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OSCILLATION OF THE SOLUTIONS OF NONLINEAR IMPULSIVE DIFFERENTIAL EQUATIONS OF THE FIRST ORDER WITH ADVANCED ARGUMENT

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Abstract. Sufficient conditions for oscillation of all solutions of a class of nonlinear impulsive differential equations of the first order with advanced argument and fixed moments of impulse effect are found.

1. Introduction

By the aid of the impulsive differential equations various models of processes and phenomena observed in physics, economics, electrotechnics, etc., can be described. Due to this reason, in the recent years they have been an object of active research. In the monographs [1], [2], [3] a number of properties of their solutions are studied and an extensive bibliography is given.

Let us notice that in spite of the great number of investigations on the impulsive differential equations their oscillation theory has not been yet

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elaborated in contrast to the oscillation theory of the differential equations with a deviating argument (see the monographs [6], [7], [9] and the bibliography therein).

We shall notice that [5] is the first work where the oscillation theory of impulsive differential equations is studied.

In the present paper sufficient conditions for oscillation of all solutions of a class of nonlinear impulsive differential equations of the first order with advanced argument and fixed moments of impulse effect are found.

2. Preliminary notes

Let $\mathbb{N}_m = \{1, 2, ..., m\}$, h_i be positive constants, $i \in \mathbb{N}_{\gtrdot}$, $\bar{h} = \max\{h_i\colon i \in \mathbb{N}_{\gtrdot}\}$, $h = \min\{h_i\colon\ i \in \mathbb{N}_{\gtrdot}\}$, $\{\tau_k\}_{k=1}^{\infty}$ be a monotone increasing, unbounded sequence of positive numbers ($\tau_1 > \bar{h}$); $\{b_k\}_{k=1}^{\infty}$ be a sequence of real numbers, $\mathbb{R}_+ = (\nvdash, \infty)$.

We consider the following impulsive differential equations and inequalities:

$$\begin{aligned} &x'(t) + a(t)x(t) - p(t)f(x(t+h_1), \dots, x(t+h_m)) \le 0,\ t \ne \tau_k \\ &\Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k - 0) = b_k x(\tau_k - 0), \end{aligned} \tag{1}$$

where $x(\tau_k - 0) = x(\tau_k)$;

$$\begin{aligned} &x'(t) + a(t)x(t) - p(t)f(x(t+h_1), \dots, x(t+h_m)) \ge 0,\ t \ne \tau_k \\ &\Delta x(\tau_k) = b_k x(\tau_k), \end{aligned} \tag{2}$$

and

$$\begin{aligned} &x'(t) + a(t)x(t) - p(t)f(x(t+h_1), \dots, x(t+h_m)) = 0,\ t \ne \tau_k \\ &\Delta x(\tau_k) = b_k x(\tau_k). \end{aligned} \tag{3}$$

Introduce the following conditions:

H1. The functions $a,\ p\colon \overline{\mathbb{R}}_+ \to \mathbb{R}_+$ are piecewise continuous in $\overline{\mathbb{R}}_+ = [0, \infty)$ with points of discontinuity $\{\tau_k\}_{k=1}^{\infty}$, where the functions a and p are continuous from the left.

H2. $f \in C(\mathbb{R}^{\gtrdot}, \mathbb{R})$.

H3. If $u_k \ne 0$ and $\operatorname{sgn} u_1 = \operatorname{sgn} u_2 = \ldots = \operatorname{sgn} u_m$, then $u_k f(u_1, u_2, \dots, u_m) > 0$, $k \in \mathbb{N}_{\gtrdot}$.

H4. There exist constants $L > 0$ and $\alpha_1, \alpha_2, \dots, \alpha_m$, $\alpha_i \ge 0$ ($i \in \mathbb{N}_{\gtrdot}$) such that $\sum_{i=1}^{m} \alpha_i = 1$ and

$$|f(u_1, u_2, \dots, u_m)| \ge L|u_1|^{\alpha_1}|u_2|^{\alpha_2} \cdots |u_m|^{\alpha_m}.$$

H5. There exists a constant $M > 0$ such that for any $k \in N$ the inequalities $b_k \ge M$ are valid.

