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LOCAL EXISTENCE OF SOLUTIONS OF A FREE BOUNDARY PROBLEM FOR EQUATIONS OF COMPRESSIBLE VISCOUS HEAT-CONDUCTING CAPILLARY FLUIDS

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Abstract. In the paper the motion of a viscous compressible heat conducting capillary fluid in a domain bounded by a free surface is considered. We prove the local existence and uniqueness of a solution to a problem describing such a motion in anisotropic Sobolev–Slobodetskii spaces. This solution is such that the velocity and temperature belong to $W_2^{2+\alpha,1+\alpha/2}(\Omega\times(0,T))$, and density to $W_2^{1+\alpha,1/2+\alpha/2}(\Omega\times(0,T))$, $\Omega\subset\mathbb{R}^3$, $\alpha\in[3/4,1)$.

1. Introduction

In this paper we examine the local motion of a drop of a compressible heat-conducting fluid.

Let $\Omega_t\subset\mathbb{R}^3$ be a bounded domain of the drop at time t. We assume that the free boundary S_t of Ω_t is governed by the surface tension. Let $v=v(x,t)$ $(v=(v_1,v_2,v_3))$ be the velocity of the fluid, $\varrho=\varrho(x,t)$ the density, $\theta=\theta(x,t)$ the temperature, $f=f(x,t)$ the external force field per unit mass, $r=r(x,t)$ the heat sources per unit mass, $\bar\theta=\bar\theta(x,t)$ the heat

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flow per unit surface, $p = p(\varrho,\theta)$ the pressure, $c_v = c_v(\varrho,\theta)$ the specific heat at constant volume, μ and ν the constant viscosity coefficients, $\varkappa$ the constant coefficient of heat conductivity, p_0 the external (constant) pressure, σ the constant surface tension. Then the motion of the drop is described by the following system of equations (see [3, 4]):

$$\begin{aligned}
&\varrho[v_t + (v\cdot\nabla)v] - \operatorname{div}\mathbb{T}(v,p) = \varrho f && \text{in } \widetilde{\Omega}^T \equiv \bigcup_{t\in(0,T)} \Omega_t\times\{t\}\\
&\varrho_t + \operatorname{div}(\varrho v) = 0 && \text{in } \widetilde{\Omega}^T,\\
&\varrho c_v(\theta_t + v\cdot\nabla\theta) - \varkappa\Delta\theta + \theta p_\theta \operatorname{div} v\\
&\quad - \frac{\mu}{2}\sum_{i,j=1}^{3}(v_{ix_j} + v_{jx_i})^2\\
&\quad - (\nu-\mu)(\operatorname{div} v)^2 = \varrho r && \text{in } \widetilde{\Omega}^T, \qquad (1.1)\\
&\mathbb{T}\bar{n} - \sigma H\bar{n} = -p_0\bar{n} && \text{on } \widetilde{S}^T \equiv \bigcup_{t\in(0,T)} S_t\times\{t\},\\
&v\cdot\bar{n} = -\frac{\varphi_t}{|\nabla\varphi|} && \text{on } \widetilde{S}^T,\\
&\frac{\partial\theta}{\partial n} = \bar{\theta} && \text{on } \widetilde{S}^T,\\
&\varrho|_{t=0} = \varrho_0,\quad v|_{t=0} = v_0,\quad \theta|_{t=0} = \theta_0 && \text{in } \Omega,
\end{aligned}$$

where $\varphi(x,t) = 0$ describes S_t (at least locally), $\bar{n}$ is the unit outward vector normal to the boundary, i.e.

$$\bar{n} = \frac{\nabla\varphi}{|\nabla\varphi|},\qquad \Omega = \Omega_t|_{t=0} = \Omega_0,\qquad S_t = \partial\Omega_t.$$

By $\mathbb{T} = \mathbb{T}(v,p)$ we denote the stress tensor of the form

$$\mathbb{T}(v,p) = \{T_{ij}\}_{i,j=1,2,3} = \{-p\delta_{ij} + D_{ij}(v)\}_{i,j=1,2,3},$$

where

$$\begin{aligned}
\mathbb{D}(v) &= \{D_{ij}(v)\}_{i,j=1,2,3} = \{\mu(v_{ix_j} + v_{jx_i}) + (\nu-\mu)\delta_{ij}\operatorname{div} v\}_{i,j=1,2,3}\\
&= \mu\mathbb{S}(v) + (\nu-\mu)I\operatorname{div} v,
\end{aligned}$$

$\mathbb{S}(v) = \{v_{ix_j} + v_{jx_i}\}_{i,j=1,2,3}$ is the velocity deformation tensor and I is the unit matrix.

Moreover, H is the double mean curvature of S_t which is negative for convex domains and can be expressed in the form

$$H\bar{n} = \Delta_{S_t}(t)x,\qquad x = (x_1,x_2,x_3),$$

where $\Delta_{S_t}(t)$ is the Laplace–Beltrami operator on S_t. Let S_t be given locally by $x = x(s_1, s_2, t)$, $(s_1, s_2) \in U \subset \mathbb{R}^2$, where U is an open set. Then

$$\begin{aligned} \Delta_{S_t}(t) &= g^{-1/2} \frac{\partial}{\partial s_\alpha} \left(g^{-1/2} \widehat{g}_{\alpha\beta} \frac{\partial}{\partial s_\beta} \right) \\ &= g^{-1/2} \frac{\partial}{\partial s_\alpha} \left(g^{1/2} g^{\alpha\beta} \frac{\partial}{\partial s_\beta} \right) \quad (\alpha, \beta = 1, 2), \end{aligned} \tag{1.2}$$

where the summation convention over repeated indices is assumed,

$$g = \det\{g_{\alpha\beta}\}_{\alpha,\beta=1,2}, \qquad g_{\alpha\beta} = x_\alpha \cdot x_\beta \quad (x_\alpha = \frac{\partial x}{\partial s_\alpha}),$$

$\{g^{\alpha\beta}\}$ is the inverse matrix to $\{g_{\alpha\beta}\}$ and $\{\widehat{g}_{\alpha\beta}\}$ is the matrix of algebraic complements of $\{g_{\alpha\beta}\}$.

Finally, thermodynamic considerations imply that $c_v > 0$, $\varkappa > 0$, $\nu > (1/3)\mu > 0$, $\sigma > 0$.

Let the domain Ω be given. Then by $(1.1)_5$, $\Omega_t = \{x \in \mathbb{R}^3 : x = x(\xi, t), \xi \in \Omega\}$, where $x = x(\xi, t)$ is the solution of the Cauchy problem

$$\frac{\partial x}{\partial t} = v(x, t), \quad x|_{t=0} = \xi \in \Omega, \quad \xi = (\xi_1, \xi_2, \xi_3). \tag{1.3}$$

Integrating (1.3) we obtain the following relation between the Eulerian x and the Lagrangian ξ coordinates of the same fluid particle:

$$x = \xi + \int_0^t u(\xi, t')\, dt' \equiv X_u(\xi, t),$$

where $u(\xi, t) = v(X_u(\xi, t), t)$. Moreover, by $(1.1)_5$, $S_t = \{x : x = x(\xi, t), \xi \in S = \partial\Omega\}$.

