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GRAPH CONVERGENCE OF SET-VALUED MAPS AND ITS RELATIONSHIP TO OTHER CONVERGENCES

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Abstract. The notion of even-outer-semicontinuity for set-valued maps is introduced and compared with related ones from [4] and [11]. The coincidence of these notions provides a new characterization of compactness and of local compactness. The following result is proved: Let X be a topological space, Y a uniform space, $\{F_\sigma : \sigma \in \Sigma\}$ be a net of set-valued maps from X to Y and F be a set valued map from X to Y. Then any two of the following conditions imply the third: (1) the net $\{F_\sigma : \sigma \in \Sigma\}$ is evenly-outer semicontinuous; (2) the net $\{F_\sigma : \sigma \in \Sigma\}$ is graph convergent to F; (3) the net $\{F_\sigma : \sigma \in \Sigma\}$ is pointwise convergent to F. This theorem generalizes some results from [4] and [11].

Graph convergence (that is Painlevé-Kuratowski convergence of graphs) of set-valued maps was studied in many books and papers (see for example [1, 2, 4, 9, 11]). In this topic we can include also graph convergence of single-valued maps [5, 12, 13], epiconvergence of lower semicontinuous functions [4, 6, 7] as well as Painlevé-Kuratowski convergence of graphs of partial maps [8] . In the books of Attouch [1], Aubin-Frankowska [2]

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and Rockafellar-Wets [14] can be found many applications of this convergence to variational and optimization problems, differential equations and approximation theory. Graph convergence of preference relations is used also in mathematical economics [3]. However graph convergence of maps, even when they are single valued, is not always easy to verify. In our paper we study the relationship between graph convergence and pointwise one introducing the notion of even-outer-semicontinuity which allows us to pass from one convergence to the other. Other notions of *joint continuity* (see for example [4, 9, 11]) are compared with ours. The relationship between the notions of continuous convergence and graph convergence is analysed too.

1. Preliminaries

If Y is a topological space and $y \in Y$, U_y (or U if there is no possibility of confusion) will denote a general nbd (neighborhood) of y and $\mathcal{U}(y)$ the set of all open nbds of y. $K(Y)$ and 2^Y will denote the sets of all compact and all closed subsets of Y respectively.

Let us recall [5] that a net $\{A_\sigma : \sigma \in \Sigma\}$ of subsets of Y is said to be:

- K^+-*convergent* to a subset A of Y if $\mathrm{Ls}\ A_\sigma \subset A$, where $\mathrm{Ls}\ A_\sigma = \{y \in Y\colon$ each nbd of y intersect A_σ for all σ in some cofinal subset of $\Sigma\}$;
- K^--*convergent* to A if $A \subset \mathrm{Li}\ A_\sigma$, where $\mathrm{Li}\ A_\sigma = \{y \in Y\colon$ each nbd of y intersect A_σ in some residual subset of $\Sigma\}$;
- *topologically convergent* (or *Painlevé-Kuratowski convergent*) to A, (and we will write $\mathrm{Lt}\ A_\sigma = A$) if $\mathrm{Li}\ A_\sigma = \mathrm{Ls}\ A_\sigma = A$.

If Y is completely regular, γ is a compatible uniformity on Y and A is closed, it is easy to prove that:

- $\mathrm{Ls}\ A_\sigma \subset A$ if and only if for every $y \in Y$ and every V in γ there is a nbd U_y of y and $\sigma_0 \in \Sigma$ such that for every $\sigma \geq \sigma_0$
$$A_\sigma \cap U_y \subset V[A]\,;$$
- $A \subset \mathrm{Li}\ A_\sigma$ if and only if for every $y \in Y$ and every V in γ there is a nbd U_y of y and $\sigma_0 \in \Sigma$ such that for every $\sigma \geq \sigma_0$
$$A \cap U_y \subset V[A_\sigma].$$

If X is a topological space, $C(X, 2^Y)$ will be the family of all continuous set-valued maps from X to Y with closed values and, if F is a set-valued map from X to Y, $\mathrm{Gr}\ F$ will denote the graph of F, that is

$$\mathrm{Gr}\ F = \{(x, y) \in X \times Y : y \in F(x)\}.$$

If F and F_σ are set-valued maps from X to Y, we will say that:

- the net $\{F_\sigma : \sigma \in \Sigma\}$ is *pointwise convergent* at $x \in X$ to F if $F(x) = \mathrm{Lt}\ F_\sigma(x)$; is *pointwise convergent* in X if this holds for every $x \in X$,
- the net $\{F_\sigma : \sigma \in \Sigma\}$ is *graph-convergent* to F if $\mathrm{Gr}\ F = \mathrm{Lt}\ \mathrm{Gr}\ F_\sigma$.

For any undefined terms see [6, 10].

It can be proved that if X is a discrete topological space, graph convergence coincides with pointwise one (see [11] for sequences of set-valued maps), while in general neither graph nor pointwise convergence implies the other. We can even characterize discreteness by this coincidence, as the following theorems show.

First we recall that $C([0,1], 2^Y)$ is non-trivial if there exists a continuous set-valued function satisfying the condition $F(0) \neq F(1)$.