H6. There exists a constant $l > 0$ such that $i[a, a+h] \geq l$ $(0 < a < \infty)$, where $i[a, a+h]$ denotes the number of the points of jumps at the interval $[a, a+h]$.

Let us construct the sequence

$$\{t_k\}_{k=1}^{\infty} = \{\tau_k\}_{k=1}^{\infty} \cup \{\tau_{ki}\}_{k=1}^{\infty}, {}_{i=1}^{m},$$

where $\tau_{ki} = \tau_k - h_i$ and $t_k < t_{k+1}$, $k \in N$.

Definition 1. By a *solution* of the equation (3) we mean any function $x\colon\ [0, \infty) \to \mathbb{R}$ for which the following conditions are valid:

1. If $0 \leq t \leq \tau_1 - h$, then x coincides with the solution of the problem

$$x'(t) + a(t)x(t) - p(t)f(x(t+h_1), \dots, x(t+h_m)) = 0.$$

2. If $t_k < t \leq t_{k+1}$, $t_k \in \{\tau_k\}_{k=1}^{\infty} \setminus \{\tau_{ki}\}_{k=1}^{\infty}, {}_{i=1}^{m}$, then x coincides with the solution of the problem

$$x'(t) + a(t)x(t) - p(t)f(x(t+h_1), \dots, x(t+h_m)) = 0$$

$$x(t_k + 0) = (1 + b_{k_i})x(t_k),$$

where the number k_i is determined from the equality $\tau_{k_i} = t_k$.

3. If $t_k < t \leq t_{k+1}$, $t_k \in \{\tau_{ki}\}_{k=1}^{\infty}, {}_{i=1}^{m} \setminus \{\tau_k\}_{k=1}^{\infty}$, then the function x coincides with the solution of the problem

$$x'(t) + a(t)x(t+0) - p(t)f(x(t+h_1+0), \dots, x(t+h_m+0)) = 0$$

$$x(t_k + 0) = x(t_k).$$

4. If $t_k < t \leq t_{k+1}$, $t_k \in \{\tau_k\}_{k=1}^{\infty} \cap \{\tau_{ki}\}_{k=1}^{\infty}, {}_{i=1}^{m}$, then the function x coincides with the solution of the problem

$$x'(t) + a(t)x(t+0) - p(t)f(x(t+h_1+0), \dots, x(t+h_m+0)) = 0$$

$$x(t_k + 0) = (1 + b_{k_i})x(t_k).$$

Remark 1. The definition of solution of the inequality (1) (inequality (2)) is analogous to Definition 1.

Definition 2. The solution x of the inequality (2) is said to be *finally positive,* if there exists a point $t_0 > 0$ such that the solution x is defined for $t \geq t_0$ and $x(t) > 0$ for $t \geq t_0$.

In an analogous way the notion of *finally negative* solution can be introduced.

Definition 3. The nonzero solution x of the equation (3) is said to be *nonoscillatory* if there exists a point $t_0 > 0$ such that the function x is defined for $t \geq t_0$ and it does not change its sign for $t \geq t_0$. Otherwise, the solution x is called *oscillatory.*

3. Main results

Theorem 1. Let the following conditions hold:

1. Conditions *H1* — *H6* are satisfied.
2. $\liminf\limits_{t\to\infty} \int\limits_t^{t+h_i} (-a(s))ds = k_i = \text{const},\ i \in \mathbb{N}_{\geqslant}.$
3. $\liminf\limits_{t\to\infty} \int\limits_t^{t+h} p(s)ds > \dfrac{1}{Le^{k+1}(1+M)^{2l}},$

where $k = \min\left\{k_i\colon\ i \in \mathbb{N}_{\geqslant}\right\}$.

Then:

1. The inequality (1) has no finally negative solutions.
2. The inequality (2) has no finally positive solutions.
3. All solutions of the equation (3) are oscillatory.

