weak H-convex) if $\Gamma_A \subset D$ (resp. $D \bigcap \Gamma_A$ is nonempty contractible) for each nonempty finite subset A of D. For a nonempty subset K of X, we define the H-convex hull of K, denoted by $H - \mathrm{co}\, K$ as

$$H - \mathrm{co}\, K = \bigcap \{D \subset X : D \text{ is } H\text{-convex and } D \supset K\}$$

(see also, [11–13]).

In this paper, all topological spaces are assumed to be Hausdorff. Let X be a nonempty set, we denote by 2^X the family of all subsets of X. If $A \subset X$, we shall denote by $\mathrm{cl}\, A$ the closure of A and by $\mathrm{int}\, A$ the interior of A. If A is a nonempty subset of a topological vector space E, we shall denote by $\mathrm{co}\, A$ the convex hull of A and by $\mathrm{cl}(\mathrm{co}\, A)$ the closed convex hull of A.

Let X, Y be two topological spaces, $T : X \to 2^Y$ a multivalued mapping and $f : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ a function.

(1) T is said to be upper semicontinuous if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X such that $T(z) \subset V$ for each $z \in U$;

(2) ([32]) T is said to have local intersection property if $x \in X$ such that $T(x) \neq \emptyset$, then there exists an open neighborhood $N(x)$ of x such that $\bigcap_{z \in N(x)} T(z) \neq \emptyset$;

(3) T is said to have open lower sections if for each $y \in Y$, the set $T^{-1}(y) = \{x \in X : y \in T(x)\}$ is open in X;

(4) ([31]) $f(x, y)$ is said to be W-lower (or, W-upper) semicontinuous in y if for each $y \in Y$ and each $r \in \mathbb{R}$ with $\{x \in X : f(x, y) > r\} \neq \emptyset$ (or, $\{x \in X : f(x, y) < r\}$), there exists $x' \in X$ such that $y \in \mathrm{int}\{z \in Y : f(x', z) > r\}$ (or, $y \in \mathrm{int}\{z \in Y : f(x', z) < r\}$).

Obviously, if $f(x, y)$ is lower (upper) semicontinuous in y, then $f(x, y)$ is W-lower (upper) semicontinuous in y. But, the converse is not true.

Example 1. Let $X = Y = (0, +\infty)$. The function $f : X \times Y \to \mathbb{R}$ is defined by

$$f(x, y) = \begin{cases} 1 & \text{if } x + y \geq 1 \\ 0 & \text{if } x + y < 1. \end{cases}$$

Example 2. Let $X = Y = (0, +\infty)$. The function $g : X \times Y \to \mathbb{R}$ is defined by

$$g(x, y) = \begin{cases} x + y + 1 & \text{if } x + y \geq 1 \\ x + y & \text{if } x + y < 1. \end{cases}$$

In the above two examples, both $f(x, y)$ and $g(x, y)$ are W-lower semicontinuous in y, but neither $f(x, y)$ nor $g(x, y)$ are lower semicontinuous in y. Moreover, if T has open lower sections, then T has local intersection property.

A family $\{D_\alpha : \alpha \in I\}$ of some subsets of a topological space X is called closed (resp. open) transfer complete if $x \in X$ such that $x \notin D_{\alpha_0}$ (resp. $x \in D_{\alpha_0}$) for some $\alpha_0 \in I$, then there exists $\alpha' \in I$ such that $x \notin \mathrm{cl}(D_{\alpha'})$ (resp. $x \in \mathrm{int}(D_{\alpha'})$). Obviously, if $\{D_\alpha : \alpha \in I\}$ is a family of some closed (resp. open) subsets of X, then it is closed (resp. open) transfer complete. But, the converse is not true.

Example 3. Let $X = (0, 1/4) \cup (1/3, 1)$, $Y = [0, 2]$ and $A = \{(x, y) \in X \times Y : x < y \leq 2\}$. For any $x \in X$, let $A[x] = \{y \in Y : (x, y) \in A\} = (x, 2]$. Obviously, $A[x]$ is not closed in Y, but the family $\{A[x] : x \in X\}$ is closed transfer complete (see also, Example 2 in [31]).