The following example shows that there is such a topological space Y for which $C([0,1], 2^Y)$ is non-trivial but $C([0,1], Y)$ is trivial.

Example 1.1. If $X = [0,1]$ is equipped with the natural topology, $Y = \{0,1,2\}$ is equipped with the topology $\boldsymbol{\mathcal{T}}$, having as open sets $\{0\}$ and $\{1,2\}$, then $C([0,1], Y)$ is trivial while $C([0,1], 2^Y)$ is non-trivial, since F defined by

$$F(x) = \begin{cases} \{0,1\} & \text{if} \quad x \in [0,1) \\ \{0,1,2\} & \text{if} \quad x = 1 \end{cases}$$

is continuous and non trivial.

Now we can prove:

Proposition 1.2. *Let X be a Tychonoff non discrete space and Y be such that $C([0,1], 2^Y)$ is not trivial. There is a net $\{F_\sigma : \sigma \in \Sigma\}$ in $C(X, 2^Y)$ pointwise convergent to $F \in C(X, 2^Y)$ which fails to be graph convergent to F.*

Proof. Let x_0 be a non-isolated point in X and for every open nbd U of x_0 choose a point $x_U \in U \backslash \{x_0\}$. There is a continuous function h_U from X to $[0,1]$ satisfying the conditions $h_U(x_U) = 1$ and $h_U(x) = 0$ for every $x \in (X \backslash U) \cup \{x_0\}$. There is a continuous set-valued map H from $[0,1]$ to 2^Y such that $H(1) \backslash H(0) \neq \emptyset$. For every open nbd U of x_0 the set-valued map $F_U = H \circ h_U$ is continuous and the net $\{F_U : U \in \mathcal{U}(x_0)\}$ is pointwise convergent to F where $F(x) = H(0)$ at every x. Indeed if $x \neq x_0$ and U is an open nbd of x_0 such that $x \notin U$, for every nbd $U' \geq U$ it results $F_{U'}(x) = H(0)$. If $x = x_0$, then $F_U(x_0) = H(0)$ for every nbd U. However $\mathrm{Lt}\ \mathrm{Gr}\ F_U \neq \mathrm{Gr}\ F$, since $(x_0, y) \in \mathrm{Li}\ \mathrm{Gr}\ F_U$ if $y \in H(1) \backslash H(0)$. $\square$

Proposition 1.3. *Let X be a Tychonoff non discrete space and Y be a normal space such that $C([0,1],2^Y)$ is not trivial. There is a net $\{F_\sigma : \sigma \in \Sigma\}$ in $C(X,2^Y)$ graph convergent to $F \in C(X,2^Y)$ which fails to be pointwise convergent to F.*

Proof. There is $h \in C([0,1],2^Y)$ such that $h(1)\backslash h(0) \neq \emptyset$. It is easy to prove that the set-valued map, from $[0,1]$ to Y, $H(x) = \overline{\bigcup_{s\leq x} h(s)}$ is continuous. Let x_0 be a non-isolated point in X. For every open nbd U of x_0, let f_U be a continuous function from X to $[0,1]$ such that $f_U(x_0) = 0$ and $f_U(X\backslash U) = 1$. If we put $F_U = H \circ f_U$, $\{F_U : U \in \mathcal{U}(x_0)\}$ is a net of continuous set-valued maps which graph converges to F, where $F(x) = H(1)$ for every $x \in X$, but does not pointwise converges to F since $\mathrm{Lt}\ F_U(x_0) = h(0)$. □

2. Continuity and equicontinuity

In this part, for nets of set-valued maps, we introduce the notion of even-outer-semicontinuity and even-inner-semicontinuity and compare them with others known in the literature.

It is well known that for closed-valued maps between topological spaces there are various notions of continuity connected with the convergence or topology supported by 2^Y. If the values belong to a uniform space (Y,γ), when 2^Y is equipped with K^+ or K^- convergence we obtain the following definitions:

Remark 2.1. A set-valued map F from X to Y, closed-valued in x, is *outer-semicontinuous at* x if for every V in γ and for every $y \in Y$ there is a nbd U_x of x, a nbd U_y of y such that, for every $z \in U_x$ it results:

$$F(z) \cap U_y \subset V[F(x)].$$

Remark 2.2. A set-valued map F from X to Y, closed-valued in x, is *lower or inner-semicontinuous at* x if for every V in γ and for every $y \in Y$ there is a nbd U_x of x, a nbd U_y of Y such that, for every $z \in U_x$ it results

$$F(x) \cap U_y \subset V[F(z)].$$

In the literature there are many definitions of *joint continuity* for single-valued or set-valued maps [9, 10, 14]. For nets of set-valued maps with values in a uniform space we give the following definitions which absorb the classical definitions of *equicontinuity* and *even-continuity* for a family of functions due to Kelley [10].