Proof of 2. Let us suppose the opposite, i.e., that $x(t)$ is a finally positive solution of the inequality (2).

Let $x(t) > 0$ for $t \geq t_0 > 0$. It is clear that $x(t+h_i) > 0$ $(i \in \mathbb{N}_{\geqslant})$ and $f(x(t+h_1), \ldots, x(t+h_m)) > 0$ for $t \geq t_0$.

Let $t > T \geq t_0$. We multiply (2) by $e^{\int_T^t a(s)ds}$ and obtain

$$\left(x(t)e^{\int_T^t a(s)ds}\right)' - p(t)e^{\int_T^t a(s)ds} f(x(t+h_1), \ldots, x(t+h_m)) \geq 0. \qquad (4)$$

We set

$$z(t) = x(t)e^{\int_T^t a(s)ds}, \quad t > T, \qquad (5)$$

and from (4) it follows that

$$z'(t) - p(t)e^{\int_T^t a(s)ds} f\left(z(t+h_1)e^{-\int_T^{t+h_1} a(s)ds}, \ldots, z(t+h_m)e^{-\int_T^{t+h_m} a(s)ds}\right) \geq 0,$$

$$\Delta z(\tau_k) = z(\tau_k + 0) - z(\tau_k) = b_k x(\tau_k)e^{\int_T^{\tau_k} a(s)ds} = b_k z(\tau_k). \qquad (6)$$

From (5) it follows that $z(t) > 0$ for $t \geq T$. Then $z(t+h_i) > 0$ ($i \in \mathbb{N}_{\geqslant}$) and

$$f\left(z(t+h_1)e^{-\int\limits_T^{t+h_1} a(s)ds}, \ldots, z(t+h_m)e^{-\int\limits_T^{t+h_m} a(s)ds}\right) > 0 \quad \text{for} \quad t > T.$$

From the above inequalities, conditions H2, H5 and from (6) it follows that $z'(t) > 0$ for $t \geq T$. Therefore, z is an increasing function in the set $J = [T, \tau_s) \cup \left[\bigcup_{i=s}^{\infty}(\tau_i, \tau_{i+1})\right]$, $\tau_{s-1} < T < \tau_s$.

We introduce the notation

$$w(t) = \frac{z(t+h)}{z(t)}, \quad t \geq T.$$

Let us fix t ($t \geq T$) and renumber the points of jump of the following way:

$$t < \tau_1 < \tau_2 < \ldots < \tau_\lambda < t+h \leq \tau_{\lambda+1}, \quad \lambda \geq l.$$

Then

$$z(t) \leq z(\tau_1) = \frac{z(\tau_1+0)}{1+b_1} \leq \cdots \leq \frac{z(t+h)}{\prod_{i=1}^{\lambda}(1+b_i)} \leq \frac{z(t+h)}{(1+M)^l},$$

i.e.,

$$w(t) = \frac{z(t+h)}{z(t)} \geq (1+M)^l, \qquad t \geq T. \tag{7}$$

We shall prove now that the function w is bounded from above for $t \geq T$. Let t^* be an arbitrary point such that $t^* \geq T$.

From condition 3 it follows that there exists a constant $N > 0$ such that

$$\int\limits_t^{t+h} p(s)ds \geq N > \frac{1}{Le^{k+1}(1+M)^{2l}}$$

for sufficiently large t ($t \geq T$). Therefore, there exists a point $t \geq T$ such that $t < t^* < t+h$ and

$$\int\limits_t^{t^*} p(s)ds \geq \frac{N}{2} \quad \text{and} \quad \int\limits_{t^*}^{t+h} p(s)ds > \frac{N}{2}.$$

Integrate (6) from t to t^* and obtain that

$$z(t^*) - z(t) - \sum_{\tau_k \in (t,t^*)} b_k z(\tau_k) \tag{8}$$

$$\geq L \int_t^{t^*} p(s) e^{\int_T^s a(u)du} \prod_{i=1}^m z^{\alpha_i}(s+h_i) \prod_{i=1}^m e^{-\alpha_i \int_T^{s+h_i} a(u)du} ds.$$