By the continuity equation $(1.1)_2$ and the kinematic condition $(1.1)_5$ the total mass is conserved, i.e.

$$\int_{\Omega_t} \varrho(x, t)\, dx = \int_\Omega \varrho_0(\xi)\, d\xi = M,$$

where M is a given constant.

In order to prove the existence of a local solution of problem (1.1) we rewrite it in the Lagrangian coordinates as follows:

$$
\begin{aligned}
&\eta u_t - \operatorname{div}_u \mathbb{T}_u(u,p) = \eta g && \text{in } \Omega^T \equiv \Omega \times (0,T),\\
&\eta_t + \eta \operatorname{div}_u u = 0 && \text{in } \Omega^T,\\
&\eta c_v(\eta,\vartheta)\vartheta_t - \varkappa \nabla_u^2 \vartheta + \vartheta p_\vartheta(\eta,\vartheta) \operatorname{div}_u u \\
&\quad - \frac{\mu}{2} \sum_{i,j=1}^3 (\xi_{x_i} \cdot \nabla_\xi u_j + \xi_{x_j} \cdot \nabla_\xi u_i)^2 \\
&\quad - (\nu - \mu)(\operatorname{div}_u u)^2 = \eta k && \text{in } \Omega^T,\\
&\mathbb{T}_u(u,p)\bar{n}_u - \sigma H \bar{n}_u = -p_0 \bar{n}_u && \text{on } S^T \equiv S \times (0,T),\\
&\bar{n}_u \cdot \nabla_u \vartheta = \bar{\vartheta} && \text{on } S^T,\\
&\eta|_{t=0} = \varrho_0, \quad u|_{t=0} = v_0, \quad \vartheta|_{t=0} = \theta_0 && \text{in } \Omega,
\end{aligned}
\tag{1.4}
$$

where $\eta(\xi,t) = \varrho(X_u(\xi,t),t)$, $\vartheta(\xi,t) = \theta(X_u(\xi,t),t)$, $p = p(\eta,\vartheta)$, $g(\xi,t) = f(X_u(\xi,t),t))$, $k(\xi,t) = r(X_u(\xi,t),t)$, $\bar{\vartheta}(\xi,t) = \bar{\theta}(X_u(\xi,t),t)$, $\nabla_u = \xi_{ix}\partial_{\xi_i}$, $\mathbb{T}_u(u,p) = -pI + \mathbb{D}_u(u)$, $I = \{\delta_{ij}\}_{i,j=1,2,3}$ is the unit matrix, $\mathbb{D}_u(u) = \{\mathbb{D}_{uij}(u)\}_{i,j=1,2,3} = \{\mu(\partial_{x_i}\xi_k\partial_{\xi_k}u_j + \partial_{x_j}\xi_k\partial_{\xi_k}u_i) + (\nu-\mu)\delta_{ij}\operatorname{div}_u u\}$, $\operatorname{div}_u u = \nabla_u \cdot u = \partial_{x_i}\xi_k\partial_{\xi_k}u_i$, $\operatorname{div}_u \mathbb{T}_u(u,p) = \{\partial_{x_j}\xi_k\partial_{\xi_k}T_{uij}(u,p)\}_{i=1,2,3}$ ($\partial_{x_i}\xi_k$ are the elements of the matrix ξ_x which is inverse to $x_\xi = I + \int_0^t u_\xi(\xi,s)ds$) and summation over repeated indices is assumed.

Let S_t be determined (at least locally) by the equation $\varphi(x,t) = 0$. Then S is described by $\varphi(x(\xi,t),t)|_{t=0} \equiv \widetilde{\varphi}(\xi) = 0$. Thus, we have

$$
\bar{n}_u = \bar{n}(X_u(\xi,t),t) = \frac{\nabla_x \varphi(x,t)}{|\nabla_x \varphi(x,t)|}\bigg|_{x=X_u(\xi,t)}
$$

and

$$
\bar{n}_0 = \bar{n}_0(\xi) = \frac{\nabla_\xi \widetilde{\varphi}(\xi)}{|\nabla_\xi \widetilde{\varphi}(\xi)|}.
$$

The aim of this paper is to prove the local-in-time existence and uniqueness of a solution to problem (1.1) in anisotropic Sobolev–Slobodetskii spaces. To do this we use the problem rewritten in the Lagrangian coordinates, i.e. problem (1.4). Applying the method of successive approximations and using Lemmas 3.1, 3.2 and 2.1 we prove the local existence and uniqueness of a solution of problem (1.4) such that $(u,\vartheta,\eta) \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T) \times W_2^{2+\alpha,1+\alpha/2}(\Omega^T) \times C(0,T;W_2^{1+\alpha}(\Omega)) \cap W_2^{1+\alpha,1/2+\alpha/2}(\Omega^T)$, where $\alpha \in [3/4,1)$ and $T > 0$ is sufficiently small (spaces $W_p^{l,l/2}(\Omega^T)$ and $W_p^l(\Omega)$, $l \in \mathbb{R}_+^1$, $1 \le p \in \mathbb{R}$, are defined by (2.1) and (2.2), respectively).

The existence of solutions to problem (1.4) follows from the existence of solutions of two parabolic equations $(1.4)_{1,3}$ and the evolution equation

$(1.4)_2$ which can be solved by simple integration. In the parabolic equations $(1.4)_{1,3}$ the first derivative with respect to time and the second derivatives with respect to ξ occur. To solve the corresponding initial-boundary value problems connected with equations $(1.4)_{1,3}$ the potential technique is applied (see [2, 5]). Therefore the existence of solutions is automatically proved in such spaces that they contain two times less derivatives with respect to t than with respect to ξ.

In this paper we prove the local existence in the Hilbert–Sobolev spaces because we need such a result for the proof of the global existence which will be done by the energy method. To simplify the proof and to decrease a number of compatibility conditions we have decided to show the existence of solutions with the lowest possible regularity which is admissible in view of the nonlinearity of the problem and L_2-approach. Since the most nonlinear expressions in equation (1.4) are involved in the transformation between the Eulerian and the Lagrangian coordinates we have to impose such a regularity that

$$\left\| \int_0^T u_\xi \, d\tau \right\|_{L_\infty(\Omega)} < \infty,$$

what is satisfied if $u \in L_2(0,T; W_2^{2+\alpha}(\Omega))$ for $\alpha > 1/2$. Then $(1.4)_1$ implies $u \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$. However, to obtain $\vartheta \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$ we have to assume $\alpha \in [3/4, 1)$. This stronger assumption is necessary only in one step of the proof, namely to obtain estimate (3.45). To derive other estimates we need $\alpha \in (1/2, 1)$. Thus, for $\alpha \in [3/4, 1)$ we obtain the lowest regularity for problem (1.4) in the L_2-approach (so in the Hilbert–Sobolev spaces). Since the spaces $W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$ contain fractional derivatives they are called the Sobolev–Slobodetskii spaces.