Definition 2.3. A net $\{F_\sigma : \sigma \in \Sigma\}$ is *evenly-outer-semicontinuous* at x if for every V in γ and for every $y \in Y$ there is a nbd U_x of x, a nbd U_y of y and a $\sigma_0 \in \Sigma$ such that, for every $\sigma \geq \sigma_0$ and for every $z \in U_x$ it results:

$$F_\sigma(z) \cap U_y \subset V[F_\sigma(x)].$$

Definition 2.4. A net $\{F_\sigma : \sigma \in \Sigma\}$ is *evenly-inner-semicontinuous* at x if for every V in γ and for every $y \in Y$ there is a nbd U_x of x, a nbd U_y of y and a $\sigma_0 \in \Sigma$ such that, for every $\sigma \geq \sigma_0$ and for every $z \in U_x$ it results:

$$F_\sigma(x) \cap U_y \subset V[F_\sigma(z)]. \tag{$*$}$$

Definition 2.5. A net $\{F_\sigma : \sigma \in \Sigma\}$ is *evenly-semicontinuous* at x if it is evenly-outer and evenly-inner-semicontinuous at x.

If the above definitions are verified for every $x \in X$ we will say that the net is *evenly-inner-semicontinuous or evenly-outer-semicontinuous or evenly-semicontinuous (in X)*.

Remark 2.6. Notice that if we require that the condition $(*)$ is satisfied for every $\sigma \in \Sigma$, our Definition 2.4 restricted on a net $\{f_\sigma : \sigma \in \Sigma\}$ of single-valued functions is equivalent to the classical notion of even continuity at x [10].

Let $\{f_\sigma : \sigma \in \Sigma\}$ be evenly continuous at x. If $V \in \gamma$ and $y \in Y$ there is $V_1 \in \gamma$ symmetric, open and such that $V_1 \circ V_1 \subset V$. By even continuity of $\{f_\sigma : \sigma \in \Sigma\}$, starting from x, y and $V_1[y]$, we can find a nbd U_x of x and a nbd U_y of y such that $f_\sigma(U_x) \subset V_1[y]$ whenever $f_\sigma(x) \in U_y$ and whenever $\sigma \in \Sigma$. We can prove that for every $z \in U_x$ and $\sigma \in \Sigma$ it results:

$$\{f_\sigma(x)\} \cap U_y \cap V_1[y] \subset V[f_\sigma(z)].$$

Indeed, if $\{f_\sigma(x)\} \cap U_y \cap V_1[y]$ is non empty, $f_\sigma(x)$ belongs to U_y and $f_\sigma(z)$ belongs to $V_1[y]$ for every $z \in U_x$. So we obtain $(f_\sigma(x), f_\sigma(z)) \in V$ since $(f_\sigma(x), y) \in V_1$ and $(f_\sigma(z), y) \in V_1$.

To prove the opposite, let $\{f_\sigma : \sigma \in \Sigma\}$ be evenly-outer-semicontinuous at x with respect to all σ. If $y \in Y$ and U_y is a nbd of y, there is V in γ with $V[y] \subset U_y$ and V_1 in γ such that $V_1 \circ V_1 \subset V$. By assumption there is a nbd U_x of x and a nbd U'_y of y such that for every $z \in U_x$ it results

$$\{f_\sigma(x)\} \cap V_1[y] \cap U'_y \subset V_1[f_\sigma(z)].$$

Now, if for a σ $f_\sigma(x) \in V_1[y] \cap U'_y$, it results $f_\sigma(x) \in V_1[f_\sigma(z)]$, therefore $(f_\sigma(x), y)$ belongs to V_1 and $f_\sigma(z) \in V[y] \subset U_y$ for every $z \in U_x$.

In the literature we can find notions stronger than even-continuity. Due to Kowalczyk [11], there are the following notions concerning a family $\mathcal{F}$ of set-valued maps with values in a uniform space (Y, γ).

Definition 2.7. $\mathcal{F}$ is *upper equicontinuous* at $x \in X$ if for every $V \in \gamma$

$(**)$ there exists a nbd U_x of x such that whenever $F \in \mathcal{F}$ and $z \in U_x$ then $F(z) \subset V[F(x)]$.

Definition 2.8. $\mathcal{F}$ is *lower equicontinuous* at $x \in X$ if for every $V \in \gamma$

there exists a nbd U_x of x such that whenever $F \in \mathcal{F}$ and $z \in U_x$ then $F(x) \subset V[F(z)]$.

Definition 2.9. $\mathcal{F}$ is *equicontinuous* at $x \in X$ if it is lower and upper equicontinuous at x.

All definitions given in 2.7, 2.8 and 2.9 applied to a family $\mathcal{F}$ of single-valued functions coincide with the classical notion of equicontinuity. It is easy to prove that if a net of set-valued maps is upper-equicontinuous then it is also even-outer-semicontinuous. We will say that a net $\{F_\sigma : \sigma \in \Sigma\}$ is *asymptotically upper equicontinuous* if the condition $(**)$ is satisfied for all σ sufficiently large. This notion can be found for example in [9] under the name *quasi equi-semicontinuity.* If Y is compact then this notion coincides with the notion of even-outer-semicontinuity. In general, however this notion turns out to be too constringent for applications purposes [4].

We have the following characterization of compact spaces.