A multivalued mapping $T : Y \to 2^X$ is said to be transfer closed valued if the family $\{T(y) : y \in Y\}$ is closed transfer complete.

Let $(X, \{\Gamma_A\})$ be a H-space and $f : X \to \mathbb{R}$ a function. f is called H-quasi–convex (or, H-quasi–concave) if for each $r \in \mathbb{R}$, the set

$$\{x \in X : f(x) \leq r\} \text{ (or, } \{x \in X : f(x) \geq r\})$$

is H-convex.

Let X, Y be two sets and A a nonempty subset of $X \times Y$. For each $x \in X$ and each $y \in Y$, we denote

$$A[x] = \{y \in Y : (x, y) \in A\}, \quad A[y] = \{x \in X : (x, y) \in A\},$$

which are called the sections of A.

2. Main results

Lemma 1. Let $(X, \{\Gamma_A\})$ and $(Y, \{\Gamma'_B\})$ are two H-spaces and K a nonempty compact subset of X. Suppose that $G, F : X \to 2^Y$, $S, T : Y \to 2^X$ are multivalued mappings such that

(i) for each $x \in K, G(x) \neq \emptyset$ and $H - \mathrm{co}\, G(x) \subset F(x)$,

(ii) G has local intersection property on K,

(iii) for each $y \in Y$, $H - \mathrm{co}\, S(y) \subset T(y) \subset K$ and S has nonempty values on each nonempty compact subset of Y,

(iv) S has local intersection property.

Then there exists a point $x_0 \in X$ and a point $y_0 \in Y$ such that $x_0 \in T(y_0)$ and $y_0 \in F(x_0)$.

Proof. By Theorem 2.2 of [28], there exists a continuous selection $f : X \to Y$ of $F|_K$ such that $f = g \circ \psi$, where $g : \Delta_n \to Y$ and $\psi : K \to \Delta_n$ are continuous mappings, n is some positive integer and Δ_n is the standard n-dimensional simplex. Since Δ_n is compact and $g : \Delta_n \to Y$ is continuous, $g(\Delta_n)$ is a compact subset of Y. Again, by Theorem 2.2 of [28], there exists a continuous selection $h : g(\Delta_n) \to K$ of $T|_{g(\Delta_n)}$. Consequently, $\psi \circ h \circ g : \Delta_n \to \Delta_n$ is continuous and hence there exists a point $u_0 \in \Delta_n$ such that $\psi \circ h \circ g(u_0) = u_0$. Let $g(u_0) = y_0$ and $x_0 = h(y_0)$. Then $x_0 \in T(y_0)$ and $y_0 = g(u_0) = g \circ \psi[h(y_0)] = f[h(y_0)] = f(x_0) \in F(x_0)$. This completes the proof. $\square$

Now, we shall prove the following minimax theorem and existence theorems of solutions for generalized quasi–variational inequalities.

Theorem 2. Let $(X, \{\Gamma_A\})$ and $(Y, \{\Gamma'_B\})$ be two H-spaces and X be compact. Suppose the functions $f, g : X \times Y \to \mathbb{R}$ such that

(i) $f(x, y) \le g(x, y)$ for all $(x, y) \in X \times Y$,
(ii) for each $x \in X, f(x, \cdot)$ is H-quasiconcave,
(iii) $f(x, y)$ is W-lower semicontinuous in x,
(iv) $g(\cdot, y)$ is H-quasiconvex,
(v) $g(x, y)$ is W-upper semicontinuous in y,

then
$$\inf_{x\in X} \sup_{y\in Y} f(x, y) \le \sup_{y\in Y} \inf_{x\in X} g(x, y).$$

