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ON THE PRODUCTS OF BOUNDED DARBOUX BAIRE ONE FUNCTIONS

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Abstract. It is shown that for each $k > 1$, if f is a Baire one function and f is the product of k bounded Darboux (quasi–continuous) functions, then f is the product of k bounded Darboux (quasi–continuous) Baire one functions as well.

1. Preliminaries

The letters $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ denote the real line, the set of integers, and the set of positive integers, respectively. The word *function* denotes a mapping from $\mathbb{R}$ into $\mathbb{R}$ unless otherwise explicitly stated. The word *interval* means a nondegenerate interval. For each $A \subset \mathbb{R}$ we use the symbols $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{fr} A$, χ_A, and $|A|$ to denote the interior, the closure, the boundary, the characteristic function, and the cardinality of A, respectively. We write $\mathfrak{c} = |\mathbb{R}|$.

Let $A \subset \mathbb{R}$ and $f\colon A \to \mathbb{R}$. For each $y \in \mathbb{R}$ let $[f < y] = \{x \in A\colon f(x) < y\}$. Similarly we define the sets $[f \le y]$, $[f > y]$, etc. If $B \subset A$ and $|B| = \mathfrak{c}$, then let $\mathfrak{c}\text{-}\inf(f, B) = \inf\{y \in \mathbb{R}\colon |[f < y] \cap B| = \mathfrak{c}\}$ and $\mathfrak{c}\text{-}\sup(f, B) = -\,\mathfrak{c}\text{-}\inf(-f, B)$. If $|A \cap (x-\varepsilon, x)| = \mathfrak{c}$ for every $\varepsilon > 0$, then let $\mathfrak{c}\text{-}\underline{\lim}\,(f, x^-) = \lim_{\varepsilon\to 0^+} \mathfrak{c}\text{-}\inf\big(f, A \cap (x-\varepsilon, x)\big)$ and $\mathfrak{c}\text{-}\overline{\lim}\,(f, x^-) = -\,\mathfrak{c}\text{-}\underline{\lim}\,(-f, x^-)$. Similarly

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we define the symbols $\mathfrak{c}\text{-}\underline{\lim}\,(f,x^+)$ and $\mathfrak{c}\text{-}\overline{\lim}\,(f,x^+)$ if $|A\cap(x,x+\varepsilon)|=\mathfrak{c}$ for every $\varepsilon>0$. If $B\subset A$ is nonempty, then let $\omega(f,B)$ be the *oscillation of* f *on* B, i.e., $\omega(f,B)=\sup\{|f(x)-f(t)|\colon x,t\in B\}$. For each $x\in A$ we write $\omega(f,x)=\lim_{\varepsilon\to0^+}\omega\big(f,A\cap(x-\varepsilon,x+\varepsilon)\big)$, i.e., $\omega(f,x)$ is the *oscillation of* f *at* x. The symbol $\mathcal{C}_f$ denotes the set of points of continuity of f.

Let $f\colon\mathbb{R}\to\mathbb{R}$. We define $\|f\|=\sup|f|[\mathbb{R}]$. We say that f is *quasi–continuous* in the sense of Kempisty [9] if $f^{-1}(V)\subset\operatorname{cl}(\operatorname{int}f^{-1}(V))$ for each open set $V\subset\mathbb{R}$. We say that f is *Darboux* if it maps intervals onto connected sets.

There are many conditions equivalent to the Darboux property of Baire one functions. (See, e.g., [2, Theorem 6.1] or [1, Theorem 1.1, p. 9].) We will use two of them.

Theorem 1.1. For each Baire one function f the following are equivalent:

(i) f is Darboux;

(ii) for each $x\in\mathbb{R}$ we have

$$\max\Big\{\underline{\lim}_{t\to x^-}f(t),\underline{\lim}_{t\to x^+}f(t)\Big\}\le f(x)\le\min\Big\{\overline{\lim}_{t\to x^-}f(t),\overline{\lim}_{t\to x^+}f(t)\Big\};$$

(iii) for each $x\in\mathbb{R}$ there are sequences $x_n\nearrow x$ and $t_n\searrow x$ such that $f(x_n)\to f(x)$ and $f(t_n)\to f(x)$.

It is evident that the problem of characterization of the products of bounded positive Darboux Baire one functions can be reduced to characterization of the sums of Darboux Baire one functions bounded below, which in turn is equivalent to characterization of the sums of nonnegative Darboux Baire one functions [11]. First we solve the latter problem[1] (Theorem 2.2), and then we deal with the products of bounded Darboux (quasi-continuous) Baire one functions.

2. Sums of nonnegative Darboux Baire one functions

To prove the main theorem of this section we will need the following technical lemma.

Lemma 2.1. Let $0<\tau<\mathit{\Gamma}\le\infty$, $k>1$, and let $g_1,\dots,g_k$ be nonnegative Baire one functions. Set $f=g_1+\dots+g_k$, and assume that for each $i\le k$ and $x\in\mathbb{R}$ the following conditions hold:

$$\min\big\{\mathfrak{c}\text{-}\overline{\lim}\,(\min\{g_i+\mathit{\Gamma},f\},x^-),\mathfrak{c}\text{-}\overline{\lim}\,(\min\{g_i+\mathit{\Gamma},f\},x^+)\big\}\ge g_i(x),\quad(1)$$

[1]The sums of nonnegative Darboux quasi-continuous Baire one functions are characterized in [12, Theorem 3.4].

$$\max\left\{\mathfrak{c}\text{-}\underline{\lim}\,(\max\{g_i - \Gamma, 0\}, x^-), \mathfrak{c}\text{-}\underline{\lim}\,(\max\{g_i - \Gamma, 0\}, x^+)\right\} \le g_i(x). \quad (2)$$

There are nonnegative Baire one functions $h_1, \dots, h_k$ such that $h_1 + \dots + h_k = f$ on $\mathbb{R}$, and that for each $i \le k$ and $x \in \mathbb{R}$ the following conditions hold:

$$\min\left\{\mathfrak{c}\text{-}\overline{\lim}\,(\min\{h_i + \tau, f\}, x^-), \mathfrak{c}\text{-}\overline{\lim}\,(\min\{h_i + \tau, f\}, x^+)\right\} \ge h_i(x), \quad (3)$$

$$\max\left\{\mathfrak{c}\text{-}\underline{\lim}\,(\max\{h_i - \tau, 0\}, x^-), \mathfrak{c}\text{-}\underline{\lim}\,(\max\{h_i - \tau, 0\}, x^+)\right\} \le h_i(x), \quad (4)$$

$$|h_i(x) - g_i(x)| < \Gamma. \quad (5)$$

Proof. For each $\eta > 0$ let $\mathcal{J}_\eta$ denote the family of all compact intervals J for which there exist nonnegative Baire one functions $h_1, \dots, h_k$ such that $h_1 + \dots + h_k = f$ on J, and for each i:

$$\text{conditions (3)–(5) hold for each } x \in J, \quad (6)$$

$$h_i(x) = g_i(x) \text{ for } x \in \operatorname{fr} J, \quad (7)$$

$$\mathfrak{c}\text{-sup}(h_i, J) \ge \mathfrak{c}\text{-sup}\big(\min\{g_i + \Gamma, f\}, J\big) - \eta, \quad (8)$$

$$\mathfrak{c}\text{-inf}(h_i, J) \le \mathfrak{c}\text{-inf}\big(\max\{g_i - \Gamma, 0\}, J\big) + \eta. \quad (9)$$