In [9] we proved the local existence and uniqueness of a solution (u, ϑ, η) of problem (1.4) such that $(u, \vartheta, \eta) \in W_2^{4,2}(\Omega^T) \times W_2^{4,2}(\Omega^T) \times C(0, T; \Gamma_0^{3,3/2}(\Omega))$ (where $\Gamma_0^{3,3/2}(\Omega)$ is the space with the norm

$$\|\eta\|_{\Gamma_0^{3,3/2}(\Omega)} = \sum_{|\alpha|+2i \le 3} \|\partial_\xi^\alpha \partial_t^i \eta\|_{L_2(\Omega)}$$

and $T > 0$ is sufficiently small).

The proof required a lot of quite involved calculations in order to obtain the estimates for the four derivatives with respect to ξ of the velocity and the temperature. Moreover, in that case a lot of compatibility conditions had to be satisfied.

We have to underline that anisotropic Sobolev spaces must be used to prove the existence of solutions to (1.4) but anisotropic Sobolev–Slobodetskii spaces were used only in order to simplify the proof.

In this paper we use Lemma 3.1 which can be proved in the same way as Theorem 1.2 of [6], which was applied by V. A. Solonnikov and A. Tani (the authors of [6]) to prove the local existence and uniqueness of a solution to a free boundary problem for a viscous compressible barotropic capillary fluid in such spaces that $u \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$, $\eta \in W_2^{1+\alpha,1/2+\alpha/2}(\Omega^T)$, $\alpha \in (1/2,1)$.

The global motion of a viscous compressible heat conducting caplliary fluid bounded by a free boundary was considered in [8], where the global existence of solutions of slightly more regularity (than that from [9]) was obtained.

In the case of incompressible motions the existence of solutions with sharp-regularity, i.e. $u \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$, $\alpha \in (1/2,1)$ is proved in [5], and in the case of the motion of a viscous barotropic fluid in a fixed domain with boundary slip conditions the analogus sharp-regularity result is obtained in [2].

Finally, papers [7] and [10] are devoted to the global motion of viscous compressible barotropic capillary fluids bounded by a free boundary.

2. Notation and auxiliary results

In the paper we use the anisotropic Sobolev–Slobodetskii spaces $W_p^{l,l/2}(\Omega^T)$, $l \in \mathbb{R}^1_+$, $1 \le p \in \mathbb{R}$, $\Omega^T = \Omega \times (0,T)$, (where Ω is a bounded domain in $\mathbb{R}^3$) with the norm

$$\begin{aligned} \|u\|_{W_p^{l,l/2}(\Omega^T)} = \bigg[& \sum_{|\beta|+2i\le[l]} \|\partial_x^\beta \partial_t^i u\|^p_{L_p(\Omega^T)} \\ & + \sum_{|\beta|+2i=[l]} \bigg(\int_0^T \int_\Omega \int_\Omega \frac{|\partial_x^\beta \partial_t^i u(x,t) - \partial_{x'}^\beta \partial_t^i u(x',t)|^p}{|x-x'|^{3+p(l-[l])}}\, dx\, dx'\, dt \\ & + \int_\Omega \int_0^T \int_0^T \frac{|\partial_x^\beta \partial_t^i u(x,t) - \partial_x^\beta \partial_{t'}^i u(x,t')|^p}{|t-t'|^{1+p(\frac{l}{2}-[\frac{l}{2}])}}\, dx\, dt\, dt' \bigg) \bigg]^{1/p}. \end{aligned} \tag{2.1}$$

In (2.1) $\partial_x^\beta = \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2} \partial_{x_3}^{\beta_3}$, $\beta = (\beta_1,\beta_2,\beta_3)$ is a multi-index, $[l]$ is the integer part of l.

We also use the isotropic spaces $W_p^l(\Omega)$, $l \in \mathbb{R}^1_+$, $1 \le p \in \mathbb{R}$ with the norm

$$\begin{aligned} \|u\|_{W_p^l(\Omega)} = \bigg[& \sum_{|\beta|\le[l]} \|\partial_x^\beta u\|^p_{L_p(\Omega)} \\ & + \sum_{|\beta|=[l]} \int_\Omega \int_\Omega \frac{|\partial_x^\beta u(x) - \partial_{x'}^\beta u(x')|^p}{|x-x'|^{3+p(l-[l])}}\, dx\, dx' \bigg]^{1/p}. \end{aligned} \tag{2.2}$$

Similarly we can introduce the spaces $W_p^{l,l/2}(S^T)$ and $W_p^l(S)$, where S is the boundary of Ω.

Moreover, we use the notation:

$$
\begin{aligned}
\|u\|_{W_2^{l,l/2}(Q^T)} &= \|u\|_{l,Q^T}, \qquad Q \in \{\Omega, S\};\\
\|u\|_{W_2^l(Q)} &= \|u\|_{l,Q}, \qquad Q \in \{\Omega, S\};\\
\|u\|_{L_p(\Omega)} &= |u|_{p,\Omega};\\
\|u\|_{L_2(Q^T)} &= \|u\|_{0,Q^T}, \qquad Q \in \{\Omega, S\};\\
\|u\|_{\Omega^T}^{(\alpha+2,\alpha/2+1)} &= \Big[\|u\|_{2+\alpha,\Omega^T}^2 + T^{-\alpha}\Big(\|u_t\|_{0,\Omega^T}^2 + \sum_{|\beta|=2} \|\partial_x^\beta u\|_{0,\Omega^T}^2\Big)\\
&\qquad + \sup_{t\le T} \|u(\cdot,t)\|_{1+\alpha,\Omega}^2\Big]^{1/2};\\
\|u\|_{Q^T}^{(\alpha,\alpha/2)} &= (\|u\|_{\alpha,Q^T}^2 + T^{-\alpha}\|u\|_{0,Q^T}^2)^{1/2}, \qquad Q \in \{\Omega, S\};\\
[u]_{\alpha,\Omega^T,x} &= \bigg(\int_0^T dt \int_\Omega \int_\Omega \frac{|u(x,y) - u(x',t)|^2}{|x-x'|^{3+2\alpha}}\, dx\, dx'\bigg)^{1/2},\\
[u]_{\alpha,\Omega^T,t} &= \bigg(\int_\Omega dx \int_0^T \int_0^T \frac{|u(x,t) - u(x,t')|^2}{|t-t'|^{1+2\alpha}}\, dt\, dt'\bigg)^{1/2}.
\end{aligned}
$$

In the above notation $\alpha \in (0,1)$.