Proof. If
$$\inf_{x\in X} \sup_{y\in Y} f(x, y) > \sup_{y\in Y} \inf_{x\in X} g(x, y)$$
then there exist real numbers α, β such that
$$\inf_{x\in X} \sup_{y\in Y} f(x, y) > \alpha > \beta > \sup_{y\in Y} \inf_{x\in X} g(x, y).$$
For each $y = Y$, let
$$S(y) = \{x \in X : g(x, y) < \beta\}, \quad T(y) = \{x \in X : g(x, y) \le \beta\}.$$
Then $S(y) \ne \emptyset$ and $H - \operatorname{co} S(y) \subset T(y)$ since $\beta > \sup_{y\in Y} \inf_{x\in X} g(x, y)$ and $g(\cdot, y)$ is H-quasiconvex. Consequently, for each $y \in Y$, the set $\{x \in X : g(x, y) < \beta\} \ne \emptyset$, and hence there is a point $x' \in X$ such that $y \in \operatorname{int}\{z \in Y : g(x', z) < \beta\}$ by (v), i.e. there exists an open neighborhood $N(y)$ of y such that $x' \in \bigcap_{z\in N(y)} S(z)$. It shows that $S : Y \to 2^X$ has local intersection property.

For each $x \in X$, let $G(x) = \{y \in Y : f(x, y) > \alpha\}, F(x) = \{y \in Y : f(x, y) \ge \alpha\}$. Then $H - \operatorname{co} G(x) \subset F(x)$ by (ii). Since $\inf_{x\in X} \sup_{y\in Y} f(x, y) > \alpha$, $G(x) \ne \emptyset$ for all $x \in X$. Consequently, for each $x \in X$, the set

$\{y \in Y : f(x,y) > \alpha\} \neq \emptyset$, and hence there exists a point $y' \in Y$ such that $x \in \operatorname{int}\{z \in X : f(z,y') > \alpha\}$ by (iii) so that there is an open neighborhood $N(x)$ of x such that $N(x) \subset \{z \in X : f(z,y') > \alpha\}$, i.e. $y' \in \bigcap_{z \in N(x)} G(z)$. It shows that the mapping $G : X \to 2^Y$ has local intersection property.

By virtue of Lemma 1, there exist a point $x_0 \in X$ and a point $y_0 \in Y$ such that $x_0 \in T(y_0)$ and $y_0 \in F(x_0)$. It implies $g(x_0,y_0) \leq \beta$ and $f(x_0,y_0) \geq \alpha$. Consequently, $\alpha \leq \beta$ by (i). It contradicts the choices of α and β. Therefore,

$$\inf_{x \in X} \sup_{y \in Y} f(x,y) \leq \sup_{y \in Y} \inf_{x \in X} g(x,y).$$

This completes the proof. $\square$

Remark 1. Since W-lower semicontinuity (respectively, W-upper semicontinuity) is weaker than lower semicontinuity (respectively, upper semicontinuity), our Theorem 2 not only generalizes Theorem 3.4 and Corollary 3.5 in Sion [26] to H-spaces, but also weakens upper semicontinuity and lower semicontinuity conditions. Moreover, Theorem 2 is different from the minimax theorems in [3], [4], [9], [16–18] and [23].

In order to research the existence of solutions for generalized quasi–variational inequalities, we give the following lemmas:

Lemma 3. Let $(X, \{\Gamma_A\})$ and $(Y, \{\Gamma'_B\})$ be two H-spaces. Let M, N be two subsets of $X \times Y$ such that

(i) for each $x \in X, H - \operatorname{co}\{y \in Y : (x,y) \notin M\} \subset \{y \in Y : (x,y) \notin N\}$,
(ii) the family $\{M[y] : y \in Y\}$ of sections of M is closed transfer complete.

Suppose that there exist a subset Q of N and a compact subset K of X such that

(iii) for each $y \in Y$, the section $Q[y] \bigcap K$ is nonempty H-convex,
(iv) the family $\{Q[x] : x \in K\}$ is open transfer complete.

Then there exists a point $x_0 \in K$ such that $\{x_0\} \times Y \subset M$.