Moreover let $\mathcal{J} = \bigcap_{\eta > 0} \mathcal{J}_\eta$, and let G denote the set of all $x \in \mathbb{R}$ for which there is a $\delta_x > 0$ such that $[a, b] \in \mathcal{J}$ whenever $a, b \in (x - \delta_x, x + \delta_x)$ and $a < b$.

The first claim is evident.

Claim 1. If $[a_0, a_1] \in \mathcal{J}$ and $[a_1, a_2] \in \mathcal{J}$, then $[a_0, a_2] \in \mathcal{J}$. ◁

Claim 2. If $a < b$ and $J = [a, b] \subset G$, then $J \in \mathcal{J}$.

Indeed, the compactness of J and the relation $J \subset \bigcup_{x \in J}(x - \delta_x, x + \delta_x)$ imply that $J \subset \bigcup_{i=1}^{m}(x_i - \delta_{x_i}, x_i + \delta_{x_i})$ for some $x_1, \dots, x_m \in J$. Hence there are nonoverlapping compact intervals $J_1, \dots, J_l \in \mathcal{J}$ with $J = \bigcup_{j=1}^{l} J_j$. So by Claim 1, $J \in \mathcal{J}$. ◁

Claim 3. Let A be an uncountable Borel measurable set such that $\sup f[A] < \infty$. For each $\eta > 0$ there are nonnegative Baire one functions $\tilde{g}_1, \dots, \tilde{g}_k$ and a perfect set $Q \subset A$ such that $\tilde{g}_1 + \dots + \tilde{g}_k = f$ on A, and for each $i \in \{1, \dots, k\}$: $\tilde{g}_i - g_i$ is a Darboux function, $g_i{\restriction}Q$ is continuous, $|\tilde{g}_i - g_i| < \Gamma$ on A, $\tilde{g}_i = g_i$ on $\mathbb{R} \setminus Q$, $\mathfrak{c}\text{-sup}(\tilde{g}_i, A) \ge d_i - \eta$, and $\mathfrak{c}\text{-inf}(\tilde{g}_i, A) \le c_i + \eta$, where $d_i = \mathfrak{c}\text{-sup}\big(\min\{g_i + \Gamma, f\}, A\big)$ and $c_i = \mathfrak{c}\text{-inf}\big(\max\{g_i - \Gamma, 0\}, A\big)$.

By [6, Lemma 2], we can find pairwise disjoint nonempty perfect sets $Q_1, \dots, Q_{2k}$ so that for each $i \le k$:

$$Q_i \quad \subset A \cap [\min\{g_i + \mathit{\Gamma}, f\} \ge d_i - \eta/3], \tag{10}$$

$$Q_{k+i} \subset A \cap [\max\{g_i - \mathit{\Gamma}, 0\} \le c_i + \eta/2], \tag{11}$$

and $g_i \restriction Q_j$ is continuous for each j. Clearly we may assume that $\omega(g_i, Q_j) < \eta/(3k)$ for each i and j. Put $Q = \bigcup_{j=1}^{2k} Q_j$.

Fix an $i \le k$. For $j \le 2k$ let $\theta_{i,j} = \min g_i[Q_j]$. Observe that for each $j \le k$, if $\zeta_j = \sum_{i \ne j} \theta_{i,j}$, then by (10),

$$\theta_{j,j} + \zeta_j = \textstyle\sum_{i=1}^{k} \big(\max g_i[Q_j] - \omega(g_i, Q_j)\big) > \max f[Q_j] - \eta/3 \ge d_j - 2\eta/3. \tag{12}$$

Let φ_i (respectively ψ_i) be a nonnegative Darboux Baire one function such that $\varphi_i = 0$ on $\mathbb{R} \setminus Q_i$ and $\|\varphi_i\| = \max\{d_i - 2\eta/3 - \theta_{i,i}, 0\}$ (such that $\psi_i = 0$ on $\mathbb{R} \setminus Q_{k+i}$ and $\|\psi_i\| = \max\{\theta_{i,k+i} - c_i - \eta/2, 0\}$, respectively). (Cf. [3, Corollary].) Define the function $\tilde{g}_i$ as follows:

- if $x \in Q_i$, then $\tilde{g}_i(x) = g_i(x) + \varphi_i(x)$;
- if $x \in Q_j$ for some $j \in \{1, \dots, k\} \setminus \{i\}$, then
$$\tilde{g}_i(x) = \begin{cases} g_i(x) - \theta_{i,j}\zeta_j^{-1}\varphi_j(x) & \text{if } \zeta_j > 0, \\ g_i(x) & \text{if } \zeta_j = 0; \end{cases}$$
- if $x \in Q_{k+i}$, then $\tilde{g}_i(x) = g_i(x) - \psi_i(x)$;
- if $x \in Q_{k+j}$ for some $j \in \{1, \dots, k\} \setminus \{i\}$, then $\tilde{g}_i(x) = g_i(x) + (k-1)^{-1}\psi_j(x)$;
- finally if $x \notin Q$, then $\tilde{g}_i(x) = g_i(x)$.

Clearly $\tilde{g}_i - g_i$ is a Darboux Baire one function. To prove that $\tilde{g}_i$ is nonnegative we consider three cases.

- If $x \in Q_j$ for some $j \in \{1, \dots, k\} \setminus \{i\}$, and $\zeta_j > 0$, then by (12), we obtain $\varphi_j(x) < \zeta_j$, so $\tilde{g}_i(x) \ge 0$.
- If $x \in Q_{k+i}$, then $\tilde{g}_i(x) \ge \min\{c_i + \eta/2, g_i(x)\} \ge 0$.
- If none of the above cases holds, then $\tilde{g}_i(x) \ge g_i(x) \ge 0$.