We have the following imbedding (see [1, Sect. 18]):

$$\partial_x^\sigma W_p^l(\Omega) \subset W_q^\varrho(\Omega), \qquad \varrho, l \in \mathbb{R}_+^1 \cup \{0\}$$

if $(n/p) - (n/q) + |\sigma| + \varrho \le l$ and if $(n/p) - (n/q) + |\sigma| + \varrho < l$ the following interpolation inequality holds:

$$\|\partial_x^\sigma u\|_{W_q^\varrho(\Omega)} \le \varepsilon^{1-\varkappa}\|u\|_{W_p^l(\Omega)} + c\varepsilon^{-\varkappa}\|u\|_{L_p(\Omega)}$$

with $\varepsilon \in (0,1)$, $\varkappa = (1/l)((n/p) - (n/q) + |\sigma| + \varrho)$.

Now, consider the problem

$$\eta_t + \eta \operatorname{div}_u u = 0 \qquad \text{in } \Omega^T, \tag{2.3}$$

$$\eta|_{t=0} = \varrho_0 \qquad \text{in } \Omega. \tag{2.4}$$

From (2.3)–(2.4) we obtain

$$\eta(\xi,t) = \varrho_0(\xi)\exp\Big(-\int_0^t \nabla_u \cdot u d\tau\Big).$$

The following lemma is proved in [2] (see Lemma 6.1).

Lemma 2.1. *Assume that* $\varrho_0 \in W_2^{1+\alpha}(\Omega)$, $\alpha \in (1/2,1)$, $\varrho_0(\xi) \ge \varrho_* > 0$ *and*

$$T^{1/2}\|u\|_{2+\alpha,\Omega^T} \le \delta,$$

where $\delta > 0$ is sufficiently small. Then

$$
\begin{aligned}
&\|\eta(\cdot,t)\|_{1+\alpha,\Omega} \le \varphi_1(a)\|\varrho_0\|_{1+\alpha,\Omega},\\
&[\eta_\xi]_{\alpha/2,\Omega^T,t} \le \|\varrho_0\|_{1+\alpha,\Omega}\varphi_2(a,T)a,\\
&\sup_t \int_\Omega \int_0^T \frac{|\eta_\xi(t)-\eta_\xi(t')|^2}{|t-t'|^{1+\alpha}}\,d\xi\,dt'\\
&\le \|\varrho_0\|^2_{1+\alpha,\Omega}\varphi_3(a)T^{1-\alpha}\int_0^T \|u\|^2_{2+\alpha,\Omega}\,dt\Big(1+\int_0^T \|u\|^2_{2+\alpha,\Omega}\,dt\Big),
\end{aligned}
$$

where $a = T^{\bar a}\|u\|_{2+\alpha,\Omega^T}$, $\bar a > 0$ is a constant and $\varphi_i (i = 1,2,3)$ are positive increasing continuous functions of their arguments.

3. Local existence

In order to prove the local existence of solution of (1.1) we consider this problem written in the Lagrangian coordinates, i.e. problem (1.4). First, we examine the following auxiliary linear problem

$$
\begin{aligned}
&\eta u_t - \operatorname{div}_w \mathbb{D}_w(u) = F && \text{in } \Omega^T,\\
&\mu\Pi_0\Pi_w\mathbb{S}_w(u)\bar n_w = \Pi_0 G_1 && \text{on } S^T,\\
&\bar n_0\cdot\mathbb{D}_w(u)\bar n_w - \sigma\bar n_0\cdot\Delta_w(t)\int_0^t u\,d\tau = G_2 && \text{on } S^T,\\
&u|_{t=0} = v_0 && \text{in } \Omega,
\end{aligned}
\tag{3.1}
$$

where

$$
\begin{aligned}
\mathbb{D}_w(u) &= \mu\mathbb{S}_w(u) + (\nu-\mu)\operatorname{div}_w u I,\\
\mathbb{S}_w(u) &= \{\partial_{x_i}\xi_k\partial_{\xi_k}u_j + \partial_{x_j}\xi_k\partial_{\xi_k}u_i\}_{i,j=1,2,3}, \qquad I = \{\delta_{ij}\}_{i,j=1,2,3},\\
\operatorname{div}_w \mathbb{D}_w(u) &= \{\partial_{x_j}\xi_k\partial_{\xi_k}D_{wij}(u)\}_{i=1,2,3},
\end{aligned}
$$

$\Delta_w(t)$ is given by (1.2) with $x = \xi + \int_0^t w(\xi,\tau)\,d\tau$, $\partial_{x_j}\xi_k (i,k = 1,2,3)$ are elements of matrix ξ_x which is inverse to

$$
x_\xi = I + \int_0^t w_\xi(\xi,\tau)d\tau, \quad \bar n_w = \bar n(X_w(\xi,t),t),
$$

$$
X_w(\xi,t) = \xi + \int_0^t w(\xi,\tau)d\tau, \qquad \Pi_0 g = g - \bar n_0(\bar n_0\cdot g)
$$

($\bar n_0$ is the unit outward vector normal to $S = \partial\Omega$), $\Pi_w g = g - \bar n_w(\bar n_w\cdot g)$. Repeating the argument of [6] (see Theorem 1.2 of [6], see also [2]) we obtain the following lemma.

Lemma 3.1. *Let* $S \in W_2^{3/2+\alpha}$, $\alpha \in (1/2,1)$, $\eta \in C^\beta(\bar\Omega^T)$, $\beta > 0$, $1/\eta \in L_\infty(\Omega^T)$ *and assume that*

$$T^{1/2}\|w\|_{\Omega^T}^{(\alpha+2,\alpha/2+1)} \le \delta, \tag{3.2}$$

where δ *is a sufficiently small constant. Let* $F \in W_2^{\alpha,\alpha/2}(\Omega^T)$, $v_0 \in W_2^{1+\alpha}(\Omega)$, $G_1 \in W_2^{1/2+\alpha,1/4+\alpha/2}(S^T)$, $G_2 = G_2^{(1)} + \sigma\int_0^t G_2^{(2)}d\tau$ *with* $G_2^{(1)} \in W_2^{1/2+\alpha,1/4+\alpha/2}(S^T)$, $G_2^{(2)} \in W_2^{\alpha-1/2,\alpha/2-1/4}(S^T)$ *and let the following compatibility conditions are satisfied:*

$$\mu\Pi_0\mathbb{S}(v_0)\bar n_0|_S = \Pi_0 G_1|_{t=0}$$

$$\bar n_0\cdot\mathbb{D}(v_0)\bar n_0|_S = G_2^{(1)}|_{t=0}.$$

Then there exists a unique solution $u \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$ *of problem* (3.1) *and*

$$\begin{aligned}\|u\|_{\Omega^T}^{(\alpha+2,\alpha/2+1)} &\le \varphi_1\bigg(T, \|\eta\|_{C^\beta(\bar\Omega^T)}, \bigg|\frac{1}{\eta}\bigg|_{\infty,\Omega^T}\bigg)(\|F\|_{\Omega^T}^{(\alpha,\alpha/2)} + \|v_0\|_{1+\alpha,\Omega}\\ &\quad + \|G_1\|_{1/2+\alpha,S^T} + \|G_2^{(1)}\|_{1/2+\alpha,S^T} + \sigma\|G_2^{(2)}\|_{S^T}^{(\alpha-1/2,\alpha/2-1/4)}),\end{aligned}$$

where φ_1 *is a positive continuous nondecreasing function of its arguments.*

Analogously, we consider the problem

$$\begin{aligned} &\eta c_v(\eta,\gamma)\vartheta_t - \varkappa\nabla_w^2\vartheta = K && \text{in } \Omega^T,\\ &\bar n_w\cdot\nabla_w\vartheta = \bar\vartheta && \text{on } S^T,\\ &\vartheta|_{t=0} = \theta_0 && \text{in } \Omega.\end{aligned} \tag{3.3}$$

