Journal of Applied Analysis
Vol. 6, No. 1 (2000), pp. 47–75

DIFFERENTIAL CALCULUS FOR COMPLEX-VALUED MULTIFUNCTIONS

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Received May 5, 1998 and, in revised form, June 17, 1999

Abstract. Using the concept of the normal cone to a multifunction we define a derivative for a complex-valued multifunction of one complex variable being a natural generalization of the ordinary complex derivative for holomorphic functions. Using results obtined by Mordukhovich, we develop a full calculus and discuss openness and Lipschitzian properties. We also prove the fundamental theorem of calculus and the Taylor expansion formula. Finally we discuss analyticity of multifunctions in the context of the normal cone.

1. Introduction

There are many notions of derivative of multifunctions, and with many applications: we must differentiate set-valued functions just as we must differentiate ordinary functions. The purpose of this paper is certainly not to survey the area (there are excellent books, e.g. [1, 11]) or to come up with a new and revolutionary notion of differentiability. Instead we shall concentrate on Calculus.

Calculus, as developed by Newton and Leibniz amongst others, might be defined in the following way (compare [7]).

1991 *Mathematics Subject Classification.* 58C06, 58C20.

Key words and phrases. Set-valued function, multifunction calculus, differential calculus, Barrow's theorem, Taylor's theorem, complex-valud.

Part of this work was done under a grant from The Royal Swedish Academy of Sciences.

ISSN 1425-6908

Definition 1.1. Calculus consists of the following rules of differentiation

1. $(f+g)' = f' + g'$;
2. $(af)' = af'$;
3. $(fg)' = fg' + gf'$;
4. $(1/f)' = -f'/f^2$;
5. $(f^{-1})' = 1/(f' \circ f)$;
6. $(f \circ g)' = (f' \circ g)g'$,

together with the fundamental theorem of calculus, or Barrow's[1] theorem:

$$f(x) = f(a) + \int_a^x f'(t)\, dt.$$

In this paper we shall consider a complex valued multifunction of one complex variable, define a pointwise derivative of this multifunction (being of course a generalization of the ordinary complex derivative for a function), and, under natural assumptions, prove generalizations of all the formulas in Definition 1.1 — *including the fundamental theorem of calculus* (see Theorem 4.13 and Theorem 8.1 respectively). As a corollary we obtain a Taylor expansion formula (see Theorem 8.2). We shall also calculate explicitly the derivative for some classes of multifunctions, discuss openness and covering properties in terms of the derivative, and briefly discuss analyticity of multifunctions in this context. The results in this paper are, at least to the author's knowledge, new in two respects: The notion of derivative have not been considered for complex-valued multifunctions and the fundamental theorem of calculus with its corollary the Taylor expansion formula have not been explicitly noted.

2. Notation

Let X and Y be sets. The symbol $F\colon X \rightrightarrows Y$ means that F is a multifunction from X to Y, i.e., a function from X to the set of all subsets of Y. We fix the following notation:

dom $F = \{x \in X;\, F(x) \neq \emptyset\}$ is the domain of F;
im $F = \{y \in Y;\, y \in F(x)$ for some $x \in X\}$ is the range of F;
gph $F = \{(x, y) \in X \times Y;\, y \in F(x)\}$ is the graph of F;
$F(A) = \{y \in Y;\, y \in F(x)$ for some $x \in A\}$ is the image of the set A under the multifunktion F;
$F^{-1}(y) = \{x \in X;\, y \in F(x)\}$ is the inverse of F;
ker $F = \{x \in X;\, 0 \in F(x)\}$ is the kernel of F.

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[1] Isaac Barrow, 1630–1677, Isaac Newton's teacher

Let X and Y be finite dimensional vector spaces and let $F\colon X \rightrightarrows Y$. F is *upper semi-continuous* if $\{x \in \operatorname{dom} F;\ F(x) \subset U\}$ is open for every open set U in Y. F is *locally bounded* if for every $x \in X$ there exists a neighborhood $N \ni x$ so that $F(N) \subset\subset Y$ (i.e., $F(N)$ is compactly enclosed in Y). An easy proposition is that F is upper semi-continuous and compact-valued if and only if F is locally bounded and gph F is closed (in dom $F \times Y$). We will use this equivalence often.

With the convention that an arithmetic operation performed with the empty set involved results in the empty set we can for real-valued and complex-valued multifunctions F and G define

$$(F+G)(x) = F(x) + G(x) = \{y + y';\ y \in F(x),\ y' \in G(x)\};$$
$$(FG)(x) = F(x)G(x) = \{yy';\ y \in F(x),\ y' \in G(x)\};$$
$$(1/F)(x) = 1/F(x) = \{1/y;\ 0 \neq y \in F(x)\}.$$

We reserve the symbol $F^n(x)$ for the multifunction with values $\{y^n;\ y \in F(x)\}$ in analogy with $nF(x) = \{ny;\ y \in F(x)\}$. Also define

$$|F(x)| = \sup\big(|y|;\ y \in F(x)\big).$$

B denotes the closed unit ball in $\mathbf{C}$ and S its boundary, i.e., $B = \{z \in \mathbf{C};\ |\mathbf{z}| \leq \mathbf{1}\}$ and $S = \{z \in \mathbf{C};\ |\mathbf{z}| = \mathbf{1}\}$. We also introduce the ball multifunction $B(z) = z + B$ and $B_r(z) = rB(z/r) = z + rB$. As a convention, capital letters will denote multifunctions whereas small letters will denote functions.

3. The normal and the coderivative

Given a closed set Γ in $\mathbf{R^n}$ we follow Mordukhovich in [9] and define the normal cone at $x_0 \in \Gamma$ as the set

$$N(x_0) = N(x_0;\Gamma) = \operatorname*{Lim\,sup}_{x \to x_0} \operatorname{cone}(x - \operatorname{proj}(x;\Gamma)), \tag{3.1}$$

where

$$\operatorname{proj}(x;\Gamma) = \{y \in \Gamma;\ |x-y| = \operatorname{dist}(x,\Gamma)\}$$

is the set of closest points in Γ to x, cone (A) is the conic hull of the set A (i.e., $\{cx;\ x \in A,\ c \geq 0\}$), and for any multifunction Φ,

$$\operatorname*{Lim\,sup}_{x \to x_0} \Phi(x) = \{y;\ \exists x_n \to x_0\ \exists y_n \to y \text{ so that } y_n \in \Phi(x_n)\}$$

is the Kuratowski–Painlevé upper limit of Φ as $x \to x_0$. Directly from the definition one can conclude that the normal is robust in the sense that

$$\operatorname*{Lim\,sup}_{\Gamma \ni x \to x_0} N(x;\Gamma) = N(x_0;\Gamma).$$

It is also easy to see that given two closed sets $\Gamma_1 \subset \mathbb{R}$ and $\Gamma_2 \subset \mathbb{R}^m$, we have

$$N((x_0, y_0); \Gamma_1 \times \Gamma_2) = N(x_0; \Gamma_1) \times N(y_0; \Gamma_2) \tag{3.2}$$

for all points $(x_0, y_0) \in \Gamma_1 \times \Gamma_2$.

If Γ is a convex set then the above normal cone coincides with the classical normal cone of convex analysis, but for a general Γ the normal cone (3.1) may be non-convex. Thus it is not a dual object to any tangent cone.

There are other ways of defining the normal to Γ. For each $x_0 \in \Gamma$ and $\epsilon \geq 0$ consider the so-called set of Fréchet ϵ-normals to Γ at x_0 defined by

$$\hat{N}_\epsilon(x_0; \Gamma) = \{x \in \mathbb{R};\ \limsup_{\Gamma \ni x' \to x_0} \langle x, x' - x_0 \rangle / |x' - x_0| \leq \epsilon\}$$

and put $\hat{N}_\epsilon(x_0; \Gamma) = \emptyset$ if $x_0 \notin \Gamma$. The following representation of the normal cone was first obtained in [8] and also in [6]:

$$N(x_0; \Gamma) = \operatorname*{Lim\,sup}_{x \to x_0} \hat{N}_0(x; \Gamma) = \operatorname*{Lim\,sup}_{x \to x_0,\, \epsilon \to 0} \hat{N}_\epsilon(x; \Gamma).$$

Hörmander (see for instance [5]) defines a normal set to a closed set in $\mathbf{R^n}$ (in fact for closed sets of a C^2-manifold, but it will not concern us here) thus: given a closed set $\Gamma \subset \mathbf{R^n}$ let $(x_0, \xi) \in N_e(\Gamma) \subset T^*(\mathbf{R^n})$ if and only if there exist $f \in C^2(\mathbf{R^n})$ such that

$$\begin{cases} 0 \neq \xi = \nabla f(x_0) \\ \max f|_\Gamma \leq f(x_0). \end{cases}$$

This is not defined everywhere on $\partial\Gamma$, but the projection onto the first coordinate of $N_e(\Gamma)$ is dense in $\partial\Gamma$. It turns out that this construction is more or less the same as the construction above. In fact if we define $N(\Gamma) = \{(x_0, \xi) \in T^*\mathbf{R^n};\ \xi \in \mathbf{N(x_0)}\}$, we have the following

Theorem 3.1. *$N(\Gamma) = \text{clos}\ N_e(\Gamma)$, where the closure is to be taken in the cotangent bundle $T^*\mathbf{R^n}$.*

The proof is not very hard and could be found in [3].

Remarks.

1. The definition of N_e is local since if f is defined only near x_0 we can replace it by $\phi(x)f(x) + (1 - \phi(x))f(x_0)$, with $0 \leq \phi \in C^2$ being equal to 1 near x_0. Moreover, if we replace f by $f_2(x) - |x - x_0|^2$, where f_2 is the second order Taylor expansion at x_0, we see that we can assume f to be (real) analytic, and strictly smaller than $f(x_0)$ when $x \neq x_0$.