Moreover,

$$e^{\int_T^s a(u)du} \prod_{i=1}^m e^{-\alpha_i \int_T^{s+h_i} a(u)du} = e^{\sum_{i=1}^m \alpha_i \int_s^{s+h_i} (-a(u))du} \tag{9}$$

$$\geq e^{\sum_{i=1}^m \alpha_i \liminf_{s\to\infty} \int_s^{s+h_i} (-a(u))du} \geq e^k.$$

From the fact that z is an increasing function in the set J it follows that

$$z(s+h) \leq \frac{z(s+h_i)}{(1+b_k)^{i[s+h,s+h_i]}} \leq \frac{z(s+h_i)}{(1+M)^l}. \tag{10}$$

From (8), (9), and (10) we obtain that

$$z(t^*) - z(t) - \sum_{\tau_k \in (t,t^*)} b_k z(\tau_k) \geq Le^k \int_t^{t^*} p(s) \prod_{i=1}^m z^{\alpha_i}(s+h_i) ds \tag{11}$$

$$\geq Le^k \int_t^{t^*} p(s)(1+M)^l z^{\sum_{i=1}^m \alpha_i}(s+h) ds$$

$$= Le^k (1+M)^l \int_t^{t^*} p(s) z(s+h) ds \geq Le^k (1+M)^l \frac{N}{2} \inf_{s\in[t,t^*]} z(s+h)$$

$$\geq Le^k (1+M)^l \frac{N}{2} z(t+h).$$

Integrate (6) from t^* to $t+h$ and obtain that

$$z(t+h) - z(t^*) - \sum_{\tau_k \in (t^*,t+h)} b_k z(\tau_k) \tag{12}$$

$$\geq Le^k (1+M)^l \frac{N}{2} \inf_{s\in[t^*,t+h]} z(s+h) = Le^k (1+M)^l \frac{N}{2} z(t^*+h).$$

It follows from (11) that

$$z(t^*) \geq Le^k(1+M)^l \frac{N}{2} z(t+h). \tag{13}$$

From (12) it follows that

$$z(t^*+h) \leq \frac{2}{NLe^k(1+M)^l} z(t+h). \tag{14}$$

From (13) and (14) there follows the inequality

$$z(t^*+h) \leq \left(\frac{2}{LNe^k(1+M)^l} \right)^2 z(t^*).$$

It follows from the above inequality that the function w is bounded from above at an arbitrary chosen point $t^* \geq T$. Hence, it is bounded for $t \geq T$.

For each $t \geq T$, there exist numbers $p = p(t)$ and $q = q(t)$ such that

$$\tau_p < t \leq \tau_{p+1} < \tau_{p+2} < \ldots < \tau_{p+q} < t+h \leq \tau_{p+q+1}.$$

It follows from condition H6 the validity of the following estimate

$$q = q(t) \geq l, \quad t \geq T.$$

Divide (6) by $z(t) > 0$ $(t \geq T)$, integrate from t to $t+h$ and obtain that

$$Le^k(1+M)^l \int\limits_t^{t+h} p(s)w(s)ds = Le^k(1+M)^l \int\limits_t^{t+h} p(s)\frac{z(s+h)}{z(s)}ds$$

$$\leq \int\limits_t^{t+h} \frac{z'(s)}{z(s)}ds = \int\limits_t^{\tau_{p+1}} \frac{z'(s)}{z(s)}ds + \int\limits_{\tau_{p+1}}^{\tau_{p+2}} \frac{z'(s)}{z(s)}ds + \ldots + \int\limits_{\tau_{p+q-1}}^{\tau_{p+q}} \frac{z'(s)}{z(s)}ds + \int\limits_{\tau_{p+q}}^{t+h} \frac{z'(s)}{z(s)}ds$$

$$= (\ln z(\tau_{p+1}) - \ln z(t)) + (\ln z(\tau_{p+2}) - \ln z(\tau_{p+1}+0)) + \ldots$$