The following lemma holds

Lemma 3.2. *Let* $S \in W_2^{3/2+\alpha}$, $\alpha \in (1/2,1)$, $\eta \in C^\beta(\bar\Omega^T)$, $\gamma \in C^\beta(\bar\Omega^T)$, $\beta > 0$, $1/\eta \in L_\infty(\Omega^T)$, $c_v > 0$, $c_v \in C^1(\mathbb{R}^2)$ *and assume that condition* (3.2) *is satisfied. Let* $K \in W_2^{(\alpha,\alpha/2)}(\Omega^T)$, $\theta_0 \in W_2^{1+\alpha}(\Omega)$, $\bar\vartheta \in W_2^{1/2+\alpha,1/4+\alpha/2}(S^T)$ *and let the following compatibility condition hold:*

$$\bar n_0\cdot\nabla\theta_0|_S = \bar\vartheta|_{t=0}.$$

Then there exists a unique solution $\vartheta \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$ *of problem* (3.3) *and*

$$\begin{aligned}\|\vartheta\|_{\Omega^T}^{(\alpha+2,\alpha/2+1)} &\le \varphi_2\bigg(T, \|\eta\|_{C^\beta(\bar\Omega^T)}, \|\gamma\|_{C^\beta(\bar\Omega^T)}, \bigg|\frac{1}{\eta}\bigg|_{\infty,\Omega^T}, \bigg|\frac{1}{c_v(\eta,\gamma)}\bigg|_{\infty,\Omega^T}\bigg)\\ &\quad\cdot(\|K\|_{\Omega^T}^{(\alpha,\alpha/2)} + \|\theta_0\|_{1+\alpha,\Omega} + \|\bar\vartheta\|_{1/2+\alpha,S^T}),\end{aligned}$$

where φ_2 *is a positive continuous nondecreasing function of its arguments.*

Now, we apply the method of successive approximations. We consider the problems

$$
\begin{aligned}
&\eta_m\partial_t u_{m+1}-\mu\nabla^2_{u_m}u_{m+1}-\nu\nabla_{u_m}\nabla_{u_m}\cdot u_{m+1} && && (3.4)\\
&=-\nabla_{u_m}p(\eta_m,\vartheta_m)+\eta_m g_m\equiv l_1+l_2 && \text{in } \Omega^T,\\
&\Pi_0\Pi_{u_m}\mathbb{S}_{u_m}(u_{m+1})\bar n_{u_m}=0 && \text{on } S^T,\\
&\bar n_0\cdot\mathbb{D}_{u_m}(u_{m+1})\bar n_{u_m}-\sigma\bar n_0\cdot\Delta_{u_m}(t)\int_0^t u_{m+1}(\tau)d\tau\\
&=\bar n_0\cdot(p(\eta_m,\vartheta_m)-p_0)\bar n_{u_m}+\sigma H(\xi,0)\\
&+\sigma\int_0^t\bar n_0\cdot\dot\Delta_{u_m}(\tau)\xi d\tau\equiv\sum_{i=3}^4 l_i+\int_0^t l_5d\tau, && \text{on } S^T,\\
&u_{m+1}|_{t=0}=v_0 && \text{in } \Omega,\\
&\eta_m c_v(\eta_m,\vartheta_m)\partial_t\vartheta_{m+1}-\varkappa\nabla^2_{u_m}\vartheta_{m+1} && && (3.5)\\
&=-\vartheta_m p_\vartheta(\eta_m,\vartheta_m)\operatorname{div}_{u_m}u_m\\
&+\frac{\mu}{2}\sum_{i,j=1}^3(\xi_{x_i}\cdot\nabla_\xi u_{mj}+\xi_{x_j}\cdot\nabla_\xi u_{mi})^2\\
&+(\nu-\mu)(\operatorname{div}_{u_m}u_m)^2+\eta_m k_m\equiv\sum_{i=6}^9 l_i && \text{in } \Omega^T,\\
&\bar n_{u_m}\cdot\nabla_{u_m}\vartheta_{m+1}=\bar\vartheta_m\equiv l_{10} && \text{on } S^T,\\
&\vartheta_{m+1}|_{t=0}=\theta_0 && \text{in } \Omega,\\
&\eta_{mt}+\eta_m\operatorname{div}_{u_m}u_m=0 && \text{in } \Omega^T, && (3.6)\\
&\eta_m|_{t=0}=\varrho_0 && \text{in } \Omega,
\end{aligned}
$$

where ξ_x is the inverse matrix to $x_\xi=I+\int_0^t u_{m\xi}(\xi,\tau)d\tau$ and

$$
\begin{aligned}
g_m(\xi,t)&=f(X_{u_m}(\xi,t),t)=f\Big(\xi+\int_0^t u_m(\xi,\tau)d\tau,t\Big),\\
k_m(\xi,t)&=r(X_{u_m}(\xi,t),t)=r\Big(\xi+\int_0^t u_m(\xi,\tau)d\tau,t\Big),\\
\bar\vartheta_m(\xi,t)&=\bar\theta(X_{u_m}(\xi,t),t)=\bar\vartheta\Big(\xi+\int_0^t u_m(\xi,\tau)d\tau,t\Big),\\
\dot\Delta_{u_m}(t)&=\frac{d}{dt}\Delta_m(t).
\end{aligned}
$$