Theorem 2.10. *If X is a non discrete topological space and Y is a Tychonoff space, the following conditions are equivalent:*

a) *Y is compact;*
b) *every net of set-valued maps from X to Y is asymptotically upper equicontinuous iff it is evenly-outer-semicontinuous.*

Proof. If Y is compact, it is easy to verify that both of the definitions coincide. Let us prove the converse. Suppose Y is not compact, then there is a net $\{y_\sigma : \sigma \in \Sigma\}$ of points of Y having no cluster point and a point x_0 in X which is not isolated, therefore in every nbd U of x_0 we can choose a point $x_U \in U \backslash \{x_0\}$. Without loss of generality we can suppose that there is a point $y \in Y$ different from all y_σ. Consider the set $\mathcal{U}(x_0) \times \Sigma$ with the natural order (that is $(U, \sigma) \geq (U', \sigma')$ iff $U \subset U'$ and $\sigma \geq \sigma'$). The net $\{F_{(U,\sigma)} : (U, \sigma) \in (\mathcal{U}(x_0) \times \Sigma)\}$ defined by

$$F_{(U,\sigma)}(x) = \begin{cases} \{y\} & \text{if} \quad x \neq x_U \\ \{y, y_\sigma\} & \text{if} \quad x = x_U \end{cases}$$

is not upper equicontinuous at x_0 with respect to any compatible uniformity γ on Y. Indeed, if $V \in \gamma$ there is σ_0 in Σ such that $y_\sigma \notin V[y]$ for all $\sigma \geq \sigma_0$;

thus $F_{(U,\sigma)}(x_U) \not\subset V[F_{(U,\sigma)}(x_0)]$ for every nbd U of x_0 and every $\sigma \geq \sigma_0$. We can prove that this net is evenly-outer-semicontinuous. Indeed, if $V \in \gamma$, $z \in Y$ and U_z is a nbd of z which does not contain y_σ for all $\sigma \geq \sigma_0$, we have

$$U_z \cap F_{(U,\sigma)}(x) \subset V[F_{(U,\sigma)}(x_0)]$$

for all $x \in X$ and for all $\sigma \geq \sigma_0$. □

Due to [4] are the following definitions concerning set-valued maps from a topological space X to a metric space Y. If $\mathcal{I}$ is an index space, $\mathcal{H}$ a filter on $\mathcal{I}$ and $\{F_i : i \in \mathcal{I}\}$ a collection of set-valued maps it states:

Definition 2.11. $\{F_i : i \in \mathcal{I}\}$ is *equi-outer-semicontinuous* at x_0 if for every compact set $B \subset Y$ and every $\varepsilon \geq 0$ there exists a nbd V of x_0 and $H \in \mathcal{H}$ such that for every $x \in V$ and every $i \in H$

$$F_i(x) \cap B \subset \varepsilon F_i(x_0) \ .$$

Definition 2.12. $\{F_i : i \in \mathcal{I}\}$ is *equi-inner-semicontinuous* at x_0 if for every compact set $B \subset Y$ and every $\varepsilon \geq 0$ there exists a nbd V of x_0 and $H \in \mathcal{H}$ such that for every $x \in V$ and every $i \in H$

$$F_i(x_0) \cap B \subset \varepsilon F_i(x) \ .$$

Definition 2.13. $\{F_i : i \in \mathcal{I}\}$ is *equi-semicontinuous* at x_0 if it is both equi-upper and equi-inner semicontinuous at x_0.

Obviously if the above conditions are verified for every $x \in X$, $\{F_i : i \in \mathcal{I}\}$ will be said *equi-outer or equi-inner or equi-semicontinuous.* Of course we can naturally extend the above Definitions 2.11, 2.12, 2.13 for any net with values in uniform spaces. It is easy to verify that if a net of set-valued maps is evenly-outer (inner)-semicontinuous then it is also equi-outer (inner)-semicontinuous and if Y is locally compact then these notions coincide. We can even characterize local compactness by this coincidence as the following theorem states.

Theorem 2.14. *If X is a non discrete topological space and Y is a Tychonoff space, the following conditions are equivalent:*

a) *Y is locally compact;*
b) *every net of set-valued maps from X to Y is equi-outer-semicontinuous iff it is evenly-outer-semicontinuous.*

Proof. If Y is locally compact, it is easy to verify that condition b) is verified. Now, suppose Y is not locally compact. There is $y \in Y$ which has no compact nbd. Thus, for every open nbd U of y and every $K \in K(Y)$, there is $y_{U,K} \in U \backslash K$. Let $x \in X$ be a non-isolated point and $z \in Y$ be a point different from y. For every open nbd W of x choose $x_W \in W \backslash \{x\}$.