Observe that

$$\sup \tilde{g}_i[Q_i] \ge \theta_{i,i} + \sup \varphi_i[Q_i] \ge d_i - 2\eta/3 > d_i - \eta.$$

Since $g_i \restriction Q_i$ is continuous and φ_i is a Darboux function which vanishes outside of Q, so $\mathfrak{c}\text{-sup}(\tilde{g}_i, A) > d_i - \eta$. (Cf. [10, Corollary 6.2].) Similarly

$$\inf \tilde{g}_i[Q_{k+i}] \le \max g_i[Q_{k+i}] - \sup \psi_i[Q_{k+i}] \le \omega(g_i, Q_{k+i}) + c_i + \eta/2 < c_i + \eta,$$

whence $\mathfrak{c}\text{-inf}(\tilde{g}_i, A) < c_i + \eta$. The other conditions are evident. (Notice that by (10) and (11), $\max\{\varphi_i, \psi_i\} < \mathit{\Gamma}$ on $\mathbb{R}$.) $\lhd$

Claim 4. We have $G = \mathbb{R}$.

Notice that G is an open set. By way of contradiction suppose that the set $P=\mathbb{R}\setminus G$ is nonempty. Let $x_0\in\bigcap_{i=1}^k\mathcal{C}_{g_i\restriction P}$. We will show that $x_0\in G$, which is impossible.

Choose a $\delta>0$ so that

$$\max\bigl\{\omega\bigl(g_i,P\cap(x_0-\delta,x_0+\delta)\bigr)\colon i\le k\bigr\}<\tau. \tag{13}$$

Let $J\subset(x_0-\delta,x_0+\delta)$ be a compact interval, and let $\eta\in(0,\tau)$. If $J\subset G$, then by Claim 2, we obtain $J\in\mathcal{J}$. So suppose $A=P\cap J\neq\emptyset$. We will assume $|A|=\mathfrak{c}$, the other case being simpler.[2] Write $G\cap J$ as the union of a family of nonoverlapping compact intervals, $\{I_n\colon n\in N\}$, such that each $x\in G\cap J$ is an interior point of $I_n\cup I_m\cup(\mathbb{R}\setminus J)$ for some $n,m\in N$. (We have $N=\emptyset$ if $G\cap J=\emptyset$, and $N=\mathbb{N}$ otherwise.) For each n let $h_{1,n},\dots,h_{k,n}$ witness $I_n\in\mathcal{J}_{\eta/n}$. (Cf. Claim 2.) Construct nonnegative Baire one functions $\tilde g_1,\dots,\tilde g_k$ and a perfect set $Q\subset A$ according to Claim 3.

Fix an $i\in\{1,\dots,k\}$. Define $h_i(x)=h_{i,n}(x)$ if $x\in I_n$ for some $n\in N$, let $h_i(x)=\tilde g_i(x)$ if $x\in A$, and let h_i be constant on $(-\infty,\min J]$ and $[\max J,\infty)$. Clearly h_i is a nonnegative Baire one function, $|h_i-g_i|<\mathit{\Gamma}$ on J, and $h_i=g_i$ on $\operatorname{fr} J$. Moreover

$$\begin{aligned}
&\mathfrak{c}\text{-}\sup(h_i,J)=\sup\bigl(\{\mathfrak{c}\text{-}\sup(h_i,I_n)\colon n\in N\}\cup\{\mathfrak{c}\text{-}\sup(h_i,A)\}\bigr)\\
&\ge\sup\bigl(\bigl\{\mathfrak{c}\text{-}\sup(\min\{g_i+\mathit{\Gamma},f\},I_n)\colon n\in N\bigr\}\cup\bigl\{\mathfrak{c}\text{-}\sup(\min\{g_i+\mathit{\Gamma},f\},A)\bigr\}\bigr)-\eta\\
&=\mathfrak{c}\text{-}\sup\bigl(\min\{g_i+\mathit{\Gamma},f\},J\bigr)-\eta.
\end{aligned}$$

Similarly we can show that condition (9) holds.

Fix an $x\in J$. We consider four cases.

- If $x\in G$, then evidently condition (3) is satisfied.
- If x is a limit point of Q from the left, then since $g_i\restriction Q$ is continuous, and $\tilde g_i-g_i$ is a Darboux function which vanishes outside of Q, so

$$\mathfrak{c}\text{-}\overline{\lim}\bigl(\min\{h_i+\tau,f\},x^-\bigr)\ge\mathfrak{c}\text{-}\overline{\lim}\,(h_i\restriction Q,x^-)\ge h_i(x).$$

- If $|A\cap(x-\varepsilon,x)|=\mathfrak{c}$ for each $\varepsilon>0$, and x is not a limit point of Q from the left, then $\tilde g_i=g_i$ on $(x-\varepsilon_0,x]$ for some $\varepsilon_0>0$, so by (13) and (1), we obtain

$$\begin{aligned}
\mathfrak{c}\text{-}\overline{\lim}\bigl(\min\{h_i+\tau,f\},x^-\bigr)&\ge\mathfrak{c}\text{-}\overline{\lim}\bigl(\min\{g_i(x),f\}\restriction A,x^-\bigr)\\
&\ge g_i(x)=h_i(x).
\end{aligned}$$

[2] We define $\tilde g_1=\dots=\tilde g_k=f/k$ on $\mathbb{R}$, and $Q=\emptyset$.

- Finally if $x \in A \setminus \{\min J\}$ and $|A \cap (x-\varepsilon_0, x)| < \mathfrak{c}$ for some $\varepsilon_0 > 0$, then by (8) and (1), we obtain

$$\begin{aligned}
\mathfrak{c}\text{-}\overline{\lim}\left(\min\{h_i+\tau, f\}, x^-\right) &\geq \lim_{\varepsilon\to 0^+} \mathfrak{c}\text{-}\sup\bigl(h_i, (x-\varepsilon, x)\bigr) \\
&\geq \lim_{\varepsilon\to 0^+} \sup\bigl\{\mathfrak{c}\text{-}\sup(h_i, I_n)\colon I_n \subset (x-\varepsilon, x)\bigr\} \\
&\geq \lim_{\varepsilon\to 0^+} \sup\bigl\{\mathfrak{c}\text{-}\sup(\min\{g_i+\Gamma, f\}, I_n) - \eta/n\colon I_n \subset (x-\varepsilon, x)\bigr\} \\
&= \mathfrak{c}\text{-}\overline{\lim}\left(\min\{g_i+\Gamma, f\}, x^-\right) \geq g_i(x) = h_i(x).
\end{aligned}$$

Similarly we can show that $\mathfrak{c}\text{-}\overline{\lim}\left(\min\{h_i+\tau, f\}, x^+\right) \geq h_i(x)$, and that condition (4) holds for each $x \in J$. The condition "$h_1 + \cdots + h_k = f$ on J" is evident. $\triangleleft$

Using Claims 4 and 2 one can easily prove the assertion of the lemma. $\square$

Theorem 2.2. Let $k > 1$. For each Baire one function f the following conditions are equivalent:

(i) f is the sum of k nonnegative Darboux functions;

(ii) f is nonnegative and f fulfills the following condition:

$$\min\bigl\{\mathfrak{c}\text{-}\overline{\lim}\,(f, x^-), \mathfrak{c}\text{-}\overline{\lim}\,(f, x^+)\bigr\} \geq f(x)/k \text{ for each } x \in \mathbb{R}; \tag{14}$$

(iii) f is the sum of k nonnegative Darboux Baire one functions.