Proof. For each $x \in X$, let $G(x) = Y \backslash M[x], F(x) = Y \backslash N[x]$. Then $G : K \to 2^Y$ has local intersection property and $H - \operatorname{co} G(x) \subset F(x)$ for each $x \in X$ by (i) and (ii). For each $y \in Y$, let $T(y) = Q[y] \bigcap K$. Then $T : Y \to 2^K$ is a multivalued mapping with nonempty H-convex values. Consequently, for each $y \in Y$, there exists a point $x \in Q[y] \bigcap K$, and hence $y \in Q[x]$. By (iv) there exists a point $x' \in K$ such that $y \in \operatorname{int} Q[x']$, and thus there is an open neighborhood $N(y)$ of y such that $N(y) \subset Q[x']$, i.e. $x' \in \bigcap_{z \in N(y)} T(z)$. It shows that $T : Y \to 2^K$ has local intersection property. Suppose $G(x) \neq \emptyset$ for all $x \in K$. By virtue of Lemma 1, there exist a point $x_0 \in K$ and a point $y_0 \in Y$ such that $x_0 \in T(y_0)$ and $y_0 \in F(x_0)$, i.e.

$(x_0, y_0) \in Q \subset N$ and $(x_0, y_0) \notin N$. This is a contradiction. Therefore, there exists a point $x_0 \in K$ such that $G(x_0) = \emptyset$, and hence $M[x_0] = Y$, i.e. $\{x_0\} \times Y \subset M$. This completes the proof. $\square$

Lemma 4. Let $(X, \{\Gamma_A\})$ and $(Y, \{\Gamma'_B\})$ be two H-spaces and Y is compact. If $F, G : X \to 2^Y$ are two multivalued mappings such that

(i) $G(x) \subset F(x)$ for each $x \in X$,
(ii) for each $y \in Y$, $X \backslash F^{-1}(y)$ is H-convex,
(iii) for each $x \in X$, $F(x)$ is transfer closed valued,
(iv) G has local intersection property and $G(x)$ is nonempty H-convex for all $x \in X$,

then $\bigcap_{x \in X} F(x) \neq \emptyset$.

Proof. Let $M = \{(y, x) \in Y \times X : y \in F(x)\}$, $N = \{(y, x) \in Y \times X : y \in G(x)\}$. Then $N \subset M$ by (i) and for each $y \in Y$, the set

$$\{x \in X : (y, x) \notin M\} = \{x \in X : y \notin F(x)\} = X \backslash F^{-1}(y)$$

is H-convex by (ii). Since the section $M[x] = F(x)$ for each $x \in X$, the family $\{M[x] : x \in X\}$ is closed transfer complete by (iii). By (iv) we know that $N[x] = G(x)$ is nonempty H-convex for each $x \in X$ and for each $y \in Y$ and each $x \in N[y] = G^{-1}(y)$, there exists an open neighborhood $O(x)$ of x and a point $y' \in Y$ such that $y' \in \bigcap_{z \in O(x)} G(z)$ and thus $x \in \operatorname{int} G^{-1}(y') = \operatorname{int} N[y']$ so that the family $\{N[y] : y \in Y\}$ is open transfer complete. By virtue of Lemma 3, there exists a point $y_0 \in Y$ such that $\{y_0\} \times X \subset M$, i.e. $y_0 \in \bigcap_{x \in X} F(x)$. This completes the proof. $\square$

Remark 2. Lemma 4 is different from other H-KKM-type theorems (see also, [10–13]). The following we apply it to study quasi–variational inequalities.

Theorem 5. Let $(X, \{\Gamma_A\})$ and $(Y, \{\Gamma'_B\})$ be two H-spaces and X is compact, $T : X \to 2^Y$ is an upper semicontinuous multivalued mapping with nonempty compact weak H-convex values. If the function $\varphi : X \times Y \times X \to \mathbb{R}$ such that

(i) for each $(x, y) \in X \times Y$,$\varphi(x, y, \cdot)$ is lower semicontinuous and H-quasiconvex,
(ii) for each $(x, z) \in X \times X$, $\varphi(x, \cdot, z)$ is H-quasiconcave,
(iii) for each $x \in X$, there exists a point $y \in T(x)$ such that $\varphi(x, y, x) \geq 0$,
(iv) for each $z \in X$, $\varphi(x, y, z)$ is upper semicontinuous in (x, y),

then there exist a point $\bar{x} \in X$ and a point $\bar{y} \in T(\bar{x})$ such that

$$\varphi(\bar{x}, \bar{y}, x) \geq 0$$

for all $x \in X$.