$$+ (\ln z(\tau_{p+q}) - \ln z(\tau_{p+q-1}+0)) + (\ln z(t+h) - \ln z(\tau_{p+q}+0))$$

$$= \ln \frac{z(t+h)}{z(t)} + \sum_{j=1}^{q} \ln \frac{z(\tau_{p+j})}{z(\tau_{p+j}+0)}$$

$$= \ln w(t) + \sum_{j=1}^{q} \ln \frac{z(\tau_{p+j})}{(1+b_{p+j})z(\tau_{p+j})}$$

$$\leq \ln w(t) + \sum_{j=1}^{q} \ln \frac{1}{1+M} = \ln \frac{w(t)}{(1+M)^q} \leq \ln \frac{w(t)}{(1+M)^l}.$$

Introduce the notation $w_0 = \liminf_{t\to\infty} w(t)$. It follows from the boundedness of w that $0 < w_0 < \infty$.

By virtue of the last inequality, we arrive at

$$\liminf_{t\to\infty} \int_t^{t+h} p(s)ds \le \frac{1}{Le^k(1+M)^l} \frac{\ln(w_0(1+M)^{-l})}{w_0}. \tag{15}$$

Finally, bearing in mind the fact that for each $x > 0$ we have the inequality $\ln x \le x/e$, we obtain

$$\frac{\ln(w_0(1+M)^{-l})}{w_0} \le \frac{(1+M)^{-l}}{e},$$

whence, with the aid of (15) we get into

$$\liminf_{t\to\infty} \int_t^{t+h} p(s)ds \le \frac{1}{Le^k(1+M)^l} \frac{(1+M)^{-l}}{e} = \frac{1}{Le^{k+1}(1+M)^{2l}}.$$

The last inequality contradicts condition 3 of Theorem 1.

In order to prove that (1) has no finally negative solution, it is sufficient to notice that if x is a solution of (2), then $-x$ is a solution of (1).

It follows from assertions 1 and 2 of Theorem 1 that the equation (3) has neither finally positive, nor finally negative solutions, i.e., each solution of the equation (3) is oscillatory. □

Theorem 2. Let the following conditions hold:

1. Conditions *H1* — *H6* are satisfied.
2. $\liminf_{t\to\infty} \int_t^{t+h_i} (-a(s))ds = k_i,$

$k_i = \text{const},\ i \in \mathbb{N}_{\geqslant}$.

3. $\limsup_{t\to\infty} \int_t^{t+h} p(s)ds > \frac{1}{Le^k(1+M)^{2l}},$

where $k = \min\left\{k_i :\ i \in \mathbb{N}_{\geqslant}\right\}$.

Then:

1. The inequality (1) has no finally negative solutions.
2. The inequality (2) has no finally positive solutions.
3. All solutions of the equation (3) are oscillatory.

Proof of 2. Let $x(t)$ be a finally positive solution of the inequality (2) for $t \ge t_0 > 0$.

It is clear that $x(t+h_i) > 0$ $(i \in \mathbb{N}_{\geqslant})$ and

$$f(x(t+h_1), \dots, x(t+h_m)) > 0 \quad \text{for} \quad t \ge t_0.$$

Analogously to the proof of Theorem 1 we arrive at (6).

It follows from (6), condition H5 as well as from condition 2 of Theorem 2 that the inequality

$$\begin{aligned} &z'(t) \ge Le^k(1+M)^l p(t) \prod_{i=1}^{m} z^{\alpha_i}(t+h_i) \\ &\Delta z(\tau_k) = b_k z(\tau_k) \end{aligned} \tag{16}$$

is valid.

Now, the Theorem on the impulsive differential inequalities ([1], Theorem 2.3) as well as (16) imply the inequality

$$\begin{aligned} z(t+h) \ge & z(t) \prod_{t \le \tau_k < t+h} (1+b_k) \\ & + Le^k(1+M)^l \int_t^{t+h} p(s) \prod_{s \le \tau_k < t+h} (1+b_k) z(s+h) ds. \end{aligned} \tag{17}$$

On the other hand, z is an increasing function on the set $J = (T, \tau_s) \cup \left[\bigcup_{i=s}^{\infty} (\tau_i, \tau_{i+1})\right]$, $\tau_{s-1} < T < \tau_s$, $t \ge T$.