Proof. The implication (i) $\Rightarrow$ (ii) follows by [11, Proposition 3.2], and the implication (iii) $\Rightarrow$ (i) is obvious.

(ii) $\Rightarrow$ (iii)

Set $\tau_0 = \infty$ and $\tau_n = 2^{-n}$ for $n \geq 1$. Let $g_{1,0} = \cdots = g_{k,0} = f/k$. For each n use Lemma 2.1 to construct nonnegative Baire one functions $g_{1,n}, \ldots, g_{k,n}$ such that $g_{1,n} + \cdots + g_{k,n} = f$ on $\mathbb{R}$, and that for each $i \leq k$ and $x \in \mathbb{R}$ we have:

$$\min\bigl\{\mathfrak{c}\text{-}\overline{\lim}\,(\min\{g_{i,n}+\tau_n, f\}, x^-), \mathfrak{c}\text{-}\overline{\lim}\,(\min\{g_{i,n}+\tau_n, f\}, x^+)\bigr\} \geq g_{i,n}(x), \tag{15}$$

$$\max\bigl\{\mathfrak{c}\text{-}\underline{\lim}\,(\max\{g_{i,n}-\tau_n, 0\}, x^-), \mathfrak{c}\text{-}\underline{\lim}\,(\max\{g_{i,n}-\tau_n, 0\}, x^+)\bigr\} \leq g_{i,n}(x), \tag{16}$$

$$|g_{i,n}(x) - g_{i,n-1}(x)| < \tau_{n-1}. \tag{17}$$

Fix an $i \le k$. Put $g_i = \lim_{n\to\infty} g_{i,n}$. By (17), the sequence $(g_{i,n})$ is uniformly convergent, so g_i belongs to Baire class one. Meanwhile by (15) and (16), we obtain

$$\min\bigl\{\mathfrak{c}\text{-}\overline{\lim}\,(g_i, x^-), \mathfrak{c}\text{-}\overline{\lim}\,(g_i, x^+)\bigr\} \ge g_i(x) \ge \max\bigl\{\mathfrak{c}\text{-}\underline{\lim}\,(g_i, x^-), \mathfrak{c}\text{-}\underline{\lim}\,(g_i, x^+)\bigr\}.$$

So by Theorem 1.1, g_i is Darboux. Clearly $f = g_1 + \cdots + g_k$ on $\mathbb{R}$. □

3. Products of bounded Darboux Baire one functions

Theorem 3.1. Let $k > 1$. For each Baire one function f the following conditions are equivalent:

(i) f is the product of k bounded Darboux functions;

(ii) f is bounded,

$$\text{for all } x, t \in \mathbb{R}, \text{ if } x < t \text{ and } f(x)f(t) < 0, \quad \text{then } [f = 0] \cap (x, t) \ne \emptyset, \tag{18}$$

and there is a $T \ge \sqrt[k]{\|f\|}$ such that for each $x \in \mathbb{R}$ we have

$$\max\bigl\{\mathfrak{c}\text{-}\underline{\lim}\,(|f|, x^-), \mathfrak{c}\text{-}\underline{\lim}\,(|f|, x^+)\bigr\} \le T^{k-1}\sqrt[k]{|f(x)|}; \tag{19}$$

(iii) f is the product of k bounded Darboux Baire one functions.

Proof. The implication (iii) ⇒ (i) is obvious.

(i) ⇒ (ii)

Let $f = g_1 \dots g_k$, where $g_1, \dots, g_k$ are bounded Darboux functions. The boundedness of f is obvious, and condition (18) follows by [4]. Put $T = \max\{\|g_i\| \colon i \le k\}$. Fix an $x \in \mathbb{R}$. There is an $i \le k$ such that $|g_i(x)| \le \sqrt[k]{|f(x)|}$. Using the fact that g_i is Darboux, we obtain

$$\mathfrak{c}\text{-}\underline{\lim}\,(|f|, x^-) \le \textstyle\prod_{j\ne i} \|g_j\| \cdot \mathfrak{c}\text{-}\underline{\lim}\,(|g_i|, x^-) \le T^{k-1} \cdot |g_i(x)| \le T^{k-1}\sqrt[k]{|f(x)|}.$$

Similarly $\mathfrak{c}\text{-}\underline{\lim}\,(|f|, x^+) \le T^{k-1}\sqrt[k]{|f(x)|}$.

(ii) ⇒ (iii)

Let $\mathcal{J}$ denote the family of all intervals $J = [a, b]$ for which there exist Darboux Baire one functions $g_1, \dots, g_k$ such that $g_1 \dots g_k = f$ on J, $T \operatorname{sgn} f(a) \in g_1[J]$, $T \in |g_1|[J] \cap g_2[J] \cap \dots \cap g_k[J]$, and for each i: $g_i(x) = \sqrt[k]{|f(x)|} \cdot (\operatorname{sgn} f(x))^{1+\operatorname{sgn}(i-1)}$ for $x \in \operatorname{fr} J$, and $|g_i| \le T$ on J. Moreover let G denote the set of all $x \in \mathbb{R}$ for which there is a $\delta_x > 0$ such that $[a, b] \in \mathcal{J}$ whenever $a, b \in (x - \delta_x, x + \delta_x)$ and $a < b$.

Claim 1. *Let K be a nonempty perfect set such that $K \cap [f = 0]$ is dense in K. There exist Baire one functions $g_1, \dots, g_k$ such that $f = g_1 \dots g_k$ on K, and for each i:*

$$g_i[K] = [-T, T], \tag{20}$$

$$g_i(x) = \sqrt[k]{|f(x)|} \cdot (\operatorname{sgn} f(x))^{1+\operatorname{sgn}(i-1)}$$
whenever $x \in K$ and x is not a bilateral limit point of K, (21)

$$\mathfrak{c}\text{-}\underline{\lim}\left(|g_i - g_i(x)|\restriction K, x^-\right) = 0$$
whenever $x \in K$ and x is not isolated in K from the left, (22)

$$\mathfrak{c}\text{-}\underline{\lim}\left(|g_i - g_i(x)|\restriction K, x^+\right) = 0$$
whenever $x \in K$ and x is not isolated in K from the right. (23)

Let $\widetilde{K}$ be the set of all points of continuity of $f\restriction K$ which are bilateral limit points of K. Observe that $\widetilde{K}$ is a dense G_δ subset of $K \cap [f = 0]$. For each $n \in \mathbb{N} \cup \{0\}$ define $\tau_n = T/2^n$. Put $F_0 = \emptyset$.