2. It is not hard to prove (see [5] again) that if $f \in C^1$, $\nabla f(x_0) = x$ and $\max f|_\Gamma \leq f(x_0)$ then $(x_0, x) \in \operatorname{clos} N_e(\Gamma)$, so for the purpose of Theorem 3.1, we could have assumed that f is only C^1 in the definition of N_e.
3. Similar constructions like that of Hörmander were studied by Crandall and Lions in [4]. They also lead to the normal cone (3.1) via a limiting procedure.

Given a multifunction $\Phi\colon \mathbf{R^n} \rightrightarrows \mathbf{R^k}$ with closed graph Γ, Mordukhovich defines the *coderivative* at a point $(x_0, y_0) \in \Gamma$ evaluated at $y \in \mathbf{R^k}$ as

$$D^*\Phi(x_0, y_0)(y) = \big\{x \in \mathbf{R^n};\ (\mathbf{x}, -\mathbf{y}) \in \mathbf{N}\big((\mathbf{x_0}, \mathbf{y_0}); \Gamma\big)\big\}.$$

The coderivative could be thought of as a generalization of the transpose of the Jacobian for functions, since from [9] we have the following

Theorem 3.2. If $\Phi\colon \mathbf{R^n} \rightrightarrows \mathbf{R^k}$ is single valued and strictly differentiable, then

$$D^*\Phi(x_0, \Phi(x_0))(y) = \{(\nabla\Phi(x_0))^* y\},$$

where $\nabla\Phi(x_0)$ is the Jacobian of Φ at x_0 and $*$ denotes the transpose.

Strictly differentiable means that

$$\lim_{x, x' \to x_0} \frac{\Phi(x) - \Phi(x') - \nabla\Phi(x_0)(x - x')}{|x - x'|} = 0.$$

The coderivative has a nice calculus and is also used to characterize certain openness and Lipschitzian properties of multifunctions. For a very rich theory concerning this and more see [9] and [10]. Many of the results in this paper rests on the results in [9] and [10].

Let $\Psi\colon \mathbf{R^n} \rightrightarrows \mathbf{R^j}$ and $\Phi\colon \mathbf{R^j} \rightrightarrows \mathbf{R^k}$ be two multifunctions and let their *composition* be defined by

$$\Phi \circ \Psi(x) = \Phi(\Psi(x)) = \bigcup_{y \in \Psi(x)} \Phi(y).$$

The following will be a key theorem in Section 4. For the proof we refer to [9].

Theorem 3.3 (Theorem 5.1 in [9])**.** Let Φ and Ψ have closed graphs and let $z_0 \in (\Phi \circ \Psi)(x_0)$. Assume that the multifunction $M\colon \mathbf{R^n} \times \mathbf{R^k} \rightrightarrows \mathbf{R^j}$ defined by

$$M(x, z) = \Psi(x) \cap \Phi^{-1}(z) = \{y \in \Psi(x);\ z \in \Phi(y)\}$$

is locally bounded around (x_0, z_0) and the qualification condition

$$D^*\Phi(y, z_0)(0) \cap \ker D^*\Psi(x_0, y) = \{0\} \text{ for all } y \in \Psi(x_0) \cap \Phi^{-1}(z_0) \quad (3.3)$$

is fulfilled. Then one has

$$D^*(\Phi \circ \Psi)(x_0, z_0) \subset \bigcup_{y \in \Psi(x_0) \cap \Phi^{-1}(z_0)} [D^*\Psi(x_0, y) \circ D^*\Phi(y, z_0)].$$

Let us introduce some notation.

Given two multifunctions $\Phi\colon \mathbf{R^n} \rightrightarrows \mathbf{R^k}$ and $\Psi\colon \mathbf{R^m} \rightrightarrows \mathbf{R^j}$ define $\Phi \otimes \Psi\colon \mathbf{R^{n+m}} \rightrightarrows \mathbf{R^{k+j}}$ by $(\Phi \otimes \Psi)(x, y) = \Phi(x) \times \Psi(y)$. If Φ and Ψ are defined in the same space let $(\Phi \times \Psi)(x) = \Phi(x) \times \Psi(x)$. If Φ and Ψ have values in the same space, define $(\Phi \oplus \Psi)(x, y) = \Phi(x) + \Psi(y)$.

Theorem 3.4. Let Φ and Ψ have closed graphs. Then

$$D^*(\Phi \otimes \Psi) = D^*\Phi \otimes D^*\Psi.$$

Proof. This is direct from (3.2) but notice the "twist" in the coordinates: gph $\Phi \otimes \Psi$ is not gph $\Phi \times$ gph Ψ but rather $\{(x, y, z, w);\ (x, z) \in \text{gph } \Phi$ and $(y, w) \in \text{gph } \Psi\}$. ☐

Theorem 3.5. Let Φ and Ψ have closed graphs and let $(y_0, z_0) \in (\Phi \times \Psi)(x_0)$. If

$$D^*\Phi(x_0, y_0)(0) \cap (-D^*\Psi(x_0, z_0)(0)) = \{0\}, \tag{3.4}$$

then

$$D^*(\Phi \times \Psi)(x_0, y_0, z_0) \subset D^*\Phi(x_0, y_0) \oplus D^*\Psi(x_0, z_0).$$

Proof. First note that $\Phi \times \Psi = (\Phi \otimes \Psi) \circ \Delta$, where Δ is the diagonal embedding $\Delta(x) = (x, x)$. We want to use Theorem 3.3 so we must check the assumptions in that theorem. Since $\Delta(x) = (x, x)$ it is clear that $M(x, y, z) = \Delta(x) \cap (\Phi \otimes \Psi)^{-1}(y, z)$ is locally bounded near every point $(x_0, (y_0, z_0)) \in \text{gph } \Phi \times \Psi$, so the first condition in the theorem is fulfilled. In fact, $\Delta(x) \cap (\Phi \otimes \Psi)^{-1}(y, z) = (x, x)$ if $(x, (y, z)) \in \text{gph } \Phi \times \Psi$ and is empty otherwise. Next the transpose of the Jacobian of the diagonal embedding is just addition of coordinates, i.e.,

$$D^*\Delta(x_0, (x_0, x_0))(a, b) = a + b,$$

so ker $D^*\Delta(x_0, (x_0, x_0)) = \{(x, -x)\}$. Therefore the negation of condition (3.3) means exactly that there is a non-zero x so that

$$(x, -x) \in D^*(\Phi \otimes \Psi)((x_0, x_0), (y_0, z_0))(0).$$

But

$$D^*(\Phi \otimes \Psi)((x_0, x_0), (y_0, z_0))(0) = D^*\Phi(x_0, y_0)(0) \times D^*\Psi(x_0, z_0)(0),$$

so condition (3.3) is exactly condition (3.4) in this setting. Hence Theorem 3.3 applies, and we get

$$D^*(\Phi\times\Psi)(x_0,y_0,z_0)\subset D^*\Delta(x_0,(x_0,x_0))\circ D^*(\Phi\otimes\Psi)((x_0,x_0),(y_0,z_0)) =$$

$$D^*\Delta(x_0,(x_0,x_0))\circ\big(D^*\Phi(x_0,y_0)\times D^*\Psi(x_0,z_0)\big) = D^*\Phi(x_0,y_0)\oplus D^*\Psi(x_0,z_0).$$

This proves the theorem. □

4. The derivative

Our aim is to use the coderivative (or the normal if you like) to define a generalization of the ordinary complex derivative of a complex-valued function of one complex variable. After this has been done, we will use Theorem 3.3 to develop calculus rules for this derivative.

Identifying $\mathbf{C}$ with $\mathbf{R^2}$ in the usual way allows us to speak about the normal and the coderivative in the above sense. For a holomorphic function $f\colon \mathbf{C}\to\mathbf{C}$ we know that the Jacobian is multiplication with f', i.e.,

$$(\nabla f(z))\mu = \mu f'(z).$$

Hence it follows from Theorem 3.2 that

$$D^*f(z,f(z))(\mu)=\mu\overline{f'(z)},$$

for all $\mu\in\mathbf{C}$, i.e., given a non-zero $\mu\in\mathbf{C}$ we have $\overline{f'(z)}=D^*f(z,f(z))(\mu)/\mu$. Thus we can write

$$f'(z)=\{y\in\mathbf{C};\ \exists\mu\neq\mathbf{0},\ \exists\mathbf{w}\in\mathbf{f}(\mathbf{z})\text{ with }\mu\overline{\mathbf{y}}\in\mathbf{D}^*\mathbf{f}(\mathbf{z},\mathbf{w})(\mu)\}. \tag{4.1}$$

Moreover, the coderivative is positively homogeneous so we need only consider μ with norm one. Formula (4.1)[2] makes sense for multifunctions also, so we define F' in this way and note that the definition is consistent with the definition of the complex derivative of a holomorphic function.

Definition 4.1. Given a closed graph multifunction $F\colon\mathbf{C}\rightrightarrows\mathbf{C}$ we define the derivative of F as

$$F'(z)=\{y\in\mathbf{C};\ \exists\mu\in\mathbf{S},\ \exists\mathbf{w}\in\mathbf{F}(\mathbf{z})\text{ with }\mu\overline{\mathbf{y}}\in\mathbf{D}^*\mathbf{F}(\mathbf{z},\mathbf{w})(\mu)\}. \tag{4.2}$$

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[2] In (4.1) we could have used "$\forall\mu\neq 0$" instead of "$\exists\mu\neq 0$", and this is in a way more natural because then the definition of f' says exactly that f' is $\mathbf{C}$-linear and this is the whole idea of having a complex derivative. However, having "$\forall\mu\neq 0$" in the definition of the derivative for multifunctions makes the derivative almost always empty. Thus we stick to the given definition.