Proof. First, we prove that there exists a point $\bar{x} \in X$ such that

$$\sup_{y \in T(\bar{x})} \varphi(\bar{x}, y, x) \geq 0, \quad \forall x \in X.$$

If this conclusion is false, then for each $u \in X$, there a point $z \in X$ such that

$$\sup_{w \in T(u)} \varphi(u, w, z) < 0.$$

Let $S(u) = \{v \in X : \sup_{w \in T(u)} \varphi(u, w, v) < 0\}$. Then $S : X \to 2^X$ is a multivalued mapping with nonempty values. For each $u \in X$ and each finite subset $A = \{v_1, v_2, \cdots, v_n\}$ of $S(u)$, we have

$$\sup_{w \in T(u)} \varphi(u, w, v_i) < 0, \quad i = 1, 2, \cdots, n.$$

Hence there is a real number such that

$$\sup_{w \in T(u)} \varphi(u, w, v_i) < r < 0, \quad i = 1, 2, \cdots, n.$$

Consequently, for each $v \in \Gamma_A$ and each $w \in T(u)$, by (i),

$$(u, w, v) \leq \max_{1 \leq i \leq n} \varphi(u, w, v_i) < r,$$

and thus,

$$\sup_{w \in T(u)} \varphi(u, w, v) \leq r < 0,$$

i.e. $v \in S(u)$. It shows that $S(u)$ is H-convex. Since again $T : X \to 2^Y$ is an upper semicontinuous multivalued mapping with nonempty compact values and $\varphi(u, w, v)$ is upper semicontinuous in (u, w), by virtue of Proposition 21 in [1, P119] we know that $\sup_{w \in T(u)} \varphi(u, w, v)$ is upper semicontinuous in u. Consequently, for each $v \in X$,

$$\begin{aligned} S^{-1}(v) &= \{u \in X : v \in S(u)\} \\ &= \{u \in X : \sup_{w \in T(u)} \varphi(u, w, v) < 0\} \end{aligned}$$

is open. By Corollary 2.3 in [28], there exists a point $\bar{u} \in X$ such that $\bar{u} \in S(\bar{u})$, i.e. $\sup_{w \in T(\bar{u})} \varphi(\bar{u}, w, \bar{u}) < 0$. It contradicts (iii). Therefore, there exists a point $\bar{x} \in X$ such that

$$\sup_{y \in T(\bar{x})} \varphi(\bar{x}, y, x) \geq 0, \quad \forall x \in X.$$

By (iv) and the compactness of $T(\bar{x})$, for each $x \in X$, there exists a point $y(x) \in T(\bar{x})$ such that

$$\varphi(\bar{x}, y(x), x) \geq 0.$$

For each fixed $\varepsilon<0$, set $G(x) = \{y \in T(\bar{x}) : \varphi(\bar{x}, y, x) > \varepsilon\}, F(x) = \{y \in T(\bar{x}) : \varphi(\bar{x}, y, x) \geq \varepsilon\}, \forall x \in X$. Then $G, F : X \to 2^{T(\bar{x})}$ are two multivalued mappings with nonempty values and $G(x) \subset F(x)$ for each $x \in X$. Since $T(\bar{x})$ is a compact weak H-convex subset of Y, $(T(\bar{x}), \{T(\bar{x}) \cap \Gamma'_B\})$ is a compact H-space and the multivalued mappig $G : X \to 2^{T(\bar{x})}$ has H-convex values by (ii). For each $y \in T(\bar{x})$,

$$G^{-1}(y) = \{x \in X : y \in G(x)\} = \{x \in X : \varphi(\bar{x}, y, x) > \varepsilon\}$$