Assume that for some $n \in \mathbb{N}$ we have constructed a nowhere dense in K closed set F_{n-1}. Put $A_n = \{x \in K : \omega(|f|\restriction K, x) \ge \tau_n^k\}$. Notice that A_n is nowhere dense in K and closed. Let $I_{n,1}, I_{n,2}, \dots$ be nonoverlapping compact intervals disjoint from A_n, such that each $x \notin A_n$ belongs to $\operatorname{int}(I_{n,m} \cup I_{n,p})$ for some $m, p \in \mathbb{N}$. For each $m \in \mathbb{N}$, if $\widetilde{K} \cap I_{n,m} \neq \emptyset$, then find pairwise disjoint nonempty nowhere dense in K perfect sets $P_{1,n,m}, \dots, P_{k,n,m} \subset \widetilde{K} \cap \operatorname{int} I_{n,m} \setminus F_{n-1}$, and for $i \le k$ construct a Darboux Baire one function $\bar{g}_{i,n,m}$ such that $\bar{g}_{i,n,m} = 0$ outside of $P_{i,n,m}$, and $\bar{g}_{i,n,m}[P_{i,n,m}] = [-\tau_{n-1}, \tau_{n-1}]$; otherwise let $P_{i,n,m} = \emptyset$ and $\bar{g}_{i,n,m} = 0$ for each i. Observe that the set $F_n = F_{n-1} \cup A_n \cup \bigcup_{m\in\mathbb{N}} \bigcup_{i=1}^k P_{i,n,m}$ is nowhere dense in K and closed.

Fix an $i \le k$. It is clear that $\bar{g}_{i,n} = \sum_{m\in\mathbb{N}} \bar{g}_{i,n,m}$ is a Baire one function for each n. So $\bar{g}_i = \sum_{n\in\mathbb{N}} \bar{g}_{i,n}$ belongs to the first class of Baire, too.

Define $\tilde{g}_i = \sqrt[k]{|f|} \cdot (\operatorname{sgn} f)^{1+\operatorname{sgn}(i-1)} \cdot \chi_K$. Obviously $\tilde{g}_i$ is Baire one if $i > 1$. Let $y \in \mathbb{R}$. If $y \le 0$, then $[\tilde{g}_1 < y] = [f - y\tilde{g}_2^{k-1} < 0]$, and $y > 0$ implies $[\tilde{g}_1 < y] = [f\chi_K < y^k]$. Similarly we can express the set $[\tilde{g}_1 > y]$. Thus $\tilde{g}_1$ belongs to Baire class one. (See also [5, p. 82].)

Define $g_i = \bar{g}_i + \tilde{g}_i$. Clearly g_i is a Baire one function, and conditions (20) and (21) hold. (Notice that $\bar{g}_i(x) = 0$ whenever $x \notin \bigcup_{n\in\mathbb{N}} F_n$.) To prove (22) suppose that $x \in K$ and x is not isolated in K from the left. We consider three cases.

- If $g_i(x) = 0$, then since $\bigcup_{n\in\mathbb{N}} F_n$ is a meager subset of $\widetilde{K}$, so for each $\varepsilon > 0$ we have $|\widetilde{K} \cap (x-\varepsilon, x) \cap [g_i = 0]| = \mathfrak{c}$. Hence $\mathfrak{c}\text{-}\underline{\lim}\,\big(|g_i|\restriction K, x^-\big) = 0$.
- If $x \in P_{j,n,m}$ for some $j \le k$ and $n, m \in \mathbb{N}$, and $g_i(x) \neq 0$, then x is not isolated in $P_{j,n,m}$ from the left. Consequently, by [10, Lemma 6.1], we have $\mathfrak{c}\text{-}\underline{\lim}\,\big(|g_i - g_i(x)|\restriction K, x^-\big) = 0$.
- Finally if $x \in A_n \setminus A_{n-1}$ for some $n \in \mathbb{N}$, then $\omega\big(|f|\restriction K, x\big) \le \tau_{n-1}^k$ and $|g_i(x)| = \sqrt[k]{|f(x)|} \le \tau_{n-1}$. Meanwhile for each $\varepsilon > 0$ there is an $m \in \mathbb{N}$ with $\emptyset \neq P_{i,n,m} \subset (x-\varepsilon, x)$, whence $g_i[K \cap (x-\varepsilon, x)] \supset [-\tau_{n-1}, \tau_{n-1}]$. Thus $\mathfrak{c}\text{-}\underline{\lim}\,\big(|g_i - g_i(x)|\restriction K, x^-\big) = 0$.

Similarly we can show that condition (23) is fulfilled.

Observe that if $x \in \widetilde{K}$, then $\tilde{g}_1(x) = \cdots = \tilde{g}_k(x) = 0$, and $x \notin \widetilde{K}$ yields $\bar{g}_1(x) = \cdots = \bar{g}_k(x) = 0$. Thus $f = g_1 \ldots g_k =$ on K. ◁

Claim 2. If $a < b$, and $(a,b) \subset [f > 0]$ or $(a,b) \subset [f < 0]$, then $[a,b] \in \mathcal{J}$.

Without loss we may suppose that $(a,b) \subset [f > 0]$. Define $\tilde{f}(x) = k \ln T - \ln f(x)$ if $x \in (a,b)$, and $\tilde{f}(x) = 0$ if $x \in \mathbb{R} \setminus (a,b)$. By (19), for each $x \in (a,b)$ we have

$$\mathfrak{c}\text{-}\overline{\lim}\,\big(\tilde{f}, x^-\big) = k \ln T - \mathfrak{c}\text{-}\underline{\lim}\,(\ln f, x^-) \ge \ln T - k^{-1} \ln f(x) = \tilde{f}(x)/k,$$

and similarly $\mathfrak{c}\text{-}\overline{\lim}\,\big(\tilde{f}, x^+\big) \ge \tilde{f}(x)/k$. Since $\tilde{f}$ is nonnegative and $\tilde{f}$ vanishes outside of (a,b), so $\tilde{f}$ fulfills condition (14).