Proposition 4.2. $F'(z) = \emptyset$ if and only if $N((z,w); \text{gph } F) \subset \mathbf{C}(\mathbf{1},\mathbf{0})$ for all $w \in F(z)$.

Proof. This is clear from the definition of F'. ☐

Proposition 4.3. If F is single-valued and strictly differentiable, then F' is single-valued if and only if F is holomorphic.

Proof. The "if" part is clear. For the other direction we notice that F is holomorphic if and only if the Jacobian is $\mathbf{C}$-linear, i.e., if and only if the transpose of the Jacobian is $\mathbf{C}$-linear. Thus if F is not holomorphic, there exists a $\mu \in \mathbf{C}$ such that $(\nabla F)^*(\mu) \neq \mu(\nabla F)^*(1)$. Since the Jacobian is $\mathbf{R}$-linear we can assume $\mu \in S$. Define z_0 and z_1 by

$$\overline{z_0} = \frac{(\nabla F)^*(\mu)}{\mu}, \ \overline{z}_1 = (\nabla F)^*(1),$$

where the bar denotes complex conjugate. Then $\mu\overline{z}_0 = (\nabla F)^*(\mu)$, and $1\overline{z}_1 = (\nabla F)^*(1)$, so since $\mu, 1 \in S$, both z_0 and z_1 are in F'. But if $z_0 = z_1$, then $\mu\overline{z}_0 = (\nabla F)^*(\mu) = \mu\overline{z}_1 = \mu(\nabla F)^*(1)$, which is a contradiction. Hence $z_0 \neq z_1$ and since both z_0 and z_1 are in F', F' is not single-valued. ☐

Proposition 4.4. If F has closed graph and is locally bounded then F' has closed graph in dom $F' \times \mathbf{C}$.

Proof. Let $(z_n, y_n) \in \text{gph } F'$ be such that $(z_n, y_n) \to (z, y)$ with $z \in \text{dom } F'$. Thus there exists $\mu_n \in S$ and $w_n \in F(z_n)$ such that $(\mu_n\overline{y_n}, -\mu_n) \in N((z_n, w_n); \text{gph } F)$. S is compact and F is locally bounded so by extracting subsequences if necessary, we may assume $\mu_n \to \mu$ and $w_n \to w$. By robustness we get $(\mu\overline{y}, -\mu) \in N((z,w); \text{gph } F)$ so $y \in F'(z)$ which proves that gph F' is closed. ☐

However, F' need not be locally bounded, not even for very smooth (in the sense of having smooth graph) and simple multifunctions.

Example 4.5. Let $F(z) = \{z, \pm\sqrt{z}\}$ where the ± indicates that we take both branches of the square-root function. F is upper semi-continuous and compact valued. If $z \neq 0$, F is the union of three holomorphic functions so we get the three normals

$$N_1 = \mathbf{C}(\frac{\mathbf{1}}{\mathbf{2}\overline{\sqrt{\mathbf{z}}}}, -\mathbf{1}), \ \mathbf{N_2} = \mathbf{C}(-\frac{\mathbf{1}}{\mathbf{2}\overline{\sqrt{\mathbf{z}}}}, -\mathbf{1}) \text{ and } \mathbf{N_3} = \mathbf{C}(\mathbf{1}, -\mathbf{1}).$$

Taking Lim sup of these sets when $z \to 0$ we get $\mathbf{C(1,0)} \cup \mathbf{C(1,-1)}$. Thus $F'(0)$ is defined and we have

$$F'(z) = \begin{cases} \{\pm 1/(2\sqrt{z}), 1\} & \text{if } z \neq 0 \\ \{1\} & \text{if } z = 0 \end{cases}.$$

Clearly this is not locally bounded at $z = 0$.

Let us calculate an example to acquire more feeling for the derivative.

Example 4.6. Let

$$F(z) = \{w;\ |w - f(z)| \leq e^{\phi(z)}\} = f(z) + Be^{\phi(z)}\ (f,\, \phi \in C^1).$$

The graph is C^1 so the normal cone is just the ordinary normal obtained from differential calculus. If $f = u + iv$ and $z = x + iy$ we can parameterize the boundary of the graph around the point $(z, f(z) + e^{\phi}e^{it})$ thus:

$$(x, y, t) \mapsto (x, y, u + e^{\phi}\cos t, v + e^{\phi}\sin t).$$

We differentiate and find a vector n orthogonal to all three derivatives. We get

$$n = (-f_x \cdot e^{it} - e^{\phi}\phi_x, -f_y \cdot e^{it} - e^{\phi}\phi_y, \cos t, \sin t).$$

(Here $\cdot$ denotes the scalar product in $\mathbf{R^2}$.) Since the scalar product of n with the outward direction $(0, e^{it})$ is > 0, n points out of the graph. Thus, with $w = f(z) + e^{\phi(z)}e^{it}$, we have

$$N(z, w) = \operatorname{cone}(-f_x \cdot e^{it} - e^{\phi}\phi_x, -f_y \cdot e^{it} - e^{\phi}\phi_y, \cos t, \sin t).$$

Now $w \in F'(z)$ if and only if $\mu\overline{w} \in D^*F(z,a)(\mu)$ for some $\mu \in S$ and some $a \in F(z)$. In the interior of gph F the normal cone is zero, so we need only to consider $a \in \partial F(z)$. Hence $w \in F'(z)$ if and only if $(\mu\overline{w}, -\mu) \in N(z, f(z) + e^{\phi}e^{it})$ for some $t \in [0, 2\pi)$ and some $\mu \in S$. If we define

$$c(t) = (f_x \cdot e^{it} + e^{\phi}\phi_x, f_y \cdot e^{it} + e^{\phi}\phi_y) = \Re e(f_x e^{-it}) + i\Re e(f_y e^{-it}) + \phi_x e^{\phi} + i\phi_y e^{\phi},$$

we see that $(\mu\overline{w}, -\mu) \in N(z, f(z) + e^{\phi}e^{it})$ if and only if $-\mu = e^{it}$ and $\mu\overline{w} = -c(t)$. We thus have $w \in F'(z)$ if and only if $w = e^{it}\overline{c(t)}$ for some $t \in [0, 2\pi)$, i.e.,

$$F'(z) = \{w;\ w = e^{it}\overline{c(t)} \text{ for some } t \in [0, 2\pi)\}.$$

If f is holomorphic, $\Re e(f_x e^{-it}) - i\Re e(f_y e^{-it}) = e^{-it}f'$, so in this case

$$F'(z) = f'(z) + \phi_z(z)e^{\phi(z)}S,$$

where $\phi_z = \phi_x - i\phi_y$.

We will now use Theorem 3.3 and Theorem 3.5 to obtain calculus rules for the derivative just defined. This will be possible since an arithmetic operation performed on a pair of multifunctions is just a composition of the crossproduct of the multifunctions with the operation in question, e.g., $x \mapsto (F+G)(x)$ could be written $x \mapsto (F\times G)(x) \ni (y,z) \mapsto y+z$.

Theorem 4.7. Let $F, G\colon \mathbf{C} \rightrightarrows \mathbf{C}$ be two closed graph multifunctions. If

$$D^*F(x_0,y_0)(0) \cap -D^*G(x_0,z_0)(0) = \{0\} \quad \text{for all } (y_0,z_0) \in F(x_0)\times G(x_0)$$

and at least one of F and G is locally bounded around x_0, then

$$(F+G)'(x_0) \subset F'(x_0) + G'(x_0).$$

Proof. Let $p\colon \mathbf{C}\times\mathbf{C} \to \mathbf{C}\colon (\mathbf{y},\mathbf{z}) \mapsto \mathbf{y}+\mathbf{z}$ be the plus operation. Then $F+G = p\circ(F\times G)$ and we have $D^*p(y,z,y+z)(\mu) = (\mu,\mu)$. Thus $\ker\, D^*p(y,z,y+z) = \{0\}$ so the second condition in Theorem 3.3 is fulfilled. Moreover, if one of F and G is locally bounded around x_0, then $M(x,y,z) = (F\times G)(x)\cap p^{-1}(y+z) = \{(y',z') \in (F\times G)(x);\ y'+z' = y+z\}$ becomes locally bounded around every point (x_0,y_0,z_0) such that $(y_0,z_0) \in (F\times G)(x_0)$. We can thus use Theorem 3.3 to conclude that

$$\begin{aligned}
&(F+G)'(x_0) = \\
&= \{y;\ \exists\,(y_1,z_1) \in (F{\times}G)(x_0),\ \exists\mu\in S \text{ with } \mu\overline{y} \in D^*(F{+}G)(x_0,y_1{+}z_1)(\mu)\} \\
&\qquad\qquad \subset \{y;\ \exists\,(y_1,z_1) \in (F\times G)(x_0),\ \exists\mu\in S \text{ with} \\
&\qquad \mu\overline{y} \in \bigcup_{\substack{y_2+z_2=y_1+z_1\\(y_2,z_2)\in(F\times G)(x_0)}} D^*(F\times G)(x_0,y_2,z_2)\circ D^*p(y_2,z_2,y_1+z_1)(\mu)\}.
\end{aligned}$$

Now, $D^*p(y_2,z_2,y_1+z_1)(\mu) = D^*p(y_2,z_2,y_2+z_2)(\mu) = (\mu,\mu)$, so, using Theorem 3.5, we conclude that

$$\begin{aligned}
&D^*(F\times G)(x_0,y_2,z_2)\circ D^*p(y_2,z_2,y_1+z_1)(\mu) \\
&\qquad\qquad\qquad\qquad \subset D^*F(x_0,y_2)(\mu) + D^*G(x_0,z_2)(\mu).
\end{aligned}$$