If we consider $\mathcal{L} = \mathcal{U}(x) \times \mathcal{U}(y) \times K(Y)$ with the natural direction (i.e. $(W,U,K) \leq (W',U',K')$ iff $W' \subset W$, $U' \subset U$, $K \subset K'$) and for every (W,U,K) we put

$$F_{(W,U,K)}(t) = \begin{cases} \{z\} & \text{if} \quad t \neq x_W \\ \{z, y_{U,K}\} & \text{if} \quad t = x_W \end{cases}$$

we can prove that the net $\{F_{(W,U,K)} : (W,U,K) \in \mathcal{L}\}$ is equi-outer-semi-continuous at every point with respect to any compatible uniformity γ. It is sufficient to verify this only at the point x. Let $V \in \gamma$ and $K \in K(Y)$. Then for every $v \in X$ and $(W,U,C) \geq (X,Y,K)$ we have

$$F_{(W,U,C)}(v) \cap K \subset V[F_{(W,U,C)}(x)] \, .$$

However $\{F_{(W,U,K)} : (W,U,K) \in \mathcal{L}\}$ is not evenly-outer-semicontinuous at x. Let $G \in \gamma$ be such that $G[z] \cap G[y] = \emptyset$ and G is open and symmetric. Now for every nbd W of x, every nbd U of y and every $(C,D,K) \in \mathcal{L}$ we have

$$F_{(W \cap C,\, U \cap D \cap G[y],\, K)}(x_{W \cap C}) \cap U = \{z, y_{U \cap D \cap G[y], K}\} \cap U \not\subset G[z] \, .$$

□

If X and Y are first countable topological spaces then a sequence of set-valued maps is equi-outer-semicontinuous iff it is evenly-outer semicontinuous.

Theorem 2.15. *Let X and Y be first countable topological spaces and (Y,γ) be a uniform space. A sequence $\{F_n : n \in \mathbb{Z}^+\}$ of set-valued maps from X to Y, is equi-outer-semicontinuous iff it is evenly-outer-semicontinuous.*

Proof. Of course it is sufficient to prove that if $\{F_n : n \in \mathbb{Z}^+\}$ is equi-outer-semicontinuous at x then it is also even-outer-semicontinuous at x. Suppose that this is not true. This means that there is $V \in \gamma$ and $y \in Y$ such that for every nbd W of x, U of y and for every $m \in \mathbb{Z}^+$ there exists $n(W,U,m) \geq m$ and a point $x_{n(W,U,m)}$ in W such that

$$F_{n(W,U,m)}(x_{n(W,U,m)}) \cap U \not\subset V[F_{n(W,U,m)}(x)] \, .$$

Let $\{W_n\}$ and $\{U_n\}$ be a countable base of nbds of x and y respectively. Thus for every $n \in \mathbb{Z}^+$ there is $m_n \geq n$, x_n and y_n such that $x_n \in W_n$, $y_n \in F_{m_n}(x_n) \cap U_n \backslash V[F_{m_n}(x_n)]$. Of course $\{x_n : n \in \mathbb{Z}^+\}$ converges to x

and $\{y_n : n \in \mathbb{Z}^+\}$ converges to y. Consider the compact set $B = \{y_n : n \in \mathbb{Z}^+\} \cup \{y\}$. By assumption there must exist a nbd W' of x and $n_0 \in \mathbb{Z}^+$ with

$$F_n(r) \cap B \subset V[F_n(x)]$$

for every $n \geq n_0$ and for every $r \in W'$, but this is a contradiction since, if $n \geq n_0$ is such that $W_n \subset W'$, we have

$$y_n \in F_{m_n}(x_n) \cap B \backslash V[F_{m_n}(x)] \, .$$

□

3. Pointwise and graph convergence

In this part we prove the main result of our paper. We show that even-outer-semicontinuity allows us to pass from the pointwise convergence to the graph convergence and vice versa. We also prove that if a net is simultaneously pointwise and graph convergent then it has to be evenly-outer-semicontinuous.

Theorem 3.1. *Let X be a topological space and (Y, γ) be a uniform space. Suppose that $\{F_\sigma : \sigma \in \Sigma\}$ is evenly-outer-semicontinuous and pointwise convergent to F. Then it is also graph convergent to F.*

Proof. Of course Gr $F \subset$ Li Gr F_σ. To prove that Ls Gr $F_\sigma \subset$ Gr F, we will show that if $(x, y) \in$ Ls Gr F_σ then $y \in$ Ls $F_\sigma(x)$. If U_y is a nbd of y and σ an element of Σ, we will find $\eta \geq \sigma$ such that $F_\eta(x) \cap U_y \neq \emptyset$. There is V_1 (open and symmetric) in γ satisfying $V_1[y] \subset U_y$ and V_2 (open and symmetric) in γ satisfying $V_2 \circ V_2 \subset V_1$. Since $\{F_\sigma : \sigma \in \Sigma\}$ is evenly-outer-semicontinuous there is a nbd U_x of x, a nbd U'_y of y and $\sigma_0 \in \Sigma$ with

$$F_\sigma(z) \cap U'_y \subset V_2[F_\sigma(x)] \qquad (\square)$$

for every $z \in U_x$ and every $\sigma \geq \sigma_0$. Without loss of generality we can suppose that $U'_y \subset V_2[y]$. Since $U_x \times U'_y$ is a nbd of $(x, y) \in$ Ls Gr F_σ, there is $\eta \geq \sigma$, $\eta \geq \sigma_0$ with