Let $\{a_z\colon z \in \mathbb{Z}\}$ be an arbitrary strictly increasing sequence with limit points a and b. Construct nonnegative Darboux Baire one functions $\tilde{g}_1, \ldots, \tilde{g}_k$ such that $\tilde{f} = \tilde{g}_1 + \cdots + \tilde{g}_k$ on $\mathbb{R}$, and

$$\tilde{g}_i\big[[a_z, a_{z+1}]\big] \supset \Big[0, \ln\!\Big(T^k/\big(\mathfrak{c}\text{-}\inf(f, [a_z, a_{z+1}]) + 2^{-|z|}\big)\Big)\Big] \tag{24}$$

for each i. (Cf. condition (8) in the proof of Lemma 2.1.)

Fix an $i \le k$. Define $g_i(x) = T/\exp\big(\tilde{g}_i(x)\big)$ if $x \in (a,b)$, let $g_i(x) = \sqrt[k]{f(x)}$ if $x \in \{a,b\}$, and let g_i be constant on $(-\infty, a]$ and $[b, \infty)$. Then by (24), we have

$$\begin{aligned} g_i\big[[a_n, b]\big] &\supset \Big[\inf\Big\{\big(\mathfrak{c}\text{-}\inf(f, [a_z, a_{z+1}]\big) + 2^{-z})/T^{k-1}\colon z \ge n\Big\}, T\Big] \\ &\supset \Big[\big(\mathfrak{c}\text{-}\inf(f, [a_n, b]) + 2^{-n}\big)/T^{k-1}, T\Big] \end{aligned}$$

for each $n \in \mathbb{N}$. So, the left cluster set of g_i at b contains $\big[\mathfrak{c}\text{-}\underline{\lim}\,(f, b^-)/T^{k-1}, T\big]$. Similarly the right cluster set of g_i at a contains $\big[\mathfrak{c}\text{-}\underline{\lim}\,(f, a^+)/T^{k-1}, T\big]$. By (18), we have $\{a, b\} \subset [f \ge 0]$. So by (19) and Theorem 1.1, g_i is Darboux. ◁

Claims 3 and 4 are easy to prove. (Cf. also Claim 2 in the proof of Lemma 2.1.)

Claim 3. If $[a_0, a_1] \in \mathcal{J}$ and $[a_1, a_2] \in \mathcal{J}$, then $[a_0, a_2] \in \mathcal{J}$. ◁

Claim 4. If $a < b$ and $[a, b] \subset G$, then $[a, b] \in \mathcal{J}$. ◁

Claim 5. If $a < b$ and $(a, b) \subset G$, then $[a, b] \in \mathcal{J}$.

Let $c \in (a, b)$. By Claim 3, it suffices to show that $[a, c] \in \mathcal{J}$ and $[c, b] \in \mathcal{J}$. We will show only that $[a, c] \in \mathcal{J}$, the proof of the other case being similar.

If $[f = 0] \cap (a, d] = \emptyset$ for some $d \in (a, c)$, then either $(a, d] \subset [f > 0]$, or $(a, d] \subset [f < 0]$, whence $[a, c] \in \mathcal{J}$. (Cf. Claim 2.) So suppose $f(c) = 0$ and $a \in \operatorname{cl}([f = 0] \cap (a, c))$. Let $(a_n) \subset [f = 0] \cap (a, c)$ be such that $a_n \searrow a$. Put $a_0 = c$. For each n let $g_{1,n}, \dots, g_{k,n}$ witness $[a_n, a_{n-1}] \in \mathcal{J}$. (See Claim 4.) One can easily construct sequences $(t_{1,n}), \dots, (t_{k,n}) \subset \{-1, 1\}$ so that $t_{1,n} \dots t_{k,n} = 1$ for each n, and that for each i and each n we have

$$\{-T, T\} \subset (t_{i,n} g_{i,n})\big[[a_n, a_{n-1}]\big] \cup \dots \cup (t_{i,n+3} g_{i,n+3})\big[[a_{n+3}, a_{n+2}]\big]. \tag{25}$$

Fix an $i \le k$. Define $g_i(x) = t_{i,n} g_{i,n}(x)$ if $x \in [a_n, a_{n-1}]$ for some $n \in \mathbb{N}$, let $g_i(a) = \sqrt[k]{|f(a)|} \cdot (\operatorname{sgn} f(a))^{1+\operatorname{sgn}(i-1)}$, and let g_i be constant on $(-\infty, a]$ and $[b, \infty)$. By (25), the right cluster set of g_i at a equals $[-T, T]$. Hence by Theorem 1.1, g_i is Darboux. ◁

Claim 6. If $P = \mathbb{R} \setminus G$, then $P \subset \operatorname{cl}[f = 0]$.

Indeed, if $x \notin \operatorname{cl}[f = 0]$, then by (18), we have either $(x - \delta, x + \delta) \subset [f > 0]$ or $(x - \delta, x + \delta) \subset [f < 0]$ for some $\delta > 0$. Hence by Claim 2, $x \in G = \mathbb{R} \setminus P$. ◁

Claim 7. The set P is perfect.

Clearly G is open. If $P \cap (s - \delta, s + \delta) = \{s\}$ for some $s \in P$ and $\delta > 0$, then by Claims 5 and 3, we obtain $s \in G$, an impossibility. ◁

Claim 8. The set $P \cap [f = 0]$ is dense in P.

By way of contradiction suppose $\emptyset \ne P \cap (s, t) \subset [f \ne 0]$ for some $s < t$. Since f is Baire one, we may assume that either $P \cap (s, t) \subset [f > 0]$ or $P \cap (s, t) \subset [f < 0]$. We will consider the first case only, the other one being analogous. We will show that $(s, t) \subset G$, which is impossible.

Fix an interval $J = [a, b] \subset (s, t)$. By Claims 7, 5, and 3, we may assume that $P \cap J$ is perfect, and $a, b \in P$. Let $\mathcal{I}$ be the family of all components of $J \setminus P$.

Let $I = (c, d) \in \mathcal{I}$. If $[f = 0] \cap I \ne \emptyset$, then by Claims 5 and 3, there are Darboux Baire one functions $g_{1,I}, \dots, g_{k,I}$ such that $f = g_{1,I} \dots g_{k,I}$ on I, and for each i: $|g_i| \le T$ on I, $g_i(x) = \sqrt[k]{f(x)}$ for $x \in \{c, d\}$, and $\{0, T\} \subset g_i\big[[c, d]\big]$. Otherwise let $g_{1,I}, \dots, g_{k,I}$ witness $[c, d] \in \mathcal{J}$.

Fix an $i \le k$. Define $g_i(x) = g_{i,I}(x)$ if $x \in I$ for some $I \in \mathcal{I}$, let $g_i(x) = \sqrt[k]{f(x)}$ if $x \in P \cap J$, and let g_i be constant on $(-\infty, a]$ and $[b, \infty)$. If $x \in P \cap J$ and x is not isolated in P from the left (from the right), then by Claim 6, the left (right) cluster set of g_i at x contains $[0, T]$. Thus by Theorem 1.1, g_i is Darboux. Hence $J \in \mathcal{J}$ and $P \cap G \supset P \cap (s,t) \neq \emptyset$, an impossibility. $\triangleleft$

Claim 9. We have $G = \mathbb{R}$.