Thus

$$\begin{aligned}
(F+G)'(x_0) &\subset \{y;\ \exists\,(y_1,z_1) \in (F\times G)(x_0),\ \exists\mu\in S \text{ with} \\
&\quad \mu\overline{y} \in \bigcup_{\substack{y_2+z_2=y_1+z_1\\(y_2,z_2)\in(F\times G)(x_0)}} D^*F(x_0,y_2)(\mu) + D^*G(x_0,z_2)(\mu)\} \\
&= \{y;\ \exists\,(y_1,z_1) \in (F\times G)(x_0),\ \exists\mu\in S \text{ with} \\
&\qquad\qquad \mu\overline{y} \in D^*F(x_0,y_1)(\mu) + D^*G(x_0,z_1)(\mu)\}.
\end{aligned}$$

Let y in this set be given. Then $\mu\overline{y} = a+b$ with $a \in D^*F(x_0,y_1)(\mu)$ and $b \in D^*G(x_0,y_1)(\mu)$ for some $y_1 \in F(x_0)$, $z_1 \in G(x_0)$ and some $\mu \in S$. Let

$y' = \overline{a}/\overline{\mu}$ and $y'' = \overline{b}/\overline{\mu}$. Then $y' \in F'(x)$ and $y'' \in G'(x)$ so their sum is in $F'(x) + G'(x)$. But $y' + y'' = \overline{a+b}/\overline{\mu} = y$. We have thus proved that $(F+G)'(x) \subset F'(x) + G'(x)$. $\square$

Theorem 4.8. Let $F, G\colon \mathbf{C} \rightrightarrows \mathbf{C}$ be two closed graph multifunctions. If

$$D^*F(x_0, y_0)(0) = D^*G(x_0, z_0)(0) = \{0\} \qquad \text{for all } (y_0, z_0) \in F(x_0) \times G(x_0) \tag{4.3}$$

and at least one of F and G is locally bounded around x_0 then

$$(FG)'(x_0) \subset F'(x_0)G(x_0) + F(x_0)G'(x_0).$$

Proof. We proceed more or less as in the proof of Theorem 4.7. Thus let $m\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}\colon (\mathbf{y}, \mathbf{z}) \mapsto \mathbf{yz}$ be the multiplication operation. Then $FG = m \circ (F \times G)$ and we have $D^*m(y, z, yz)(\mu) = (\mu\overline{z}, \mu\overline{y})$. Theorem 3.3 and Theorem 3.5 are applicable and we get

$$\begin{aligned} &D^*(FG)(x_0, y_1z_1)(\mu) \\ &\quad\subset \bigcup_{\substack{y_2z_2 = y_1z_1 \\ (y_2,z_2)\in(F\times G)(x_0)}} D^*(F\times G)(x_0, y_2, z_2) \circ D^*m(y_2, z_2, y_1z_1)(\mu) \\ &\qquad\subset \bigcup_{\substack{y_2z_2 = y_1z_1 \\ (y_2,z_2)\in(F\times G)(x_0)}} D^*F(x_0, y_2)(\mu\overline{z_2}) + D^*G(x_0, z_2)(\mu\overline{y_2}). \end{aligned}$$

Thus,

$$\begin{aligned} &(FG)'(x_0) \\ &= \{y;\ \exists\,(y_1, z_1) \in (F\times G)(x_0),\ \exists \mu \in S \text{ with } \mu\overline{y} \in D^*(FG)(x_0, y_1z_1)(\mu)\} \\ &\qquad\subset \{y;\ \exists\,(y_1, z_1) \in (F\times G)(x_0),\ \exists \mu \in S \text{ with} \\ &\quad \mu\overline{y} \in \bigcup_{\substack{y_2z_2 = y_1z_1 \\ (y_2,z_2)\in(F\times G)(x_0)}} D^*F(x_0, y_2)(\mu\overline{z_2}) + D^*G(x_0, z_2)(\mu\overline{y_2})\} \\ &\qquad= \{y;\ \exists\,(y_1, z_1) \in (F\times G)(x_0),\ \exists \mu \in S \text{ with} \\ &\qquad\qquad \mu\overline{y} \in D^*F(x_0, y_1)(\mu\overline{z_1}) + D^*G(x_0, z_1)(\mu\overline{y_1})\} \end{aligned}$$

Let y be given from this set. Then $\mu\overline{y} = a + b$ with $a \in D^*F(x_0, y_1)(\mu\overline{z_1})$ and $b \in D^*G(x_0, z_1)(\mu\overline{y_1})$ for some $\mu \in S$, some $y_1 \in F(x_0)$ and some $z_1 \in G(x_0)$. We must show that $y = y' + y''$ for some $y' \in F'(x_0)G(x_0)$ and some $y'' \in F(x_0)G'(x_0)$. If $z_1 \neq 0$ we let $\mu_1 = \overline{z_1}\mu/|z_1|$. Then $\mu_1 \in S$ and $D^*F(x_0, y_1)(\mu\overline{z_1}) = |z_1| D^*F(x_0, y_1)(\mu_1)$ by positive homogeneity. Hence $\overline{a}/(|z_1|\overline{\mu_1}) \in F'(x_0)$, so

$$\frac{\overline{a}}{|z_1|\overline{\mu_1}} z_1 = \frac{\overline{a}}{\overline{\mu}} \in F'(x_0)G(x_0).$$

If $z_1 = 0$ then, by (4.3), $a = 0$. Hence $0 = \overline{a}/\overline{\mu} \in G(x_0)$. But $F'(x_0) \neq \emptyset$ because of (4.3) and Proposition 4.2 so $\overline{a}/\overline{\mu} \in F'(x_0)G(x_0)$ also if $z_1 = 0$. The proof that $\overline{b}/\overline{\mu} \in F(x_0)G'(x_0)$ is completely analogous, so we have shown

$$y = \frac{1}{\overline{\mu}}\overline{a+b} \in F'(x_0)G(x_0) + F(x_0)G'(x_0).$$

□

Theorem 4.9. Let $F\colon \mathbf{C} \rightrightarrows \mathbf{C}$ be a closed graph multifunction. Then

$$\left(\frac{1}{F}\right)'(x_0) \subset -\frac{F'(x_0)}{F^2(x_0)}.$$

Proof. Let $r(y) = 1/y$ so that $1/F = r \circ F$. The function r is locally bounded and $D^*r(y, 1/y)(\mu) = -\mu/\overline{y^2}$ so all assumptions in Theorem 3.4 are fulfilled. Thus we conclude

$$D^*\frac{1}{F}(x_0, 1/y)(\mu) \subset [D^*F(x_0, y) \circ D^*r(y, 1/y)](\mu) = D^*F(x_0, y)\left(\frac{-\mu}{\overline{y^2}}\right),$$

for every y such that $1/y \in (1/F)(x_0)$.[3] We get

$$\begin{aligned}\left(\frac{1}{F}\right)'(x_0) &= \left\{y;\ \exists\mu \in S,\ \exists 0 \neq v \in F(x_0) \text{ with } \mu\overline{y} \in D^*\frac{1}{F}(x_0, 1/v)(\mu)\right\}\\ &\subset \left\{y;\ \exists\mu \in S,\ \exists 0 \neq v \in F(x_0) \text{ with } \mu\overline{y} \in D^*F(x_0, v)\left(\frac{-\mu}{\overline{v^2}}\right)\right\}.\end{aligned}$$

Defining $\mu_1 = -\mu|v|^2/v^2$, we see that $\mu\overline{y} = b/|v|^2$ for some $b \in D^*F(x_0, v)(\mu_1)$ and also that $\overline{b}/\overline{\mu}_1 \in F'(x_0)$. Thus

$$y = \frac{\overline{b}}{\overline{\mu}|v|^2} = -\frac{\overline{b}}{\overline{\mu}_1}\frac{1}{v^2}$$

will be in $-F'(x_0)/F^2(x_0)$ as required. □

Corollary 4.10. Let $F, G\colon \mathbf{C} \rightrightarrows \mathbf{C}$ be two closed graph multifunctions. If

$$D^*F(x_0, y)(0) = D^*G(x_0, z)(0) = \{0\} \ \text{ for all } (y, z) \in F(x_0) \times G(x_0)$$

and at least one of F and $1/G$ is locally bounded around x_0, then

$$\left(\frac{F}{G}\right)'(x_0) \subset \frac{F'(x_0)}{G(x_0)} - \frac{F(x_0)G'(x_0)}{G^2(x_0)}.$$

Proof. Use Theorem 4.8 and Theorem 4.9. □

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[3] Recall that such a y is always non-zero

Theorem 4.11. Let F, $G : \mathbf{C} \rightrightarrows \mathbf{C}$ be two closed graph multifunctions. Assume that $G(x) \cap F^{-1}(z)$ is locally bounded around (x_0, z_0) for every $z_0 \in F \circ G(x_0)$, that $D^*F(x_0, y)(0) \cap \ker\ D^*G(y, z) = \{0\}$ for all $z \in F \circ G(x_0)$ and all $y \in G(x_0) \cap F^{-1}(z)$ and that for those y's, $D^*G(x_0, y)(0) = \{0\}$. Then

$$(F \circ G)'(x_0) \subset (F' \circ G)(x_0)G'(x_0).$$

Proof. The first and second assumption make it possible to use Theorem 3.3. We get

$$(F \circ G)'(x_0) = \{y;\ \exists z \in F \circ G(x_0),\ \exists \mu \in S \text{ with } \mu\overline{y} \in D^*(F \circ G)(x_0, z)(\mu)\}$$

$$\subset \{y;\ \exists z \in F \circ G(x_0),\ \exists \mu \in S \text{ with } \mu\overline{y} \in \bigcup_{\substack{v \in G(x_0) \\ z \in F(v)}} D^*G(x_0, v) \circ D^*F(v, z)(\mu)\}$$

$$= \{y;\ \exists v \in G(x_0),\ \exists z \in F(v),\ \exists \mu \in S \text{ with } \mu\overline{y} \in D^*G(x_0, v) \circ D^*F(v, z)(\mu)\}.$$