is open in X (and hence G has local intersection property) and

$$X \backslash F^{-1}(y) = \{x \in X : \varphi(\bar{x}, y, x) < \varepsilon\}$$

is H-convex by (i). Obviously, $F(x)$ is closed in $T(\bar{x})$ by (iv). By virtue of Lemma 4, there exists a point $y_\varepsilon \in \bigcap_{x \in X} F(x)$, i.e. $y_\varepsilon \in T(\bar{x})$ and $\inf_{x \in X} \varphi(\bar{x}, y_\varepsilon, x) \geq \varepsilon$. Since $T(\bar{x})$ is compact, we may assume $y_\varepsilon \to \bar{y} \in T(\bar{x}) (\varepsilon \to 0)$. Consequently, $\inf_{x \in X} \varphi(\bar{x}, \bar{y}, x) \geq 0$ by (iv), i.e. $\varphi(\bar{x}, \bar{y}, x) \geq 0$ for all $x \in X$. $\square$

Remark 3. In Theorem 5, if X is a nonempty compact convex subset of a Hausdorff locally convex topological vector space $E, Y = E^*$ (the conjugate space of E) and $\varphi(x, y, z) = \langle y, x - z \rangle$ (by Lemma B of [20], φ is continuous), then Theorem 5 reduces to Theorem 6 in Browder [2].

Theorem 6. Let $(X, \{\Gamma_A\})$ and $(Y, \{\Gamma'_B\})$ be two H-spaces and X is compact, $T : X \to 2^Y$ is an upper semicontinuous multivalued mapping with nonempty compact weak H-convex values. If functions $\varphi, \psi : X \times Y \to \mathbb{R}$ such that

(i) $\varphi(x, y)$ is lower semicontinuous and H-quasiconvex in x,
(ii) $\varphi(x, \cdot)$ is upper semicontinuous and H-quasiconcave,
(iii) for each $x \in X$, there exists a point $y \in T(x)$ such that $\psi(x, y) \geq c$ (c is a constant),
(iv) $\psi(x, y) \leq \varphi(x, y)$ for all $(x, y) \in X \times Y$,

then there exist a point $\bar{x} \in X$ and a point $\bar{y} \in T(\bar{x})$ such that

$$\varphi(x, \bar{y}) \geq c$$

for all $x \in X$.

Proof. For each $x \in X$, let

$$S(x) = \{z \in X : \max_{y \in T(x)} \varphi(z,y) < c\},$$

$$H(x) = \{z \in X : \sup_{y \in T(x)} \psi(z,y) < c\},$$

Then $S, H : X \to 2^X$ are two multivalued mappings with $S(x) \subset H(x)$ for all $x \in X$ by (iv). Since $\varphi(z,y)$ is H-quasiconvex in z, the set

$$S(x) = \bigcap_{y \in T(x)} \{z \in X : \varphi(z,y) < c\}$$

is H-convex. Let $f(z,x) = \max_{y \in T(x)} \varphi(z,y)$. For each fixed $z \in X$ and each $r \in \mathbb{R}$, let

$$D = \{x \in X : f(z,x) \geq r\}.$$

If $\{x_\alpha : \alpha \in I\}$ is a net in D such that $x_\alpha \to u$, then

$$f(z, x_\alpha) \geq r, \quad \forall \alpha \in I,$$

i.e.

$$\max_{y \in T(x_\alpha)} \varphi(z,y) \geq r, \quad \forall \alpha \in I.$$

Consequently, for each $\alpha \in I$, there exists a point $y_\alpha \in T(x_\alpha)$ such that $\varphi(z, y_\alpha) \geq r$. By Proposition 1 in [21], there exists a point $v \in T(u)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}_{\alpha \in I}$ such that $y_\beta \to v$. By (ii) $\varphi(z,v) \geq r$, and hence $\max_{y \in T(u)} \varphi(z,y) \geq r$, i.e. $f(z,u) \geq r$, i.e. $u \in D$. Hence D is closed. Consequently, $f(z,x)$ is upper semicontinuous in x. Hence for each $z \in X$, $S^{-1}(z) = \{x \in X : f(z,x) < c\}$ is open.