By way of contradiction suppose that P is nonempty. We will show that $P \subset G$, which is impossible.

Let $J = [a, b]$ be a compact interval. We may assume that $P \cap J$ is perfect, and $a, b \in P$. Apply Claim 1 with $K = P \cap J$ to construct Baire one functions $\tilde{g}_1, \dots, \tilde{g}_k$ with $f = \tilde{g}_1 \dots \tilde{g}_k$ on K, fulfilling conditions (20)–(23). (Cf. Claims 7 and 8.) Let $\mathcal{I}$ be the family of all components of $J \setminus P$. For each $I \in \mathcal{I}$ let $g_{1,I}, \dots, g_{k,I}$ witness $\operatorname{cl} I \in \mathcal{J}$.

Fix an $i \le k$. Define $g_i(x) = g_{i,I}(x)$ if $x \in I$ for some $I \in \mathcal{I}$, let $g_i(x) = \tilde{g}_i(x)$ if $x \in P \cap J$, and let g_i be constant on $(-\infty, a]$ and $[b, \infty)$. By (22) and (23), the graph of g_i is bilaterally dense in itself, so by Theorem 1.1, g_i is Darboux. Hence $J \in \mathcal{J}$ and $\emptyset \neq P \subset G$, an impossibility. $\triangleleft$

Using Claims 9 and 4 one can easily prove the assertion of the theorem. $\square$

Theorem 3.2. Let $k > 1$. For each Baire one function f the following conditions are equivalent:

(i) f is the product of k bounded Darboux quasi-continuous functions;

(ii) f is bounded, f fulfills condition (18), the set $[f = 0] \setminus \operatorname{int}[f = 0]$ is nowhere dense, and there is a $T \ge \sqrt[k]{\|f\|}$ such that for each $x \in \mathbb{R}$ we have

$$\lim_{\varepsilon \to 0^+} \max\bigl\{\inf |f|[\mathcal{C}_f \cap (x - \varepsilon, x)], \inf |f|[\mathcal{C}_f \cap (x, x + \varepsilon)]\bigr\} \le T^{k-1} \sqrt[k]{|f(x)|}; \quad (26)$$

(iii) f is the product of k bounded Darboux quasi-continuous Baire one functions.

Proof. The implication (iii) $\Rightarrow$ (i) is obvious.

(i) $\Rightarrow$ (ii)

Let $f = g_1 \dots g_k$, where $g_1, \dots, g_k$ are bounded Darboux quasi-continuous functions. The boundedness of f is obvious, condition (18) follows by [4], and the set $[f = 0] \setminus \operatorname{int}[f = 0]$ is nowhere dense by [13]. Put $T = \max\{\|g_i\| \colon i \le k\}$. Fix an $x \in \mathbb{R}$. There is an $i \le k$ such that $|g_i(x)| \le$

$\sqrt[k]{|f(x)|}$. By [14, Lemma 2], there is a sequence $(x_n) \subset \mathcal{C}_f$ such that $x_n \nearrow x$ and $g_i(x_n) \to g_i(x)$. Hence for each $\varepsilon > 0$ we have

$$\begin{aligned}\inf |f|[\mathcal{C}_f \cap (x-\varepsilon, x)] &\leq \prod_{j\neq i} \|g_j\| \cdot \inf |g_i|[\mathcal{C}_f \cap (x-\varepsilon, x)] \\ &\leq T^{k-1} \cdot |g_i(x)| \leq T^{k-1} \sqrt[k]{|f(x)|}.\end{aligned}$$

Similarly we can prove that $\inf |f|[\mathcal{C}_f \cap (x, x+\varepsilon)] \leq T^{k-1} \sqrt[k]{|f(x)|}$ for each $\varepsilon > 0$.

(ii) $\Rightarrow$ (iii)

For each $\eta > 0$ let $\mathcal{J}_\eta$ denote the family of all compact intervals J for which there exist Darboux quasi-continuous Baire one functions $g_1, \dots, g_k$ and a $\theta \in \{-1, 1\}$ such that $g_1 \dots g_k = f$ on J, and for each i: $g_i(x) = \sqrt[k]{|f(x)|} \cdot (\operatorname{sgn} f(x))^{1+\operatorname{sgn}(i-1)}$ for $x \in \operatorname{fr} J$, $|g_i| \leq T$ on J, and

$$(\theta^{1+\operatorname{sgn}(i-1)} g_i)[\mathcal{C}_{g_i} \cap J] \supset \left[\inf |f|[\mathcal{C}_f \cap J]/T^{k-1} + \eta, T\right].$$

Moreover let $\mathcal{J} = \bigcap_{\eta>0} \mathcal{J}_\eta$, and let G denote the set of all $x \in \mathbb{R}$ for which there is a $\delta_x > 0$ such that $[a, b] \in \mathcal{J}$ whenever $a, b \in (x - \delta_x, x + \delta_x)$ and $a < b$.

Claim 1 is evident, Claim 2 follows easily by (18) and [12, Lemma 4.4], and Claims 3–7 are easy to prove. (Cf. also Claims 2–8 in the proof of Theorem 3.1.)

Claim 1. If $a < b$ and $(a, b) \subset [f = 0]$, then $[a, b] \in \mathcal{J}$. $\lhd$

Claim 2. If $a < b$ and $(a, b) \subset [f > 0]$ or $(a, b) \subset [f < 0]$, then $[a, b] \in \mathcal{J}$. $\lhd$

Claim 3. If $[a_0, a_1] \in \mathcal{J}$ and $[a_1, a_2] \in \mathcal{J}$, then $[a_0, a_2] \in \mathcal{J}$. $\lhd$

Claim 4. If $a < b$ and $(a, b) \subset G$, then $[a, b] \in \mathcal{J}$. $\lhd$

Claim 5. If $P = \mathbb{R} \setminus G$, then $P \subset \operatorname{fr}[f = 0]$. $\lhd$