Let y in this set be given. Then $\mu\overline{y} \in D^*G(x_0, v)(\xi)$ for some $\xi \in D^*F(v, z)(\mu)$ and for some $\mu \in S$, some $v \in G(x_0)$ and some $z \in F(v)$. Thus $a = \overline{\xi}/\overline{\mu} \in F'(v)$ i.e., $a \in (F' \circ G)(x_0)$. If $\xi = 0$ then $y = 0$ by the third assumption in the theorem, so then $0 = y = \xi/\overline{\mu} \in F'(v) \subset (F' \circ)G(x_0)$ and hence $y \in (F' \circ G)(x_0)G'(x_0)$. Otherwise $\xi \neq 0$, so $\mu\overline{y} = |\xi|b$ for some $b \in D^*G(x_0, y)(\xi/|\xi|)$ and thus $|\xi|\overline{b}/\overline{\xi} \in G'(x_0)$. Hence

$$y = \frac{|\xi|\overline{b}}{\overline{\mu}} = \frac{\overline{\xi}}{\overline{\mu}} \frac{|\xi|\overline{b}}{\overline{\xi}} = a\frac{|\xi|\overline{b}}{\overline{\xi}} \in (F' \circ G)(x_0)G'(x_0)$$

as required. □

It is easy to see that

$$(x, y) \in N((x_0, y_0); \text{gph } F) \text{ if and only if } (y, x) \in N((y_0, x_0); \text{gph } F^{-1}),$$

so we have

$$y \in D^*F^{-1}(y_0, x_0)(x) \text{ if and only if } -x \in D^*F(x_0, y_0)(-y).$$

Writing down the derivative we get

$$y \in (F^{-1})'(w) \Rightarrow \frac{1}{y} \in F'(z) \text{ for some } z \in F^{-1}(w).$$

Let us state this in a theorem.

Theorem 4.12. If $F\colon \mathbf{C} \rightrightarrows \mathbf{C}$ is a closed graph multifunction, then

$$(F^{-1})'(x) \subset \frac{1}{F' \circ F^{-1}(x)}.$$

Let us look at the condition $D^*F(x_0,y_0)(0)=\{0\}$ in terms of the normal at (x_0,y_0). Recall that $(x,y)\in N((x_0,y_0);\mathrm{gph}\ F)$ if and only if $x\in D^*F(x_0,y_0)(-y)$. Thus the above condition means that

$$(x,0)\in N((x_0,y_0);\mathrm{gph}\ F)\Rightarrow x=0$$

i.e., that the normal does not contain any "horizontal" vectors. It is therefore reasonable to call a multifunction F *non-singular at* x_0 if $F(x_0)\neq\emptyset$ and the normal $N((x_0,y_0)\,\mathrm{gph}\ F)$ does not contain any vector of the form $(x,0)$ with $x\neq 0$ for any $y_0\in F(x_0)$. With these notions we can summarize the above theorems. Compare this with the definition of calculus in the introduction.

Theorem 4.13. Let $F,G\colon \mathbf{C}\rightrightarrows\mathbf{C}$ be two closed graph multifunctions. Then

1. $(1/F)'\subset -F'/F^2$;
2. $(F^{-1})'\subset 1/(F'\circ F^{-1})$.

If F is locally bounded and non-singular, then

3. $(F+G)'\subset F'+G'$;

If, moreover, G is locally bounded and non-singular, then

4. $(FG)'\subset F'G+FG'$;
5. $(F/G)'\subset F'/G-FG'/G^2$;
6. $(F\circ G)'\subset (F'\circ G)G'$.

In certain situations we have equality in some of the inclusions in the theorem above.

Theorem 4.14. Let $f\colon\mathbf{C}\to\mathbf{C}$ a function analytic near z_0. Then

$$(F+f)'(z_0)=F'(z_0)+f'(z_0).$$

If, moreover, F is locally bounded and non-singular near $f(z_0)$ and $f'(z_0)\neq 0$, then

$$(F\circ f)'(z_0)=(F'\circ f(z_0))f'(z_0).$$

Proof. For the first part we use Theorem 4.13 (1) on $F+f$ and $F=F+f-f$ to get

$$(F+f)'\subset F'+f'\subset (F+f)'-f'+f'=(F+f)'.$$

For the second part note that $f'(z_0)\neq 0$ tells us that f has a local inverse f^{-1} which is holomorphic near z_0 We can thus use Theorem 4.13 (6) to conclude that

$$F'=(F\circ f\circ f^{-1})'\subset\big((F\circ f)'\circ f^{-1}\big)(f^{-1})$$

so that

$$F' \circ f \subset (F \circ f)'\big((f^{-1})' \circ f\big) = (F \circ f)'\frac{1}{f'}.$$

Hence it follows that

$$(F \circ f)' \subset (F' \circ f)f' \subset (F \circ f)'\frac{1}{f'}f = (F \circ f)'.$$

□

The next proposition shows us that the property of being upper semi-continuous and compact-valued is preserved under the operation of taking derivative if the multifunction is non-singular.

Proposition 4.15. Let $F\colon \mathbf{C} \rightrightarrows \mathbf{C}$, be upper semi-continuous, compact-valued and non-singular in an open set Ω. Then F' is upper semi-continuous and compact-valued in Ω.

Proof. Using Proposition 4.2 we see that F non-singular at z implies in particular that $F'(z)$ exists. Thus dom $F' = \Omega$. Since by Proposition 4.4 the graph of F' is then closed in $\Omega \times \mathbf{C}$ it suffices to show that F' is locally bounded in Ω. If this is not the case, then we can find a point $z_0 \in \Omega$ and two sequences, $\Omega \ni z_n \to z_0$ and $y_n \in F'(z_n)$, so that $y_n \to \infty$ as $n \to \infty$. Thus there exists $\mu_n \in S$ and $w_n \in F(z_n)$ such that $\mu_n\overline{y_n} \in D^*F(z_n, w_n)(\mu_n)$ i.e., such that $\mu_n\overline{y_n}/|y_n| \in D^*F(z_n, w_n)(\mu_n/|y_n|)$. Using compactness of S and upper semi-continuity of F we may assume that $w_n \to w_0 \in F(z_0)$ and that $\mu_n\overline{y_n}/|y_n| \to \xi \in S$. Since $\mu_n/|y_n| \to 0$ we get by robustness that $\xi \in D^*F(z_0, w_0)(0)$. But since $\xi \neq 0$ this means exactly that F is singular at z_0. □

5. Openness and Lipschitz

In [10] Mordukhovich uses the coderivative to characterize certain openness and Lipschitzian properties of multifunctions. These properties are localized with respect to both the domain and the image. Considering our derivative it is only meaningful to localize with respect to the domain simply because we have "$\exists w \in F(z)$" in the definition of the derivative, i.e., we take the union over all $w \in F(z)$.

Definition 5.1. A closed graph multifunction Φ is said to enjoy *the covering property around* x_0 if there exists an $a > 0$ and a neighborhood U of x_0 such that $B_r(x) \subset U$ implies $B_{ar}(\Phi(x)) \subset \Phi(B_r(x))$. Each such number

a (corresponding to different neighborhoods U) is called *a covering modulus* for Φ around x_0. The supremum of the covering moduli is called *the covering bound* for Φ around x_0 and is denoted by $(\mathrm{cov}\ \Phi)(x_0)$.

With the number

$$a(\Phi, x_0) = \inf\big(|x|;\ x \in D^*\Phi(x_0, y_0)(y),\ |y| = 1,\ y_0 \in \Phi(x_0)\big)$$

the following is true.

Theorem 5.2 (Theorem 3.3 in [10])**.** Let Φ be locally bounded at x_0. Then the following conditions are equivalent:

1. Φ enjoys the covering property around x_0;
2. $a(\Phi, x_0) > 0$;
3. $\ker D^*\Phi(x_0, y_0) = \{0\}$ for all $y_0 \in \Phi(x_0)$.

When these properties hold, one has

$$(\mathrm{cov}\ \Phi)(x_0) = a(\Phi, x_0).$$

It is clear that if Φ has the covering property around x_0 it is open there, i.e., the images under Φ of small enough open neighborhoods around x_0 are open. The converse is not true: Consider the function $\mathbf{R^2} \ni (\mathbf{x}, \mathbf{y}) \mapsto \Phi(\mathbf{x}, \mathbf{y}) = (\mathbf{x^2} - \mathbf{y^2}, \mathbf{2xy}) \in \mathbf{R^2}$. This is open since writing $z = x + iy$, Φ is the holomorphic function $z \mapsto z^2$. But Φ does not enjoy the covering property around $(x, y) = 0$. To see this notice that $B_{ar}(\Phi(0)) = B_{ar}(0)$ and $\Phi(B_r(0)) = B_{r^2}(0)$. Thus it is not true that for some $a > 0$ and every small enough r, $B_{ar}(\Phi(0)) \subset \Phi(B_r(0))$.

Definition 5.3. A closed graph multifunction $\Phi\colon \mathbf{R^n} \rightrightarrows \mathbf{R^k}$ is said to be *locally Lipschitzian* around $x_0 \in \mathrm{dom}\ \Phi$ with modulus $l > 0$ if there exists a neighborhood U of x_0 such that

$$\Phi(x') \subset \Phi(x) + l|x' - x|B \text{ for any } x', x \in U.$$

Theorem 5.4 (Theorem 5.11 in [10])**.** Let Φ be locally bounded around x_0. Then Φ is locally Lipschitzian around x_0 with modulus l if and only if there exists a neighborhood U of x_0 such that

$$\sup\big(|x'|;\ x' \in D^*\Phi(x, y)(y')\big) \le l|y'|$$

for all $x \in U$, $y \in \Phi(x)$ and $y' \in \mathbf{R^k}$.