$$\text{Gr } F_\eta \cap (U_x \times U'_y) \neq \emptyset \, .$$

If we choose $(z_\eta, y_\eta) \in$ Gr $F_\eta \cap (U_x \times U'_y)$, from $(\square)$ we obtain

$$y_\eta \in F_\eta(z_\eta) \cap U'_y \subset V_2[F_\eta(x)] \, .$$

Thus there is $v_\eta \in F_\eta(x)$ with $(y_\eta, v_\eta) \in V_2$, and, being $(y_\eta, y) \in V_2$, it results that

$$(v_\eta, y) \in V_2 \circ V_2 \subset V_1 \, ,$$

i.e. $V_1[y] \cap F_\eta(x) \neq \emptyset$. □

Theorem 3.2. *Let X be a topological space and (Y,γ) be a uniform one. Suppose that $\{F_\sigma : \sigma \in \Sigma\}$ is evenly-outer-semicontinuous and graph convergent to F. Then it is also pointwise convergent to F.*

Proof. Of course Ls $F_\sigma(x) \subset F(x)$. Now let $y \in F(x)$ and let U_y be a nbd of y. We will show that there is $\sigma_0 \in \Sigma$ for which

$$F_\sigma(x) \cap U_y \neq \emptyset$$

for every $\sigma \geq \sigma_0$. There are V_1 and V_2 (both open and symmetric) in γ satisfying $V_1[y] \subset U_y$ and $V_2 \circ V_2 \subset V_1$. Since $\{F_\sigma : \sigma \in \Sigma\}$ is evenly-outer-semicontinuous, there is a nbd U_x of x, a nbd U'_y of y and $\sigma_0 \in \Sigma$ with

$$F_\sigma(z) \cap U'_y \subset V_2\,[F_\sigma(x)]$$

for every $z \in U_x$ and every $\sigma \geq \sigma_0$. Without loss of generality we can suppose that $U'_y \subset V_2[y]$. Since $U_x \times U'_y$ is a nbd of $(x,y) \in$ Li Gr F_σ, there is $\sigma_1 \in \Sigma$ such that for every $\sigma \geq \sigma_1$

$$\text{Gr } F_\sigma \cap (U_x \times U'_y) \neq \emptyset\ .$$

For $\sigma \geq \sigma_0$ and $\sigma \geq \sigma_1$, choose

$$(x_\sigma, y_\sigma) \in \text{Gr } F_\sigma \cap (U_x \times U'_y)\ .$$

We obtain that $(y_\sigma, y) \in V_2$ and

$$y_\sigma \in F_\sigma(x_\sigma) \cap U'_y \subset V_2\,[F_\sigma(x)]\ .$$

We can conclude that there is $v_\sigma \in F_\sigma(x)$ satisfying $(y_\sigma, v_\sigma) \in V_2$ that is $v_\sigma \in V_1[y] \subset U_y$. □

Theorem 3.3. *Let X be a topological space and (Y,γ) be a uniform one. Suppose that $\{F_\sigma : \sigma \in \Sigma\}$ graph and pointwise converges to F. Then it is evenly-outer-semicontinuous.*

Proof. Suppose this is not true for a point $x \in X$. There is V in γ and $y \in Y$ such that for every nbd U of x, every nbd O of y and every $\sigma \in \Sigma$, there is $\eta(U,O,\sigma) \in \Sigma$, $\eta(U,O,\sigma) \geq \sigma$ and $x_{\eta(U,O,\sigma)} \in U$ for which

$$F_{\eta(U,O,\sigma)}(x_{\eta(U,O,\sigma)}) \cap O \not\subset V[F_{\eta(U,O,\sigma)}(x)]\ .$$

This allows us to choose a net

$$y_{\eta(U,O,\sigma)} \in F_{\eta(U,O,\sigma)}(x_{\eta(U,O,\sigma)}) \cap O \setminus V[F_{\eta(U,O,\sigma)}(x)] \qquad (\triangle)$$

converging to y. It is easy to verify that

$$\boldsymbol{\mathcal{L}} = \{\eta(U,O,\sigma) : U \in \mathcal{U}(x),\, O \in \mathcal{U}(y),\, \sigma \in \Sigma\}$$

is a cofinal family in Σ, thus

$$\{F_{\eta(U,O,\sigma)} : U \in \mathcal{U}(x),\, O \in \mathcal{U}(y),\, \sigma \in \Sigma\}$$

is a subnet of $\{F_\sigma : \sigma \in \Sigma\}$.