Hence

$$\inf_{s \in [t, t+h)} z(s+h) = z(t+h+0). \tag{18}$$

It follows from (17) and (18) the inequality

$$z(t) \prod_{t \le \tau_k < t+h} (1+b_k) + z(t+h) \left[Le^k(1+M)^{2l} \int_t^{t+h} p(s) ds - 1 \right] \le 0.$$

The last inequality contradicts condition 3 of Theorem 2. The proofs of the assertions 1 and 3 are analogous to the corresponding proofs in Theorem 1. □

4. An application

In 1984, H. Onose obtained sufficient conditions for oscillation of all solutions of the equation without impulses

$$x'(t) + a(t)x(t) - p(t) f(x(t-h_1), \dots, x(t-h_m)) = 0, \tag{19}$$

where the functions $a,\ p:\ \overline{\mathbb{R}}_+ \to \mathbb{R}_+$, $f: \mathbb{R}^{\geqslant} \to \mathbb{R}$ and $h_1 > 0, \dots, h_m > 0$ (see [8]).

The equation (3), considered by us, differs from (19) in two directions. First of all, f in (3) depends on the unknown function, calculated in m different advanced arguments. Concerning (19), the unknown function depends on delaying arguments. Next, in the case of missing delay or advance, that is, $h_1 = \ldots = h_m = 0$, the equation considered here generalizes (19). In fact, the solution of (3) is subject of impulsive perturbations. It is easy to see that if the impulsive perturbations equal zero, i.e., $b_1 = \ldots = b_m = 0$, then (3) reduces to the equation (19) without impulses and delay.

A particular case of (19) is the model equation

$$x'(t) + rx(t) - \frac{\alpha\beta(x(t-h_1))^{n_1}}{\beta + (x(t-h_1))^{n_2}} = 0, \tag{20}$$

considered by K. Gopalsamy in [4] (Chapter 1, Exercise 1). The equation (20) models *haematopoiesis.* The constants r, α, β and $h_1 \in \mathbb{R}_+$; n_1 and n_2 are positive integers; $x = x(t)$, $0 \le t \le T$ represents the quantity of the blood at the moment t.

We will consider here the model equation describing the change of blood quantity:

$$x'(t) + rx(t) - \frac{\alpha\beta x(t+h_1)}{\beta + (x(t+h_1))^2} = 0. \tag{21}$$

Here $n_1 = 1$ and $n_2 = 2$. Let us note that in (21) the momentary change of the blood quantity depends not only on its present value, but also on the value at the future moment $h_1 > 0$.

We will study the change of the blood quantity at a large enough period of time $t \in [0, T]$. It will be supposed that at the initial moment $t = 0$ one has

$$x(0) = x_0, \tag{22}$$

where $x_0 \in \mathbb{R}_+$. The equality (22) is called an initial condition.

Suppose that the blood quantity in the time interval $[T, T + h_1]$ satisfies

$$x(t) = \varphi(t), \quad T \le t \le T + h_1, \tag{23}$$

where $\varphi : [T, T+h_1] \to \mathbb{R}_+$ is a given continuous function. From a practical point of view, the function φ is chosen in such a way that the blood quantity $\varphi(t)$ for $t \in [T, T + h_1]$ is the best possible for the donor. The equality (23) will be called final condition.

The model, described above, shows that the equations with advanced argument are an adequate mathematical model for simulation of processes having a future development.

The finding of analitical solutions of the final problems for equations with advanced argument (as well as of the problem (21), (23)) is impossible, in general. That is why, this kind of problems are to be solved numerically. Its solution is obtained in a similar manner as for initial problems for equations with retarded argument. The difference between the two kind of problems consists of the fact that when solving final problems for equations with avdanced arguments, the solution is approximated successively at points, initiating at the final T and tending to the origin 0. That is, the step between the successive points is negative one. When solving initial problems with retarded arguments, however, the situation is opposite.