Theorem 5.2 and Theorem 5.4 have immediate consequences for our derivative. Let us start with the covering property of F around a point z_0.

Theorem 5.2 tells us that F enjoys the covering property around z_0 if and only if

$$\ker D^*F(z_0, w) = \{0\} \text{ for all } w \in F(z_0). \tag{5.1}$$

This condition means exactly that for all $\mu \in S$ and all $w \in F(z_0)$ we have $0 \notin D^*F(z_0, w)(\mu)$ i.e., (5.1) is equivalent to $0 \notin F'(z_0)$. Moreover, the number $a(F, z_0)$ equals the smallest of the numbers $|y|$ such that $y \in F'(z_0)$. We thus get the following theorem.

Theorem 5.5. Let $F\colon \mathbf{C} \rightrightarrows \mathbf{C}$ be a closed graph multifunction. Then F enjoys the covering property around $z_0 \in \operatorname{dom} F$ if and only if $0 \notin F'(z_0)$. If this is the case, then the covering modulus $a(F, z_0) = 1/|1/F'(z_0)| = 1/\operatorname{dist}(0, F'(z_0))$.

We now turn to the characterization of local Lipschitzness of F. Let Ω be an open subset of $\mathbf{C}$ and let $F\colon \Omega \rightrightarrows \mathbf{C}$ be an upper semi-continuous, compact-valued and non-singular multifunction. If $U \subset\subset \Omega$ then $l = \sup_{z \in U} |F'(z)| < \infty$ since F' is locally bounded. Moreover, by the definition of F', we have

$$\sup\big(|x|;\, x \in D^*F(z, w)(y)\big) \le l|y|$$

for all $z \in U$ and $w \in F(z)$. Thus by Theorem 5.4 we know that F is locally Lipschitzian at every $z_0 \in U$ i.e., for all $z_0 \in U$ there exists a neighborhood V of z_0 such that

$$F(z) \subset F(z') + l|z - z'|B \text{ for all } z, z' \in V. \tag{5.2}$$

Theorem 5.6. Suppose $F\colon \Omega \rightrightarrows \mathbf{C}$ is upper semi-continuous, compact-valued and non-singular. If $U \subset\subset \Omega$ is convex, then for all $z, z' \in U$ we have

$$F(z) \subset F(z') + l|z - z'|B,$$

where $l = \sup_{z \in U} |F'(z)|$.

Proof. Let $z, z' \in U$ be given. Denote by $[z, z']$ the line segment between z and z'. Since U is convex, $[z, z'] \subset U$. Thus for all $w \in [z, z']$ there exists a neighborhood V such that (5.2) is valid. These sets cover $[z, z']$ so there is a finite sub-cover $V_1, V_2, \dots, V_n$. We can assume $z \in V_1$, $z' \in V_n$ and $z_j \in V_j \cap V_{j+1}$ with the z_j's different and consecutive on the line $[z, z']$. With this we get

$$\begin{aligned} F(z) &\subset F(z_1) + l|z - z_1|B \subset F(z_2) + l(|z - z_1| + |z_1 - z_2|)B \subset \dots \\ &\subset F(z') + l(|z - z_1| + \dots + |z_{n-1} - z'|)B = F(z') + l|z - z'|B. \end{aligned}$$

□

This theorem will be substantially improved in section below.

Corollary 5.7. Let $F\colon \Omega \rightrightarrows \mathbf{C}$ be upper semi-continuous, compact-valued and non-singular, and suppose Ω is connected. Then F is constant if and only if $F'(z) = 0$ for all $z \in \Omega$.

6. Analytic multifunctions

The theory of analytic multifunctions has its origin in two parts of mathematics: spectral theory and several complex variables. In spectral theory it is natural to ask how the eigenvalues of an analytic family of matrices, i.e., a family of matrices whose coefficients depend analytically on a parameter, behave. In fact this is an old question; it was studied for the first time by Cauchy in the 1830's. From complex analysis one can think of many problems generating multifunctions, for instance the roots of an equation involving analytic expressions. Problems like these have led to the concept of an analytic multifunction. Their main applications are in spectral theory where one has found applications to, among other things, joint spectrum, spectral interpolation, local spectral theory and even to Jordan-Banach algebras. In other parts of mathemathics one has applications to, for instance, uniform algebras in connection with problems of analytic structure and to complex dynamics. We refer to [2] for a survey of the theory of analytic multifunctions and their applications.

Following [12] we define an analytic multifunction.

Definition 6.1. An upper semicontinuous, compact-valued map

$$F\colon D \rightrightarrows \mathbf{C},$$

where D is an open subset of $\mathbf{C}$, is said to be analytic if for every open $D' \subset D$ and for every $\psi(\lambda, z)$, plurisubharmonic in a neighbourhood of $\{(\lambda, z) \in D' \times \mathbf{C};\ z \in F(\lambda)\}$, the function

$$\lambda \mapsto \sup\bigl(\psi(\lambda, z);\ z \in F(\lambda)\bigr)$$

is plurisubharmonic in D'.

Remark. For the purpose of this paper we can use the above definition of analytic multifunction. The definition of analyticity for multifuctions taking values in $\mathbf{C}^n$, $n \geq 2$ has however changed to a slightly smaller class. See for instance [14].

If F is a function, then F is analytic in the ordinary sense if and only if it is analytic in the above sense.

Słodkowski (see [13]) has shown that analyticity of a multifunction is equivalent with the impossibility of defining plurisubharmonic functions having a strict maximum visavi the graph of the multifunction. To be precise, he proves that a multifunction F is analytic if and only if there do not exist $x_0 \in \operatorname{gph} F$, $r > 0$, $\epsilon > 0$, and a strict plurisubharmonic function f in $B_r(x_0)$ such that

$$\begin{cases} f(x_0) = 0 \\ f(x) \leq -\epsilon |x - x_0|^2, \text{ for } x \in \operatorname{gph} F \cap B_r(x_0). \end{cases}$$

Recalling the way Hörmander defined the normal to a set we make the following definition.

Definition 6.2. Given $x_0 \in \operatorname{gph} F$ we say that $\xi \in N_{psh}(x_0; \operatorname{gph} F)$ if there exists a plurisubharmonic function $f \in C^2$ near x_0 such that

$$\begin{cases} \xi = \nabla f(x_0) \\ f(x) < f(x_0), \text{ when } x \neq x_0 \text{ and } x \in \operatorname{gph} F. \end{cases}$$

With this notion Słodkowski's result imply the following proposition.

Proposition 6.3. A closed graph, locally bounded multifunction $F\colon \mathbf{C} \rightrightarrows \mathbf{C}$ defined on an open set is analytic if and only if $N_{psh} = \emptyset$ everywhere.

7. An example

We shall consider multifunctions of the form

$$F(z) = \sum_{k=0}^{\infty} A_k (z - z_0)^k = \left\{ \sum_{k=0}^{\infty} a_k (z - z_0)^k;\ a_k \in A_k,\ k = 0, 1, \dots \right\},$$

where A_k are compact subsets of the complex plane fulfilling

$$\frac{1}{R} = \limsup_{k \to \infty} |A_k|^{1/k} < \infty.$$

(If $A_k = \emptyset$ for some k we put $A_k = \{0\}$.) Thus for every $r < R$, every $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ from $F(z)$ converges absolutely and uniformly on $B_r(z_0)$, and if $z \notin B_R(z_0)$ there exists $a_k \in A_k$, $k = 0, 1, \dots$ so that $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ forms a diverging series.

Let us write $f \in F$ if $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ for some $a_k \in A_k$, $k = 0, 1, \dots$ and let the statement $f(z) \in F(z)$ refer to the pointwise inclusion, i.e., for every z under consideration there exists a sequence $\{a_k\}$ with $a_k \in A_k$ so that $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$.

Proposition 7.1. Let $f_n \in F$, $n = 1, 2, \dots$ be a sequence of functions from F. Then there exists a subsequence $\{f_j\}$ converging to an $f \in F$ uniformly on compact subsets of $\operatorname{int} B_R(z_0)$.

Proof. Since the sequence $\{f_n\}$ is uniformly bounded on compact subsets it follows from the Stieltjes-Vitalis theorem that there exists a subsequence $\{f_j\}$ converging uniformly on compact subsets to a holomorphic function f. Suppose $f(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k$ and suppose $f_j(z) = \sum_{k=0}^{\infty} a_k^{(j)} (z - z_0)^k$ with $a_k^{(j)} \in A_k$. Since f_j converges together with all its derivatives it follows that, for all k, $a_k^{(j)} \to b_k$ as $j \to \infty$. But all the A_k's are closed so $b_k \in A_k$ and hence $f \in F$ $\square$

Proposition 7.2. F is upper semi-continuous, compact-valued and analytic.