We will show that $(x, y) \in \text{Ls Gr } F_{\eta(U,O,\sigma)}$. Let $G \in \mathcal{U}(x)$, $H \in \mathcal{U}(y)$ and $\eta(U, O, \sigma) \in \boldsymbol{\mathcal{L}}$. By the assumption we can find $\eta(G, H, \eta(U, O, \sigma)) \geq \eta(U, O, \sigma)$ and the related points satisfying the conditions:

$$x_{\eta(G,H,\eta(U,O,\sigma))} \in G$$

and

$$y_{\eta(G,H,\eta(U,O,\sigma))} \in H \cap F_{\eta(G,H,\eta(U,O,\sigma))}(x_{\eta(G,H,\eta(U,O,\sigma))})\ .$$

Since $(x, y) \in \text{ Ls } F_{\eta(U,O,\sigma)} \subset \text{Ls } F_\sigma$, we have $y \in F(x)$. There is $V_1 \in \gamma$ such that V_1 is open, symmetric and $V_1 \circ V_1 \subset V$. Being $F(x) = \text{Li } F_\sigma(x)$, there is $\sigma_0 \in \Sigma$ with $F_\sigma(x) \cap V_1\,[y] \neq \emptyset$ for every $\sigma \geq \sigma_0$. Then for $\eta(X, V_1\,[y], \sigma_0)$ we have

$$F_{\eta(X,V_1[y],\sigma_0)}(x)\, \cap\, V_1\,[y] \neq \emptyset$$

and

$$y_{\eta(X,V_1[y],\sigma_0)} \in F_{\eta(X,V_1[y],\sigma_0)}(x_{\eta(X,V_1[y],\sigma_0)}) \cap V_1[y] \subset\ (V_1 \circ V_1)[F_{\eta(X,V_1[y],\sigma_0)}(x)]$$

which is in contradiction with ($\triangle$). □

For nets of single-valued functions we obtain the following result ([5, 12, 13]):

Corollary 3.4. *Let X be a topological space and Y be a uniform one. Let f be a single-valued function from X to Y and $\{f_\sigma : \sigma \in \Sigma\}$ be a net of single-valued functions from X to Y. Consider the following conditions:*

a) $\{f_\sigma : \sigma \in \Sigma\}$ *converges pointwise to* f*;*

b) $\{f_\sigma : \sigma \in \Sigma\}$ *graph converges to* f*;*

c) $\{f_\sigma : \sigma \in \Sigma\}$ *is evenly-outer-semicontinuous.*

Then any two of these conditions imply the other.

The following example shows that even if we consider a sequence of continuous single-valued functions the condition c) in the above corollary cannot be replaced by

c') $\{f_\sigma : \sigma \in \Sigma\}$ *is evenly-continuous.*

Let $X = \{0\} \cup \{1/n; n \in \mathbb{Z}^+\}$ with the usual topology and, for every n, put

$$f_n(x) = \begin{cases} 0 & \text{if} \quad x \neq 1/n \\ n & \text{if} \quad x = 1/n\ . \end{cases}$$

Then $\{f_n\}$ is a sequence of continuous functions pointwise and graph convergent to f, where $f(x) = 0$ at every $x \in X$. Of course $\{f_n\}$ is not evenly continuous.

Let us observe also that for sequences of continuous single-valued functions the notions of even continuity and even-outer-semicontinuity are independent. It is easy to prove that the sequence ([13])

$$f_n(x) = \begin{cases} 0 & \text{if } \ 1/n \le x \le 1 \\ n - xn^2 & \text{if } \ 0 \le x < 1/n \end{cases}$$

is evenly continuous but not evenly-outer-semicontinuous.

4. Continuous convergence

Definition 4.1. Let X and Y be topological spaces. A net $\{F_\sigma : \sigma \in \Sigma\}$ of set-valued maps from X to Y *converges continuously* to a set-valued map F if for every $x \in X$ and every net $\{x_\alpha : \alpha \in A\}$ convergent to x, the net $\{F_\sigma(x_\alpha) : (\sigma, \alpha) \in \Sigma \times A\}$ Painlevé-Kuratowski converges to $F(x)$.

Obviuosly continuous convergence implies pointwise and graph convergence (and therefore even-outer-semicontinuity).

The following proposition shows that this convergence implies as well even-inner-semicontinuity.

Proposition 4.2. *If a net $\{F_\sigma : \sigma \in \Sigma\}$ continuously converges to F, then $\{F_\sigma : \sigma \in \Sigma\}$ is evenly-semicontinuous.*

Proof. It is sufficient to prove that the net $\{F_\sigma : \sigma \in \Sigma\}$ is evenly-inner-semicontinuous. If it is not true at a point $x \in X$, then there exists V in γ and $y \in Y$ such that for every nbd U of x, every nbd O of y and every $\sigma \in \Sigma$ there is $\eta(U,O,\sigma) \in \Sigma$, $\eta(U,O,\sigma) \ge \sigma$, $x_{(U,O,\sigma)} \in U$ for which

$$F_{\eta(U,O,\sigma)}(x) \cap O \not\subset V[F_{\eta(U,O,\sigma)}(x_{(U,O,\sigma)})] \, .$$

This allows us to choose a net

$$y_{\eta(U,O,\sigma)} \in F_{\eta(U,O,\sigma)}(x) \cap O \setminus V[F_{\eta(U,O,\sigma)}(x_{\eta(U,O,\sigma)})]$$

converging to y. It is easy to verify that

$$\boldsymbol{\mathcal{L}} = \{\eta(U,O,\sigma) : U \in \mathcal{U}(x),\, O \in \mathcal{U}(y),\, \sigma \in \Sigma\}$$

is a cofinal family in Σ, thus

$$\{F_{\eta(U,O,\sigma)} : U \in \mathcal{U}(x),\, O \in \mathcal{U}(y),\, \sigma \in \Sigma\}$$

is a subnet of $\{F_\sigma : \sigma \in \Sigma\}$.