If $S(x) \neq \emptyset$ for all $x \in X$, there exists a point $\bar{u} \in X$ such that $\bar{u} \in H(\bar{u})$ by Corollary 2.3 in [28], i.e. $\sup_{y = T(\bar{u})} \psi(\bar{u}, y) < c$. It contradicts (iii). Therefore, there exists a point $\bar{x} \in X$ such that $S(\bar{x}) = \emptyset$, i.e.

$$\max_{y = T(\bar{x})} \varphi(x,y) \geq c, \quad \forall x \in X.$$

For each $\varepsilon < c$, let $G(x) = \{y \in T(\bar{x}) : \varphi(x,y) > \varepsilon\}, F(x) = \{y \in T(\bar{x}) : \varphi(x,y) \geq \varepsilon\}, \forall x \in X$. Then $G, F : X \to 2^{T(\bar{x})}$ are two multivalued mappings with nonempty values and $G(x) \subset F(x)$ for each $x \in X$. Since $T(\bar{x})$ is a compact weak H-convex subset of Y, $(T(\bar{x}), \{T(\bar{x}) \cap \Gamma'_B\})$ is a compact H-space and the multivalued mappig $G : X \to 2^{T(\bar{x})}$ has H-convex values by (ii). For each $y \in T(\bar{x})$,

$$G^{-1}(y) = \{x \in X : y \in G(x)\} = \{x \in X : \varphi(x,y) > \varepsilon\}$$

is open in X (and hence G has local intersection property) and

$$X \backslash F^{-1}(y) = \{x \in X : \varphi(x,y) < \varepsilon\}$$

is H-convex by (i). Obviously, $F(x)$ is closed in $T(\bar{x})$ by (ii). By virtue of Lemma 4, there exists a point $y_\varepsilon \in \bigcap_{x \in X} F(x)$, i.e. $y_\varepsilon \in T(\bar{x})$ and

$\inf_{x\in X}\varphi(x,y_\varepsilon)\geq\varepsilon$. Since $T(\bar{x})$ is compact, we may assume $y_\varepsilon\to\bar{y}\in T(\bar{x})$ $(\varepsilon\to c)$. Consequently, $\inf_{x\in X}\varphi(x,\bar{y})\geq c$ by (ii), i.e. $\varphi(x,\bar{y})\geq c$ for all $x\in X$. $\square$

Remark 4. Theorem 6 improves and develops Theorem 5.7.1 of [5].

References

[1] Aubin, J.P. and Ekeland, I., *Applied Nonlinear Analysis*, John Wiley and Sons, New York, 1984.

[2] Browder, F.E., *The fixed point theory of multivalued mapping in topological vector spaces*, Math. Ann. **177** (1968), 283–301.

[3] Bardaro, C. and Ceppitelli, R., *Some further generalizations of the KKM theorem and minimax inequalities*, J. Math. Anal. Appl. **132** (1988), 484–490.

[4] Chang, S.S., Cho, Y., Wu, X. and Zhang, Y., *The topological versions of KKM theorems and Fan's matching theorem with applications*, Topol. Methods Nonlinear Anal. **1** (1993), 231–245.

[5] Chang, S.-S., *Variational Inequality and Complementarity Problem Theory and Applications*, Shanghai Scientific and Technological Literature Publishing, Shanghai, China, 1991.

[6] Chang, S.-S. and Yang, L., *Section theorems on H-spaces with applications*, J. Math. Anal. Appl. **179** (1993), 214–231.

[7] Ding, X.P. and Tarafdar, E., *Some coincidence theorems and applications*, Bull. Austral. Math. Soc. **50** (1994), 73–80.

[8] Fan, K., *A generalization of Tychonoff's fixed point theorem*, Math. Ann. **142** (1961), 305–310.

[9] Horvath, C., *Quelques theorems en theorie des mini–max*, C. R. Acad. Sci. Paris Ser. I Math. **310** (1990), 269–272.

[10] Horvath, C., *Convexite generalisee et applications, methodes topologiques en analyse convexe: partie des comptes rendus du cours d'ete OTAN*, in "Variational Method in Nonlinear Problems", ed. A. Granas, Press. Univ. Montreal, Quebec, Canada, 1990, 79–99.