Proof. Since F is clearly locally bounded it suffices to prove that the graph of F, gph F, is closed in int $B_R(z_0) \times \mathbf{C}$. Let therefore a sequence $\zeta_n \to \zeta \in$ int $B_R(z_0)$ and a sequence $w_n \to w$ be given such that $\zeta_n \in$ int $B_R(z_0)$ and $w_n \in F(\zeta_n)$. Then $w_n = \sum_{k=0}^{\infty} a_k^{(n)} (\zeta_n - z_0)^k$ for some $a_k^{(n)} \in A_k$. Let $f_n(z) = \sum_{k=0}^{\infty} a_k^{(n)} (z - z_0)^k$. By Proposition 7.1 we may assume that f_n converges to $f \in F$. We get

$$w = \lim_{n \to \infty} w_n = \lim_{n \to \infty} f_n(\zeta_n) = f(\zeta)$$

so that $(\zeta, w) \in$ gph F as required. The analyticity is then clear, since gph F is the union of graphs of analytic functions. $\square$

Let us take a look at the terms of F. Define $G(z) = A$ to be the constant multifunction having as value the (non-empty) compact set A. Then $\operatorname{gph} G = \mathbf{C} \times \mathbf{A}$ so by (3.2), $N((z,a); \operatorname{gph} G) = N(z; \mathbf{C}) \times \mathbf{N}(\mathbf{a}; \mathbf{A}) = \{\mathbf{0}\} \times \mathbf{N}(\mathbf{a}; \mathbf{A})$. Recall that $N(a; A) \neq \{0\}$ if and only if $a \in \partial A$. Therefore $(\mu \overline{y}, -\mu) \in N((z,a); \operatorname{gph} G)$ for some $\mu \in S$ if and only if $a \in \partial A$. But $\partial A \neq \emptyset$ since A is compact so it follows that $G'(z) = \{y;\ \exists \mu \in S,\ \exists a \in G(z) \text{ with } (\mu \overline{y}, -\mu) \in N((z,a); \operatorname{gph} G)\} = \{0\}$.

Next we study $G(z) = Af(z)$, where f is a holomorphic function and A is compact. The proof of Theorem 4.8 gives us that

$$D^*(Af)(z, af(z))(\mu) \subset \bigcup_{\substack{bw = af(z) \\ b \in A,\, w = f(z)}} D^*A(z,b)(\mu \overline{w}) + D^*f(z,w)(\mu B).$$

If $f(z) \neq 0$ this equals $D^*A(z,a)(\mu\overline{f(z)}) + \overline{f'(z)}\mu\overline{a}$. Thus for $a \in \partial A$ we get, using what we know of the normal of the constant multifunction $z \mapsto A$, that

$$D^*(Af)(z, af(z))(\mu) \subset \{\overline{f'(z)}\mu\overline{a}\}.$$

The set of zeroes to f is a discrete set so this inclusion holds everywhere. Finally

$$D^*(Af)(z, af(z))(\mu) \neq \emptyset$$

for $\mu \neq 0$ if and only if $a \in \partial A$, so then the inclusion is an equality. This shows that $G = Af$ is non-singular and that $G'(z) = \partial A f'(z)$. Let us record these results in a proposition.

Proposition 7.3. Let A be a compact subset of $\mathbf{C}$ and let f be a holomorphic function. Then

1. $A'(z) = \{0\}$;
2. $(Af)'(z) = \partial A f'(z)$.

Theorem 7.4. Let $F(z) = \sum_{k=0}^{\infty} A_k(z-z_0)^k$, with A_k compact, have positive radius of convergence R. Then $F'(z)$ exists for every $z \in \text{int } B_R(z_0)$ and

$$F'(z) \subset \sum_{k=0}^{\infty} k\partial A_k(z-z_0)^{k-1}.$$

Proof. We prove the existence by proving a lower bound for the derivative: For every $z \in \text{int } B_R(z_0)$ there exist an $f \in F$ such that $f'(z) \in F'(z)$. To see this let $x = (z,w) \in \partial\,\text{gph } F$. Let $x_n \to x$ be such that $x_n \notin \text{gph } F$. Since gph F is closed there exist $\epsilon_n > 0$ so that $B_{\epsilon_n}(x_n) \cap \text{gph } F = \emptyset$. Let B be such a ball. Move it towards x until it meets gph F in a point y_n. Thus $y_n \in \text{gph } f_n$ for some $f_n \in F$ and $x_n - y_n \in N(y_n; \text{gph } f_n) = \mathbf{C}(\overline{\mathbf{f}'_{\mathbf{n}}(\mathbf{z}_{\mathbf{n}})}, -\mathbf{1})$, where $y_n = (z_n, f_n(z_n))$, and it follows that $\xi_n = c_n(x_n - y_n) = (\mu_n\overline{f'_n(z_n)}, -\mu_n)$ for some positive c_n and some $\mu_n \in S$. By Proposition 7.1 and compactness of S we may assume that $f_n \to f \in F$ uniformly on compact subsets and that $\mu_n \to \mu$. Since the derivative of f_n also converges it becomes clear that ξ_n converges to some ξ which thus lies in $N(x; \text{gph } F)$. Hence $\xi = (\mu\overline{f'(z)}, -\mu) \in N(x; \text{gph } F)$ for some $\mu \in S$ which means exactly that $f'(z) \in F'(z)$.

Fix now z with $|z - z_0| < R$ and let $y \in F'(z)$. Then there exist $\mu \in S$ and $w \in F(z)$ such that $(\mu\overline{y}, -\mu) \in N\big((z,w); \text{gph } F\big)$. According to the definition of the normal cone we can find $\text{gph } F \not\ni x_n \to (z,w)$, $y_n \in \text{proj}\,(x_n; \text{gph } F)$ and $c_n \geq 0$ such that $\xi_n = c_n(x_n - y_n) \to (\mu\overline{y}, -\mu)$. Since $y_n \in \text{gph } F$, we can write $y_n = (z_n, f_n(z_n))$ for some $z_n \to z$ and some $f_n \in F$. As above we may assume that $f_n \to f \in F$ uniformly on compact subsets. Moreover, since $y_n \in \operatorname{proj}(x_n; \operatorname{gph} F)$, it is clear that $\xi_n \in N(y_n; \operatorname{gph} f_n) = \mathbf{C}(\overline{\mathbf{f'_n(z_n)}}, -\mathbf{1})$. We also know that $f'_n \to f'$ so it follows that $(\mu\overline{y}, -\mu) \in \mathbf{C}(\overline{\mathbf{f'(z)}}, -\mathbf{1})$. This means exactly that $y = f'(z) \in \sum_{k=0}^{\infty} kA_k(z-z_0)^{k-1}$ and we have thus proved that

$$F'(z) \subset \sum_{k=0}^{\infty} kA_k(z-z_0)^{k-1}.$$

Since the derivative exists, everything is nicely bounded, and every multifunction $z \mapsto A(z-z_0)^k$ is non-singular, we can freely apply Theorem 4.7. Doing this and using Proposition 7.3 we get, for every $m \geq 1$,

$$F'(z) \subset \sum_{k=0}^{m} (A_k(z-z_0)^k)' + G'_m(z) \subset \sum_{k=0}^{m} k\partial A_k(z-z_0)^{k-1} + G'_m(z), \quad (7.1)$$

where $G_m(z) = \sum_{k=m+1}^{\infty} A_k(z-z_0)^k$. By the above we have

$$G'_m(z) \subset \sum_{k=m+1}^{\infty} kA_k(z-z_0)^{k-1}$$

and this tends to zero as m tends to infinity. Thus we can go to the limit in (7.1) and this proves the theorem. ☐

Using Theorem 5.5 we get the following corollary.

Corollary 7.5. Let F be as in the above theorem. If $f'(z) \neq 0$ for all $f \in F$, then F enjoys the covering property around z. In particular F is open around z.

As we saw in the proof above, a good knowledge of the boundary of the graph of F helps us estimate the derivative from below. In one situation we know exactly what the boundary looks like and consequently we get in this case very exact estimates of the derivative. The situation we have in mind is the following.

Theorem 7.6. Let $F(z) = \sum_{k=0}^{\infty} A_k(z-z_0)^k$, with A_k compact and countable, have positive radius of convergence R. Then for every $z \in \operatorname{int} B_R(z_0)$

$$F'(z) = \sum_{k=0}^{\infty} kA_k(z-z_0)^{k-1}.$$

Proof. The inclusion to the right is clear from the theorem above. Thus we only have to prove that $g'(z) \in F'(z)$ for all $z \in \operatorname{int} B_R(z_0)$ and all $g \in F$. By considering $F - g$ we may assume $g \equiv 0$. Since the zero-set of a holomorphic function is discrete, it follows that the set of points $z \in \operatorname{int} B_R(z_0)$ *not* being a zero of any function in F different from the zero-function is dense in $\operatorname{int} B_R(z_0)$. Since F' has closed graph we need only to consider such points. Thus let $z \in \operatorname{int} B_R(z_0)$ be such that $g \in F$, $g(z) = 0$ implies $g = 0$. Every point in the graph of F is in the boundary so with $x = (z, 0) \in \partial \operatorname{gph} F$ we conclude from the first part of the proof of Theorem 7.4 the existence of a function $f \in F$ such that $f'(z) \in F'(z)$ and $x = (z, f(z))$. Hence $f(z) = 0$, so by assumption about the point z, $f = 0$ and thus $0 = f'(z) \in F'(z)$. □

Corollary 7.7. Let $F(z) = \sum_{k=0}^{\infty} A_k (z - z_0)^k$, with A_k compact and countable, have positive radius of convergence R. Then

1. The n^{th} derivative $F^{(n)}(z)$ exists for all n and all $z \in \operatorname{int} B_R(z_0)$ and we have
$$F^{(n)}(z) = \sum_{k=n}^{\infty} k(k-1)\dots(k-n+1) A_k (z - z_0)^{(k-n)};$$
2. $F^{(n)}$ is analytic for all n and have radius of convergence equal to R;
3. Taylor's formula holds, i.e., $A_k = F^{(k)}(z_0)/k!$ so
$$F(z) = \sum_{k=0}^{\infty} \frac{F^{(k)}(z_0)}{k!} (z - z_0)^k;$$
4. F enjoys the covering property around z if and only if $f'(z) \neq 0$ for all $f \in F$.