In general, the solution of the final problem (21), (23) at the moment 0 does not coincide with the initial condition x_0. From a practical point of view, it means that if at the initial moment $t = 0$ the blood quantity equals x_0, it cannot reach the value $x(t) = \varphi(t)$ for $t \in [T, T + h_1]$ without external actions.

The external actions may be of two kinds. First, they are actions on the environment, which reflect on the right-hand side of the equation. Later, they may be perturbations on the quantity of the object under consideration. This leads to discontinuities of the solution to the considered equation.

When modelling the change of blood quantity, the external actions cannot be connected to the environment. That is why, they result as "giving-in" or "getting-out" of blood to/from the donor. The duration of these external actions is very small in comparison to the whole duration of the process and therefore, we may assume that the external actions take place as impulses

$$\Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k) = b_k x(\tau_k), \quad k = 1, 2, \dots . \tag{24}$$

It is natural to assume that the successive impulsive perturbations (giving-in/getting-out of blood) take place in limited time-intervals, i.e., there exist constants $\Delta\tau$ and ΔT such that

$$0 < \Delta\tau \le \tau_{k+1} - \tau_k \le \Delta T, \quad k = 1, 2, \dots . \tag{25}$$

Moreover, we suppose that also the constants b_k are bounded. That is, there exist constants b and B such that

$$0 < b \le |b_k| \le B, \quad k = 1, 2, \dots .$$

Now, assuming

$$r < \alpha, \tag{26}$$

the equation (21) possesses a constant, positive, stable equilibrium

$$x(t) = x^* = \sqrt{\frac{\beta(\alpha - r)}{r}},$$

which results as a solution of the equation

$$rx^* - \frac{\alpha\beta x^*}{\beta + (x^*)^2} = 0.$$

We will investigate the equation (21) in a neighbourhood of the solution $x(t) = x^*$. For this goal, we substitute $x(t) = \chi(t) + x^*$. Thus, the equation (21) reads

$$\chi'(t) + r\chi(t) + rx^* - \frac{\alpha\beta(\chi(t+h_1) + x^*)}{\beta + (\chi(t+h_1) + x^*)^2} = 0, \tag{27}$$

while the equality (24) takes on the form

$$\Delta\chi(\tau_k) = b_k(\chi(\tau_k) + x^*) = \beta_k \chi(\tau_k), \quad k = 1, 2, \dots, \tag{28}$$

where the constants β_k are specific for each solution $\chi(t)$.

We will apply Theorem 1 to the impulsive equation (27), (28). To check the conditions of that theorem we set

$$a(t) = r, \quad p(t) = 1, \quad m = 1, \quad h = h_1,$$

$$f(\chi(t+h_1), \dots, \chi(t+h_m)) = f(\chi(t+h_1)) = \frac{\alpha\beta(\chi(t+h_1) + x^*)}{\beta + (\chi(t+h_1) + x^*)^2} - rx^*.$$

Conditions H1 and H2 are checked directly.

Bearing in mind the nature of the solution to the equation (27), (28), we suppose that it lies at a sufficiently small neighbourhood of 0. Otherwise, the donor would not survive. Let us notice that the solution $\chi(t)$ represents the oscillation of the blood quantity near the equilibrium. Therefore, this oscillation cannot be "large enough". In other words, there exists a constant $\Delta\chi > 0$ such that

$$-\Delta\chi < \chi(t) < \Delta\chi, \quad t \geq 0.$$

Condition 3 would be satisfied if the inequality $uf(u) > 0$ holds true for $-\Delta\chi < u < \Delta\chi$. We have

$$f(u) = \frac{\alpha\beta(u + x^*)}{\beta + (u + x^*)^2} - rx^*.$$

First of all, we will show that

$$f(0) = \left(\frac{\alpha\beta}{\beta + (x^*)^2} - r\right)x^* = \left(\frac{\alpha\beta}{\beta + \beta(\alpha - r)/r} - r\right)x^* = 0. \tag{29}$$