[11] Horvath, C., *Pinnt fixes et coincidences pour les applications multivoques sans convexite*, C. R. Acad. Sci. Paris Ser. I Math. **296** (1983), 403–406.

[12] Horvath, C., *Piint fixes et coincidences dans les espaces topologiques compacts contractiles*, C. R. Acad. Sci. Paris Ser. I Math. **299** (1984), 519–521.

[13] Horvath, C., *Some results on multivalued mappings and inequalities without convexity*, in "Nonlinear and Convex Analysis", eds. B.L. Lin and S. Simons, Marcel Dekker, New York, 1987, 99–106.

[14] Himmelberg, C.J., *Fixed points of compact multifunctions*, J. Math. Anal. Appl. **38** (1972), 205–207.

[15] Ha, C.W., *Minimax and fixed point theorems*, Math. Ann., **248** (1980), 73–77.

[16] Koning, H., *A general minimax theorem based on connectedness*, Arch. Math. **59** (1992), 55–64.

[17] Kindler, J., Trost, R., *Minimax theorems for interval spaces*, Acta Math. Hungar. **54** (1989), 39–49.

[18] Komiya, H., *On minimax theorem without linear structure*, Hiyoshi Review of Natural Science **8** (1990), 74–78.

[19] Komiya, H., *Coincidence theorem and saddle point theorem*, Proc. Amer. Math. Soc. **96**(4) (1986), 599–602.

[20] Kum, S. *A generalization of generalized quasi–variational inequalities*, J. Math. Anal. Appl. **182** (1994), 158–164.

[21] Kaczynski, T. and Zeidan, V., *An application of Ky Fan fixed point theorem to an optimization problem*, Nonlinear Anal. TMA **13**(3) (1989), 259–262.

[22] Kim, W.K. *Remark on a generalized quasi–variational inequality*, Proc. Amer. Math. Soc. **103**(2) (1989), 667–668.

[23] Lin, B.L. and Quan, X.C., *A symmetric minimax theorem without linear structure*, Arch. Math. **52** (1989), 367–370.

[24] Park, S., Sook, J. and Kang, H.K., *Geometric properties, minimax inequalities, and fixed point theorems on convex spaces*, Proc. Amer. Math. Soc. **121**(2) (1994), 429–439.

[25] Su, C.H. and Sehgal, V.M., *Some fixed point theorems for condensing multifunctions in locally convex spaces*, Proc. Amer. Math. Soc. **50** (1975), 150–154.

[26] Sion, M., *On general minimax theorems*, Pacific J. Math. **8** (1958), 171–176.

[27] Shih, M.H and Tan, K.K., *Generalized quasi–variational inequalities in locally convex topological vector spaces*, J. Math. Anal. Appl. **108** (1985), 333–343.

[28] Tarafdar, E., *Fixed point theorems in H-spaces and equilibrium points of abstract economies*, J. Austral. Math. Soc. Ser. B **53** (1992), 252–260.

[29] Tarafdar, E. and Yuan, X.-Z., *A remark on coincidence theorems*, Proc. Amer. Math. Soc. **122**(3) (1994), 957–959.

[30] Tan, K.K. and Yuan, X.Z., *Existence of equilibrium for abstract economies*, J. Math. Econom. **23** (1994), 243–251.

[31] Tian, G.Q., *Generalizations of the FKKM theorem and the Ky Fan minimax inequality, with applications to maximal elements, prine equilibrium, and complementarity*, J. Math. Anal. Appl. **170** (1992), 457–471.

[32] Wu, X. and Shen, S.K, *A further generalization of Yannelis–Prabhakar's continuous selection theorem and its applications*, J. Math. Anal. Appl. **197** (1996), 61–74.

Xian Wu
Department of Mathematics
Yunnan Normal University
Kunming, Yunnan 650092
P. R. China

Zhiming Zhang
Department of Mathematics
Yunnan Normal University
Kunming, Yunnan 650092
P. R. China