Proof. This is clear by Theorem 7.6 and Theorem 5.5. □

Corollary 7.8. Let $F(z) = \sum_{k=0}^{\infty} A_k (z - z_0)^k$ and $G(z) = \sum_{k=0}^{\infty} B_k (z - z_0)^k$, with A_k, B_k countable and compact, have positive radius of convergence. If $F(z) = G(z)$ near z_0 then
$$A_k = B_k \text{ for all } k = 0, 1, \dots.$$

Proof. This is clear by (3) in the above corollary. □

8. Barrow's theorem and Taylor's formula

Let $\gamma\colon [0,1] \to \mathbf{C}$ be a C^1 curve. We define the *path integral* of F over the curve γ as

$$\int_\gamma F(z)\,dz = \int_0^1 F(\gamma(t))\gamma'(t)\,dt.$$

With this notion the following generalization of Barrow's theorem holds.

Theorem 8.1 (Barrow's theorem)**.** Let F be non-singular and convex-valued in a neighborhood of im γ, where γ is a C^1 curve between the points a and b. Then

$$F(b) \subset F(a) + \int_\gamma F'(z)\,dz.$$

The proof below will show that if F is not convex, then

$$F(b) \subset \operatorname{cvx} F(a) + \int_\gamma F'(z)\,dz.$$

This is good since it allows us to iterate the formula with the (usually) non-convex multifunction F' in place of F to obtain the Taylor expansion formula, which we now state and prove before proving Barrow's theorem.

Theorem 8.2 (Taylor's formula)**.** Let F be non-singular and convex-valued in a convex set Ω and suppose that F has non-singular derivatives up to some order n. Then for any a and z in Ω, we have

$$F(z) \subset F(a) + \sum_{k=1}^{n} \frac{\operatorname{cvx} F^{(k)}(a)}{k!} + R(z)$$

and

$$F(z) \subset F(a) - \sum_{k=1}^{n} \frac{\operatorname{cvx} F^{(k)}(a)}{k!} + R(z),$$

where the remainder term R is of order $(z-a)^{n+1}$.

Proof. The proof goes along the same lines as in an ordinary calculus course. We shall do the case $n = 1$ and it will be clear how to do it for any n. First, however, we need some facts from the theory of integration of multifunctions. For details and proofs, we refer to [1, Chapter 8].

Let $F\colon [a,b] \rightrightarrows \mathbf{R}^d$ be an upper semi-continuous, compact valued multifunction. Define the set of *integrable selections* of F as

$$\mathcal{F} = \{f \in L^1([a,b]);\ f(x) \in F(x) \text{ almost everywhere in } [a,b]\}.$$

Then the integral of F over $[a,b]$ is defined by

$$\int_a^b F(t)\,dt = \left\{ \int_a^b f(t)\,dt;\ f \in \mathcal{F} \right\}.$$

Let us list a few properties of the integral in a proposition.

Proposition 8.3. If F and G are upper semi-continuous and compact-valued, then

i) $\displaystyle\int_a^b F(t)\,dt$ is compact and convex;

ii) $\displaystyle\int_a^b (F+G)(t)\,dt = \int_a^b F(t)\,dt + \int_a^b G(t)\,dt$;

iii) $\displaystyle\sigma(p, \int_a^b F(t)\,dt) = \int_a^b \sigma(p, F(t))\,dt$ for all $p \in \mathbf{R}^d$.

Continuing with the proof of the Taylor expansion formula, we assume that F and F' are non-singular in Ω and we let γ be the straight line between a and z: $\gamma(t) = (1-t)a + tz,\ 0 \le t \le 1$. From Barrow's theorem we then get

$$F(z) \subset F(a) + \int_\gamma F'(\zeta)\,d\zeta = \int_0^1 F'(\gamma(t))(z-a)\,dt$$

and

$$F'(\gamma(t)) \subset \operatorname{cvx} F'(a) + \int_0^t F''(\gamma(s))(z-a)\,ds.$$

Thus

$$\begin{aligned} F(z) &\subset F(a) + \int_0^1 (\operatorname{cvx} F'(a) + \int_0^t F''(\gamma(s))(z-a)\,ds)(z-a)\,dt \\ &= F(a) + \int_0^1 F'(a)(z-a)\,dt + \int_0^1 \int_0^t F''(\gamma(s))(z-a)^2\,ds\,dt. \end{aligned} \tag{8.1}$$

But F'' is locally bounded, so we can estimate it on γ with the set lB to get

$$\begin{aligned} F(z) &\subset F(a) + \int_0^1 F'(a)(z-a)\,dt + (z-a)^2 lB \int_0^1 \int_0^t ds\,dt \\ &= \operatorname{cvx} F'(a)(z-a) + lB\frac{(z-a)^2}{2}. \end{aligned} \tag{8.2}$$

□

We shall now prove Theorem 8.1. In the proof we will use the supporting functions of the values of F. We shall therefore need some facts about this. For details see [5].

Recall that the *supporting function* $\sigma(p, F)$ of a set $F \in \mathbf{R}^d$ is defined as

$$\sigma(p, F) = \sup\big(\langle p, x\rangle;\ x \in F\big),$$

where $p \in \mathbf{R}^d$. Let F_1 and F_2 be two convex, compact subsets of $\mathbf{R}^d$. Then

$$F_1 \subset F_2 \text{ if and only if } \sigma(p, F_1) \le \sigma(p, F_2) \text{ for all } p \in S. \tag{8.3}$$

Moreover, the Hausdorff distance between F_1 and F_2,

$$\mathrm{d}(F_1, F_2) = \max\big(\sup_{y \in F_2} \operatorname{dist}(F_1, y), \sup_{x \in F_1} \operatorname{dist}(x, F_2)\big),$$

is given by

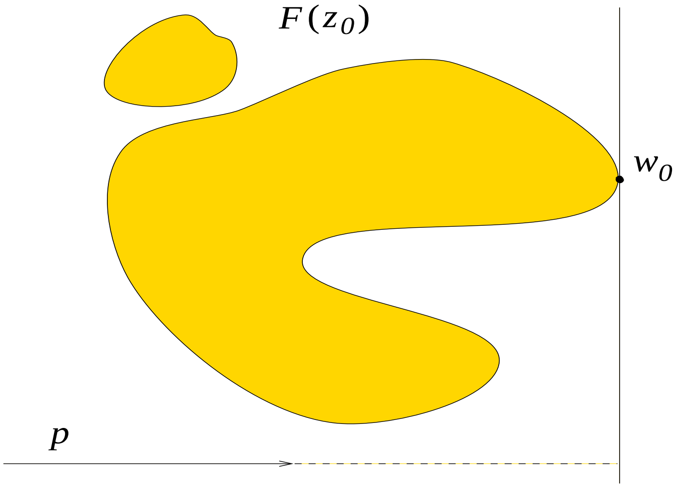
$$\mathrm{d}(F_1, F_2) = \sup_{p \in S} |\sigma(p, F_1) - \sigma(p, F_2)|. \tag{8.4}$$

The following lemma connects the derivative (in the ordinary sense) of the supporting function with our set-valued derivative of a multifunction.

Lemma 8.4. *Given $p \in S$ let $s(z) = \sigma(p, F(z))$. Whenever $\nabla s(z_0)$ exists we have*

$$(-\nabla s(z_0), p) \in N(z_0; \operatorname{gph} F).$$

Proof. Suppose $\nabla s(z_0)$ exists and let w_0 be such that $\sigma(p, F(z_0)) = \langle p, w_0 \rangle$.



We then have, for $(z, w) \in \operatorname{gph} F$,

$$\begin{aligned}\frac{\langle(-\nabla s(z_0), p), (z, w) - (z_0, w_0)\rangle}{|(z, w) - (z_0, w_0)|} &= \frac{-\langle \nabla s(z_0), z - z_0\rangle + \langle p, w - w_0\rangle}{\sqrt{|z - z_0|^2 + |w - w_0|^2}} \\ &\le \frac{-\langle \nabla s(z_0), z - z_0\rangle + s(z) - s(z_0)}{\sqrt{|z - z_0|^2 + |w - w_0|^2}} \to 0.\end{aligned}$$

□

The lemma shows that $\nabla\sigma(p, F(z)) \in \overline{pF'(z)}$ whenever the derivative exists.

Proof of Theorem 8.1. Let

$$\gamma\colon [0,1] \to \mathbf{C}$$

be a smooth curve with $\gamma(0) = a$ and $\gamma(1) = b$. Let $p \in S$ be arbitrary and define $s(z)$ as above. Since F is non-singular we know that $F \circ \gamma$ is locally Lipschitz. From formula (8.4) it is then clear that $s \circ \gamma$ is also locally Lipschitz. It is then also absolutely continuous, differentiable almost everywhere, and

$$s(b) = s(a) + \int_0^1 (s \circ \gamma)'(t)\, dt.$$

Applying the lemma to the function $s \circ \gamma$ and the multifunction $F \circ \gamma$ we see that $(s\circ\gamma)'(t) \in \langle \overline{pF'(\gamma(t))}, \gamma'(t)\rangle$ whenever the derivative $(s\circ\gamma)'$ exists. We thus get

$$\begin{aligned}\int_0^1 (s \circ \gamma)'(t)\, dt &\in \int_0^1 \langle \overline{pF'(\gamma(t))}, \gamma'(t)\rangle\, dt \\ &= \int_0^1 \langle p, F'(\gamma(t))\gamma'(t)\rangle\, dt = \langle p, \int_0^1 F'(\gamma(t))\gamma'(t)\, dt\rangle = \langle p, \int_\gamma F'(z)\, dz\rangle.\end{aligned}$$

Thus

$$\sigma(p, F(a)) \le \sigma(p, F(b)) + \sigma(p, \int_\gamma F'(z)\, dz)$$

so Barrow's theorem follows from (8.3). □

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