For u_0 we take a function which is a solution of the problem

$$\begin{aligned}
&u_{0t} - \operatorname{div} \mathbb{D}(u_0) = 0 && \text{in } \Omega^T,\\
&\Pi_0 \mathbb{D}(u_0)\bar{n}_0 = 0 && \text{on } S^T,\\
&\bar{n}_0 \cdot \mathbb{D}(u_0)\bar{n}_0 - \sigma\bar{n}_0 \cdot \Delta_S(0)\int_0^t u_0(\tau)d\tau &&\\
&\ = \bar{n}_0 \cdot [p(\varrho_0,\theta_0) - p_0]\bar{n}_0 + \sigma\bar{n}_0 \cdot \Delta_S(0)\xi && \text{on } S^T,\\
&u_0|_{t=0} = v_0 && \text{in } \Omega
\end{aligned}$$

and for ϑ_0 we take a solution of the problem

$$\begin{aligned}
&\vartheta_{0t} - \varkappa\nabla_\xi^2\vartheta_0 = 0 && \text{in } \Omega^T,\\
&\bar{n}_0 \cdot \nabla_\xi\vartheta_0 = \bar{\vartheta}_0 && \text{on } S^T,\\
&\vartheta_0|_{t=0} = \theta_0 && \text{in } \Omega,
\end{aligned}$$

where

$$\bar{\vartheta}_0(\xi,t) = \bar{\theta}(X_{u_0}(\xi,t),t) = \bar{\theta}\Big(\xi + \int_0^t u_0(\xi,\tau)d\tau, t\Big).$$

It can be proved that u_0 and ϑ_0 satisfy for $t \le T$ the estimates

$$\begin{aligned}
\|u_0\|_{\Omega^t}^{(\alpha+2,\alpha/2+1)} \le c_1(t)\big[&\|v_0\|_{1+\alpha,\Omega}\\
&+ \|\bar{n}_0 \cdot [p(\varrho_0,\theta_0) - p_0]\bar{n}_0\|_{1/2+\alpha,S}\\
&+ \|\bar{n}_0 \cdot \Delta_S(0)\xi\|_{1/2+\alpha,S}\big] \equiv F_1(t)
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
\|\vartheta_0\|_{\Omega^t}^{(\alpha+2,\alpha/2+1)} &\le c_2(t)[\|\theta_0\|_{1+\alpha,\Omega} + \|\bar{\vartheta}_0\|_{1/2+\alpha,S^t}]\\
&\equiv F_2(t),
\end{aligned} \tag{3.8}$$

where c_1 and c_2 are positive continuous increasing functions. Finally, η_0 is a solution of the problem

$$\begin{aligned}
&\eta_{0t} + \eta_0\nabla_{u_0} \cdot u_0 = 0 && \text{in } \Omega^T,\\
&\eta_0|_{t=0} = \varrho_0 && \text{in } \Omega.
\end{aligned}$$

We obtain the following lemma.

Lemma 3.3. *Let* $S \in W_2^{5/2+\alpha}$, $\varrho_0 \in W_2^{1+\alpha}(\Omega)$, $v_0 \in W_2^{1+\alpha}(\Omega)$, $\theta_0 \in W_2^{1+\alpha}(\Omega)$, $\alpha \in (1/2,1)$, $\varrho_0 \ge \varrho_* > 0$, *where* ϱ_* *is a constant. Let* $c_v \in C^2(\mathbb{R}^2)$, $c_v > 0$, $p \in C^3(\mathbb{R}^2)$; f, r *and* $\bar{\theta}$ *have continuous derivatives of order one and two;* f, f_{x_k}, r, r_{x_k} *and* $\bar{\theta}$, $\bar{\theta}_{x_k}$ *satisfy the Hölder condition* *with the exponent* $\bar{\alpha} \geq 1/2$ *and let the following compatibility conditions be satisfied:*

$$\begin{aligned}
&\Pi_0 \mathbb{D}(v_0)\bar{n}_0 = 0 && \text{on } S,\\
&\bar{n}_0 \cdot \mathbb{D}(v_0)\bar{n}_0 = \bar{n}_0 \cdot (p(\varrho_0,\theta_0) - p_0)\bar{n}_0 + \sigma \bar{n}_0 \cdot \Delta_S(0)\xi && \text{on } S,\\
&\bar{n}_0 \cdot \nabla_\xi \theta_0 = \bar{\theta}(\xi, 0) && \text{on } S.
\end{aligned}$$

Denote

$$\alpha_m(t) = \|u_m\|_{\Omega^t}^{(\alpha+2,\alpha/2+1)} + \|\vartheta_m\|_{\Omega^t}^{(\alpha+2,\alpha/2+1)}.$$

Let $A > 0$ *be a constant satisfying* (3.26) *and*

$$F_1(t) + F_2(t) \leq A \quad \text{for } t \leq T_0 \tag{3.9}$$

(where $T_0 \leq T$ *is so small that* $\det\{x_\xi\} > 0$ *with* $x = \xi + \int_0^t u_0(\xi,\tau)d\tau$*). Then there exists* $0 < T_* \leq T_0$ *such that if* $T \leq T_*$ *we have*

$$\alpha_m(t) \leq A \quad \text{for } t \leq T \tag{3.10}$$

and $m = 0, 1, 2, \ldots$.

Proof. By (3.7)–(3.8) and (3.9) we have

$$\alpha_0(t) \leq A \quad \text{for } t \leq T_0.$$

Now, we assume that for some $m > 0$ there exists $0 < T_* \leq T_0$ such that if $T \leq T_*$ then

$$\alpha_m(t) \leq A \quad \text{for } t \leq T. \tag{3.11}$$

We shall prove that $\alpha_{m+1}(t) \leq A$ for $t \leq T$. Applying Lemmas 3.1 and 3.2 to problems (3.4) and (3.5), respectively we have for $t \leq T$

$$\begin{aligned}
\alpha_{m+1}(t) \leq \varphi_3\bigg(&T, \|\eta_m\|_{C^\beta(\bar{\Omega}^t)}, \|\vartheta_m\|_{C^\beta(\bar{\Omega}^t)}, \bigg|\frac{1}{\eta_m}\bigg|_{\infty,\Omega^t}, \\
&\bigg|\frac{1}{c_v(\eta_m,\vartheta_m)}\bigg|_{\infty,\Omega^t}\bigg)\Big[\|l_1\|_{\Omega^t}^{(\alpha,\alpha/2)} + \|l_2\|_{\Omega^t}^{(\alpha,\alpha/2)} \\
&+ \sum_{i=6}^{9} \|l_i\|_{\Omega^t}^{(\alpha,\alpha/2)} + \|l_3\|_{1/2+\alpha,S^t} + \|l_4\|_{1/2+\alpha,S^t} + \|l_{10}\|_{1/2+\alpha,S^t} \\
&+ \|l_5\|_{S^t}^{(\alpha-1/2,\alpha/2-1/4)} + \|v_0\|_{1+\alpha,\Omega} + \|\theta_0\|_{1+\alpha,\Omega}\Big],
\end{aligned} \tag{3.12}$$

where φ_3 is a positive increasing continuous function of its arguments.

We estimate only the norms $\|l_i\|_{\Omega^T}^{(\alpha,\alpha/2)}$ $(i = 7, 8)$ because the other norms on the right-hand side of (3.12) are estimated similarly as the appropriate norms in the proof of Theorem 5.3 of [6] and as in the proof of Lemma 6.2 of [2].