Claim 6. The set P is perfect. $\lhd$

Claim 7. The set $P \cap [f = 0]$ is dense in P. $\lhd$

Claim 8. Let K be a nowhere dense compact perfect set such that $K \cap [f = 0]$ is dense in K, and let $\mathcal{I}$ be the family of all components of $[\min K, \max K] \setminus K$. There are pairwise disjoint families $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3 \subset \mathcal{I}$ such that:

if we put $\tilde{f}(x) = f(x)$ for $x \in \bigcup_{I \in \mathcal{I}_1 \cup \mathcal{I}_2} \operatorname{fr} I$, and $\tilde{f}(x) = 0$ otherwise, then $\tilde{f}$ belongs to Baire class one, (27)

for each $j \leq 3$ and $x \in K$, if x is not isolated in K from the left (from the right), then there is a sequence $(I_m) \subset \mathcal{I}_j$ such that $\min I_m \nearrow x$ and $f(\min I_m) \to 0$ (such that $\max I_m \searrow x$ and $f(\max I_m) \to 0$, respectively). (28)

Without loss we may assume that $K \neq \emptyset$. For $j \leq 3$ put $\mathcal{I}_{j,0} = \emptyset$. We proceed by induction. Fix an $n \in \mathbb{N}$, and assume that the family $\bigcup_{j \leq 3} \mathcal{I}_{j,n-1}$ is finite. Find pairwise disjoint open intervals $V_{n,1}, \ldots, V_{n,p_n}$ of diameter less than n^{-1} such that $K \cap V_{n,s} \neq \emptyset$ for each s, and $K \subset \bigcup_{s \leq p_n} V_{n,s}$. For each $s \leq p_n$ and $j \leq 3$, observe that the family

$$\tilde{\mathcal{I}}_{j,n,s} = \big\{ I \in \mathcal{I} \setminus \big(\bigcup_{i \leq 3} \mathcal{I}_{i,n-1} \cup \{ I_{i,n,s} : i < j \} \big) : I \subset V_{n,s} \text{ and } \operatorname{fr} I \subset \big[|f| < 2^{-n} \big] \big\}$$

is nonempty (recall that K is perfect, and $\mathcal{C}_{f \restriction K} \subset \big[f = 0 \big]$), and pick $I_{j,n,s} \in \tilde{\mathcal{I}}_{j,n,s}$. For $j \leq 3$ define $\mathcal{I}_{j,n} = \mathcal{I}_{j,n-1} \cup \big\{ I_{j,n,s} : s \leq p_n \big\}$. This completes the induction step.

For $j \leq 3$ put $\mathcal{I}_j = \bigcup_{n \in \mathbb{N}} \mathcal{I}_{j,n}$. Clearly these families are pairwise disjoint, and condition (27) is evident. To prove condition (28) fix a $j \leq 3$ and an $x \in K$, and suppose that x is not isolated in K from the left. (The other case is similar.) Fix an $m \in \mathbb{N}$. Let $t \in K \cap (x - m^{-1}/2, x)$ be a bilateral limit point of K, and let $n > (x-t)^{-1}$. Let $s \leq p_n$ be such that $t \in V_{n,s}$, and define $I_m = I_{j,n,s}$. Since the diameter of $V_{n,s}$ is less than $x - t$, so $I_m \subset (x - m^{-1}, x)$. Moreover $f(\min I_m) < 2^{-m}$. $\lhd$

Claim 9. We have $G = \mathbb{R}$.

By way of contradiction suppose that P is nonempty. We will show that $P \subset G$, which is impossible.

Let $J = [a,b]$ be a compact interval. We may assume that $P \cap J$ is perfect, and $a, b \in P$. Let $\mathcal{I} = \{ I_n : n \in \mathbb{N} \}$ be the family of all components of $J \setminus P$. Apply Claim 8 with $K = P \cap J$ to find pairwise disjoint families $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3 \subset \mathcal{I}$ fulfilling conditions (27) and (28). (Cf. Claims 5–7.) Define $\tilde{f}$ as in (27).

Let $n \in \mathbb{N}$. Let Baire one functions $g_{1,n}, \ldots, g_{k,n}$ and $\theta_n \in \{-1, 1\}$ witness $\operatorname{cl} I_n \in \mathcal{J}_{1/n}$. (Cf. Claim 4.) Set $t_{1,1,n} = \theta_n$ and $t_{1,2,n} = -\theta_n$. For $j \leq 2$ define $t_{2,j,n}, \ldots, t_{k,j,n} \in \{-1, 1\}$ so that $t_{1,j,n} \ldots t_{k,j,n} = 1$.

Fix an $i \leq k$. Define the function g_i so that:

- if $x \in \operatorname{cl} I_n$ and $I_n \in \mathcal{I}_j$ for some $n \in \mathbb{N}$ and $j \leq 2$, then $g_i(x) = t_{i,j,n} g_{i,n}(x)$,
- if $x \in I_n$ and $I_n \in \mathcal{I} \setminus (\mathcal{I}_1 \cup \mathcal{I}_2)$ for some $n \in \mathbb{N}$, then $g_i(x) = g_{i,n}(x)$,
- if $x \in P \cap J \setminus \bigcup_{I \in \mathcal{I}_1 \cup \mathcal{I}_2} \operatorname{fr} I$, then $g_i(x) = \sqrt[k]{|f(x)|} \cdot (\operatorname{sgn} f(x))^{1 + \operatorname{sgn}(i-1)}$,
- g_i is constant on $(-\infty, a]$ and $[b, \infty)$.

Observe that if $x \in K$, then

$$\big| g_i(x) - \sqrt[k]{|f(x)|} \cdot (\operatorname{sgn} f(x))^{1 + \operatorname{sgn}(i-1)} \big| \leq 2 \big| \tilde{f}(x) \big|.$$

Since f is Baire one, so the function $\sqrt[k]{|f|} \cdot (\operatorname{sgn} f)^{1 + \operatorname{sgn}(i-1)}$ belongs to Baire class one, too. (Cf. the proof of Claim 1 in Theorem 3.1.) But $\tilde{f}$ is a Baire one function which vanishes outside of a countable set, so g_i is also a Baire one function.

Let $x \in K$. If x is isolated in K from the left, then $g_i \restriction [x - \varepsilon_0, x]$ is Darboux and quasi-continuous for some $\varepsilon_0 > 0$. In the other case by (28) and the definition of g_i, there is a sequence $(x_p) \subset \mathcal{C}_{g_i}$ such that $x_p \nearrow x$ and $g_i(x_p) \to g_i(x)$. Similarly for each $x \in K$ there is a sequence $(t_p) \subset \mathcal{C}_{g_i}$ such that $t_p \searrow x$ and $g_i(t_p) \to g_i(x)$. Clearly g_i is Darboux and quasi-continuous on $\mathbb{R} \setminus K$. Thus g_i is both Darboux and quasi-continuous on $\mathbb{R}$. (Cf. Theorem 1.1 and [7] or [8, Lemma 2].) Hence $J \in \mathcal{J}$ and $\emptyset \neq P \subset G$, an impossibility. $\lhd$

Using Claims 9 and 4 one can easily prove the assertion of the theorem. $\square$

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