Since $(x,y) \in \mathrm{Ls}\ F_{\eta(U,O,\sigma)}$ we obtain $y \in F(x)$. Let V_1 be a symmetric open element from γ with $V_1 \circ V_1 \subset V$. The net

$$\{x_{(U,O,\sigma)} : (U,O,\sigma) \in \mathcal{U}(x) \times \mathcal{U}(y) \times \Sigma\}$$

converges to x (considering the natural direction on $\mathcal{U}(x) \times \mathcal{U}(y) \times \Sigma$) thus

$$\{F_\sigma(x_{(U,O,\sigma)}) : \sigma \in \Sigma,\ (U,O,\sigma) \in \mathcal{U}(x) \times \mathcal{U}(y) \times \Sigma\}$$

Painlevé-Kuratowski converges to $F(x)$ by assumption. Thus we can find $\sigma_0 \in \Sigma$ and $(U,O,\sigma) \in \mathcal{U}(x) \times \mathcal{U}(y) \times \Sigma$; such that for every $\sigma \geq \sigma_0$ and every $(U',O',\sigma') \geq (U,O,\sigma)$

$$F_\sigma(x_{(U',O',\sigma')}) \cap V_1[y] \neq \emptyset\ .$$

Let $\sigma' \in \Sigma$ be such that $\sigma' \geq \sigma$ and $\sigma' \geq \sigma_0$. If we put $\eta' = \eta(U, O \cap V_1\,[y], \sigma')$, it results that $\eta' \in \boldsymbol{\mathcal{L}}$ and

$$y_{\eta'} \in V_1\,[y] \subset V_1 \circ V_1[F_{\eta'}(x_{(U,O\cap V_1\,[y],\sigma')})] \subset V[F_{\eta'}(x_{(U,O\cap V_1\,[y],\sigma')})]$$

and this is a contradiction.

Since the continuous convergence at x implies that $\{F_\sigma : \sigma \in \Sigma\}$ is evenly-outer-semicontinuous at x, the net has to be evenly-semicontinuous. □

Proposition 4.3. *If a net $\{F_\sigma : \sigma \in \Sigma\}$ graph converges to F and is enenly-semicontinuous then $\{F_\sigma : \sigma \in \Sigma\}$ converges continuosly to F.*

Proof. Let $\{x_\alpha : \alpha \in A\}$ be a net in X convergent to x. It is sufficient to prove that $F(x) \subset \mathrm{Li}\ F_\sigma(x_\alpha)$. Let $y \in F(x)$ and O be a nbd of y (we will find $\sigma_0 \in \Sigma$ and $\alpha_0 \in A$ such that for every $\sigma \geq \sigma_0$ and $\alpha \geq \alpha_0$ it results $F_\sigma(x_\alpha) \cap O \neq \emptyset$).

Let $U \in \gamma$ be such that $U\,[y] \subset O$, let further V_1 be an open symmetric element from γ with $V_1 \circ V_1 \subset U$. Since $\{F_\sigma : \sigma \in \Sigma\}$ is evenly-inner-semicontinuous at x, there exist a nbd W of y satisfying $W \subset V_1\,[y]$, a nbd I of x, an index $\sigma_1 \in \Sigma$ such that

$$F_\sigma(x) \cap W \subset V_1\,[F_\sigma(z)] \qquad (\diamond)$$

for every $z \in I$ and every $\sigma \geq \sigma_1$.

Since $F(x) = \mathrm{Li}\ F_\sigma(x)$, there is $\sigma_2 \in \Sigma$ such that for every $\sigma \geq \sigma_2$

$$F_\sigma(x) \cap W \neq \emptyset\ .$$

There is $\alpha_0 \in A$ such that $x_\alpha \in I$ for every $\alpha \geq \alpha_0$. Let $\sigma_0 \in \Sigma$ be such that $\sigma_0 \geq \sigma_1$ and $\sigma_0 \geq \sigma_2$. We claim that for every $\alpha \geq \alpha_0$ and $\sigma \geq \sigma_0$

$$F_\sigma(x_\alpha) \cap O \neq \emptyset\ .$$

Indeed if $\alpha \geq \alpha_0$ and $\sigma \geq \sigma_0$ take $y_\sigma \in F_\sigma(x) \cap W$, by using condition $(\diamond)$ it results $y_\sigma \in V_1\,[F_\sigma(x_\alpha)]$. Let $z_{\sigma,\alpha} \in F_\sigma(x_\alpha)$ with $(y_\sigma, z_{\sigma,\alpha}) \in V_1$. Since

$y_\sigma \in V_1[y]$ we have $(z_{\sigma,\alpha}, y) \in V_1 \circ V_1$ that is $z_{\sigma,\alpha} \subset U[y] \subset O$. Thus $z_{\sigma,\alpha} \in F_\sigma(x_\alpha) \cap O$. □

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