In order to estimate l_7 and l_8 we write them in the qualitative form as $\mathcal{I} \equiv l_7 + l_8 \sim \xi_x^2 u_{m\xi}^2$, where $\xi_x^2 = g(I + \int_0^t u_{m\xi}(\xi,\tau)d\tau)$ and g is a smooth function, $I + \int_0^t u_{m\xi}(\xi,\tau)d\tau = \{\delta_{ij} + \int_0^t u_{im\xi_j}(\xi,\tau)d\tau\}_{i,j=1,2,3}$.

It suffices to estimate only the highest derivatives in the considered norm. We have

$$\begin{aligned}[\mathcal{I}]^2_{\alpha,\Omega^T,\xi} \le \psi_1(a_m)\int_0^T\int_\Omega\int_\Omega\bigg[&\frac{|\int_0^t(u_{m\xi}-u_{m\xi'})d\tau|^2}{|\xi-\xi'|^{3+2\alpha}}|u_{m\xi}|^4\\ &+\frac{|u_{m\xi}-u_{m\xi'}|^2}{|\xi-\xi'|^{3+2\alpha}}|u_{m\xi}|^2\bigg]d\xi d\xi' dt \equiv \mathcal{I}_1+\mathcal{I}_2,\end{aligned}$$

where $a_m = T^a(\int_0^T \|u_m\|^2_{2+\alpha,\Omega}d\tau)^{1/2}$, $a > 0$ is a constant, ψ_1 and all functions ψ_i occuring below are positive increasing continuous functions. First estimate $\mathcal{I}_1$. We have

$$\begin{aligned}\mathcal{I}_1 &\le T\psi_1(a_m)\int_0^T |u_{m\xi}|^2_{\infty,\Omega}dt\int_\Omega\int_\Omega\int_0^T\frac{|u_{m\xi}-u_{m\xi'}|^2}{|\xi-\xi'|^{3+2\alpha}}|u_{m\xi}|^2 d\xi d\xi' dt\\ &\le T\psi_1(a_m)\|u_m\|^2_{2+\alpha,\Omega^T}\int_0^T\bigg(\int_\Omega\int_\Omega\frac{|u_{m\xi}-u_{m\xi'}|^{2p_1}}{|\xi-\xi'|^{3+2p_1(3\mu_1/2-3/(2p_1)+\alpha)}}d\xi d\xi'\bigg)^{1/p_1}dt\\ &\cdot\sup_{t\le T}\bigg(\int_\Omega\int_\Omega\frac{|u_{m\xi}|^{2p_2}}{|\xi-\xi'|^{3\mu_2p_2}}d\xi\, d\xi'\bigg)^{1/p_2},\end{aligned}$$

where $1/p_1 + 1/p_2 = 1$, $\mu_1 + \mu_2 = 1$ and $\mu_2 p_2 < 1$. Using the imbeddings $\partial_\xi^\sigma W_2^{2+\alpha}(\Omega) \subset W_{2p_1}^{\alpha+3\mu_1/2-3/(2p_1)}(\Omega)$ and $\partial_\xi^\sigma W_2^{1+\alpha}(\Omega) \subset L_{2p_2}(\Omega)$ (where $|\sigma| = 1$) which hold simultaneously for any p_1 and p_2 such that $3/2 - 3/(2p_1) + 1 + \alpha + (3/2)\mu_1 - 3/(2p_1) \le 2 + \alpha$ and $3/2 - 3/(2p_1) + 1 \le 1 + \alpha$ if $\alpha > 1/2$, we obtain

$$\mathcal{I}_1 \le T\psi_2(a_m)\|u_m\|^4_{2+\alpha,\Omega^T}\sup_{t\le T}\|u_m\|^2_{1+\alpha,\Omega}. \tag{3.13}$$

Next, we estimate $\mathcal{I}_2$. We have

$$\begin{aligned}\mathcal{I}_2 \le &\psi_1(a_m)\int_0^T\bigg(\int_\Omega\int_\Omega\frac{|u_{m\xi}-u_{m\xi'}|^{2p_1}}{|\xi-\xi'|^{3+2p_1(3\mu_1/2-3/(2p_1)+\alpha)}}d\xi d\xi'\bigg)^{1/p_1}dt\\ &\cdot\sup_{t\le T}\bigg(\int_\Omega\int_\Omega\frac{|u_{m\xi}|^{2p_2}}{|\xi-\xi'|^{3\mu_2p_2}}\bigg)^{1/p_2},\end{aligned}$$

where $1/p_1 + 1/p_2 = 1$, $\mu_1 + \mu_2 = 1$ and $\mu_2 p_2 < 1$. Using the imbedding $\partial_\xi^\sigma W_2^{1+\alpha}(\Omega) \subset L_{2p_2}(\Omega)$ (where $|\sigma| = 1$) and the interpolation inequality $\|u_{m\xi}\|_{W_2^{\alpha+(3/2)\mu_1-3/(2p_1)}(\Omega)} \le \varepsilon_1^{1-\varkappa}\|u_m\|_{2+\alpha,\Omega} + c\varepsilon_1^{-\varkappa}|u_m|_{2,\Omega}$ (where $\varkappa = [1/(2+\alpha)](3/2 - 3/p_1 + (3/2)\mu_1 + 1 + \alpha)$) which hold simultaneously for any p_1 and p_2 such that $3/2-3/(2p_1)+1+\alpha+(3/2)\mu_1-3/(2p_1)<2+\alpha$ and $3/2-3/(2p_2)+1\le 1+\alpha$ if $\alpha>1/2$, we obtain

$$\mathcal{I}_2\le\psi_4(a_m)\Big(\varepsilon_1^{1-\varkappa}\|u_m\|^2_{2+\alpha,\Omega^T}+c\varepsilon_1^{-\varkappa}\int_0^T|u_m|^2_{2,\Omega}dt\Big)\sup_{t\le T}\|u_m\|^2_{1+\alpha,\Omega}.$$

Therefore, choosing $\varepsilon_1=T$ and using that $u_m=\int_0^t u_{m\tau}d\tau+v_0$ we have

$$\begin{aligned}\mathcal{I}_2&\le\psi_5(a_m)\Big(T^{1-\varkappa}\|u_m\|^2_{2+\alpha,\Omega^T}\\&\quad+T^{1-\varkappa}|v_0|^2_{2,\Omega}+T^{1-\varkappa}\int_0^T|u_{m\tau}|^2_{2,\Omega}d\tau\Big)\sup_{t\le T}\|u_m\|^2_{1+\alpha,\Omega}\\&\le\psi_6(a_m)T^{1-\varkappa}(\|u_m\|^2_{2+\alpha,\Omega^T}+\|v_0\|^2_{1+\alpha,\Omega})\cdot\sup_{t\le T}\|u_m\|^2_{1+\alpha,\Omega}.\end{aligned}\tag{3.14}$$

Hence, by (3.13) and (3.14) we get

$$\begin{aligned}[\mathcal{I}]^2_{\alpha,\Omega^T,\xi}\le T^{a_1}\psi_7(a_m)\sup_{t\le T}\|u_m\|^2_{1+\alpha,\Omega}(\|u_m\|^4_{2+\alpha,\Omega^T}\\+\|u_m\|^2_{2+\alpha,\Omega^T}+\|v_0\|^2_{1+\alpha,\Omega}),\end{aligned}\tag{3.15}$$

where $a_1>0$ is a constant.