If

$$r > \frac{\alpha}{2}, \tag{30}$$

then we obtain the estimate

$$f'(0) = \alpha\beta \left.\frac{\beta + (u + x^*)^2 - 2(u + x^*)^2}{(\beta + (u + x^*)^2)^2}\right|_{u=0}$$

$$= \frac{\alpha\beta^2}{r(\beta + (x^*)^2)^2}(2r - \alpha) > 0.$$

It follows from (29), (30) and the continuity of the function f, that f has the sign of u in a sufficiently small neighbourhood of 0. Without loss of generality we may suppose that if $\alpha/2 < r < \alpha$ (cf. (26) and (30)), then we have

$$uf(u) > 0, \quad -\Delta\chi < u < \Delta\chi,$$

which establishes condition H3.

Since f' is a continuous function and $f'(0) > 0$, then in a small enough neighbourhood of 0 the zero of f' preserves its sign. Without loss of generality we may suppose that there exists a constant $L > 0$ such that

$$\inf_{-\Delta\chi<u<\Delta\chi} |f'(u)| = \inf_{-\Delta\chi<u<\Delta\chi} f'(u) = L.$$

Then, for $-\Delta\chi < u < \Delta\chi$ one has

$$|f(u)| = |f(u) - f(0)| = |f'(u^*)|\,|u - 0| \geq L|u|,$$

where u^* lies between 0 and u. The last inequality implies H4.

Let us suppose that the external impulsive effects are bounded from below. This means, that the quantities of the given-in or getting-out blood at each action are greater than some given volume (called a minimal volume). This assumption can be described analitically in the following manner: there exists a constant $M > 0$ such that $\beta_k > M$, $k = 1, 2, \dots$. This proves the validity of condition H5.

It follows from the inequality (25) that

$$i[a, a+h] \geq \frac{h_1}{\Delta T} + 1 = l, \quad a \geq 0,$$

which shows condition H6.

Condition 2 of Theorem 1 follows directly:

$$\liminf_{t\to\infty} \int_t^{t+h} (-a(s))ds = \lim_{t\to\infty} - \int_t^{t+h_1} r ds = -rh_1 = k_1 = k.$$

Condition 3 of the same theorem is fulfilled if the following inequality holds true:

$$\liminf_{t\to\infty} \int_t^{t+h_1} p(s)ds > \frac{1}{Le^{k+1}(1+M)^{2l}}.$$

Since we have $p(t) = 1$, $t \geq 0$, then condition 3 holds if

$$h_1 L e^{1-rh_1}(1+M)^{2(h_1/\Delta T+1)} > 1.$$

To conclude, we have from Theorem 1 that:

1. There exist constants $\Delta\tau,\ \Delta T > 0$ such that

$$\Delta\tau \leq \tau_{k+1} - \tau_k \leq \Delta T, \quad k = 1, 2, \dots .$$

This means that the external impulsive interventions (giving-in or getting-out of blood) take place discretely in the time, without moments of accumulation of the interventions.

2. The oscillation of the blood around the equilibrium $x^* = \sqrt{\beta(\alpha - r)/r}$ is small enough. This assumption becomes natural from the biological point of view. Therefore, there exists a constant $\Delta\chi > 0$ such that the deviation $\chi(t)$ from the equilibrium satisfies the inequalities

$$-\Delta\chi < \chi(t) < \Delta\chi, \quad t \geq 0.$$

3. $\dfrac{\alpha}{2} < r < \alpha$.

4. There exists a constant $M > 0$ such that

$$\beta_k \geq M, \quad k = 1, 2, \ldots .$$

5. There exists a constant $L > 0$ such that

$$\inf_{-\Delta\chi < u < \Delta\chi} \alpha\beta \frac{\beta - (u + x^*)^2}{(\beta + (u + x^*)^2)^2} = L.$$

6. $h_1 L e^{1 - r h_1} (1 + M)^{2(h_1/\Delta T + 1)} > 1$.

Then, the solutions of the equation (21), which are sufficiently close to its equilibrium x^* and are object of the external impulsive perturbations (24), oscillate around x^*.

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