Now, we estimate

$$\begin{aligned}[\mathcal{I}]^2_{\alpha/2,\Omega^T,t}\le\psi_8(a_m)\int_\Omega\int_0^T\int_0^T\bigg[\frac{|\int_{t'}^t u_{m\xi}d\tau|^2}{|t-t'|^{1+\alpha}}|u_{m\xi}|^4\\+\frac{|u_{m\xi}(t)-u_{m\xi}(t')|^2}{|t-t'|^{1+\alpha}}|u_{m\xi}|^2\bigg]d\xi dtdt'=\mathcal{I}_3+\mathcal{I}_4.\end{aligned}\tag{3.16}$$

First, we have

$$\begin{aligned}\mathcal{I}_3&\le\psi_8(a_m)\int_0^T\int_0^T\frac{|\int_{t'}^t u_{m\xi}d\tau|^2_{\infty,\Omega}}{|t-t'|^{1+\alpha}}|u_m|^2_{\infty,\Omega}|u_{m\xi}|^2_{2,\Omega}dtdt'\\&\le T^{1-\alpha}\psi_9(a_m)\Big(\int_0^T|u_{m\xi}|^2_{\infty,\Omega}dt\Big)^2\sup_{t\le T}|u_{m\xi}|^2_{2,\Omega}\qquad(3.17)\\&\le T^{1-\alpha}\psi_{10}(a_m)\|u_m\|^4_{2+\alpha,\Omega^T}\sup_{t\le T}\|u_m\|^2_{1+\alpha,\Omega}.\end{aligned}$$

Next, we obtain

$$\mathcal{I}_4\le\psi_5(a_m)\int_0^T\int_0^T\frac{|u_{m\xi}(t)-u_{m\xi}(t')|^2_{2p_1,\Omega}}{|t-t'|^{1+\alpha}}|u_{m\xi}|^2_{2p_2,\Omega}dtdt'.$$

Using the imbedding $\partial_\xi^\sigma W_2^{1+\alpha}(\Omega)\subset L_{2p_2}(\Omega)$ (where $|\sigma|=1$) and the interpolation inequality

$$|u_{m\xi}(t)-u_{m\xi}(t')|^2_{2p_1,\Omega}\le\varepsilon_1^{1-\varkappa}\|u_m(t)-u_m(t')\|^2_{2,\Omega}+c\varepsilon_1^{-\varkappa}|u_m(t)-u_m(t')|^2_{2,\Omega}$$

(where $\varkappa = 3/4 - 3/(4p_1) + 1/2$) which hold simultaneously for any p_1 and p_2 such that $1 < p_1,\ p_2 < \infty$, $1/p_1 + 1/p_2 = 1$, $3/2 - 3/(2p_1) < 1$ and $3/2 - 3/(2p_2) \le \alpha$ if $\alpha > 1/2$, we get

$$\begin{aligned}\mathcal{I}_4 \le{} & \psi_{11}(a_m) \sup_{t\le T} \|u_m\|^2_{1+\alpha,\Omega} \\ & \cdot \int_0^T \int_0^T \frac{(\varepsilon_1^{1-\varkappa}\|u_m(t) - u_m(t')\|^2_{2,\Omega} + \varepsilon_1^{-\varkappa}|u_m(t) - u_m(t')|^2_{2,\Omega})}{|t-t'|^{1+\alpha}} dtdt'.\end{aligned}$$

Therefore, choosing $\varepsilon_1 = T$ and using that $u_m(t) - u_m(t') = \int_{t'}^t u_{m\tau} d\tau$ we have

$$\begin{aligned}\mathcal{I}_4 \le{} & \psi_{12}(a_m) \sup_{t\le T} \|u_m\|^2_{1+\alpha,\Omega} \bigg(T^{1-\varkappa}\|u_m\|^2_{2+\alpha,\Omega^T} \\ & + T^{-\varkappa} \int_0^T \int_0^T \frac{|\int_{t'}^t u_{m\tau} d\tau|^2_{2,\Omega}}{|t-t'|^{1+\alpha}} dtdt' \bigg) \\ \le{} & \psi_{13}(a_m) T^{1-\varkappa}(1+T^{1-\alpha}) \sup_{t\le T} \|u_m\|^2_{1+\alpha,\Omega} \|u_m\|^2_{2+\alpha,\Omega^T}.\end{aligned} \tag{3.18}$$

Hence, estimates (3.16)–(3.18) yield

$$\begin{aligned}[\mathcal{I}]^2_{\alpha/2,\Omega^T,t} \le{} & \psi_{14}(a_m) T^{a_2} \bar{\psi}(T) \sup_{t\le T} \|u_m\|^2_{1+\alpha,\Omega} \|u_m\|^2_{2+\alpha,\Omega^T} \\ & \cdot (1 + \|u_m\|^2_{2+\alpha,\Omega^T}),\end{aligned} \tag{3.19}$$

where $a_2 > 0$ and $\bar{\psi}$ is a positive increasing continuous function of T.

Taking into account (3.15) and (3.19) we get

$$\begin{aligned}\|l_7\|^{(\alpha,\frac{\alpha}{2})}_{\Omega^T} + \|l_8\|^{(\alpha,\frac{\alpha}{2})}_{\Omega^T} \le{} & \psi_{15}(a_m, T) T^{a_3} [\sup_{t\le T} \|u_m\|^2_{1+\alpha,\Omega} \\ & \cdot (\|u_m\|^4_{2+\alpha,\Omega^T} + \|u_m\|^2_{2+\alpha,\Omega^T} + \|v_0\|^2_{1+\alpha,\Omega})]^{1//2},\end{aligned} \tag{3.20}$$

where ψ_{15} is a positive increasing continuous function of its arguments.

Now, by (3.11) and Lemma 2.1 we have

$$\begin{aligned}& \|\eta_m(t)\|_{1+\alpha,\Omega} + \|\vartheta_m(t)\|_{1+\alpha,\Omega} \\ & \le [A + \|\varrho_0\|_{1+\alpha,\Omega} \varphi_1(T_*^{\overline{a}} A)]\end{aligned} \tag{3.21}$$

for $t \le T$, where φ_1 is the function from Lemma 2.1. Assume that $B > 0$ is a constant such that $T_*^{\overline{a}} A \le B$ and denote

$$\bar{B} = A + \|\varrho_0\|_{1+\alpha,\Omega} \varphi_1(B).$$

Then (3.21) implies

$$|\eta_m|_{\infty,\Omega^T} + |\vartheta_m|_{\infty,\Omega^T} \le c_3 \bar{B}, \tag{3.22}$$

where $c_3 > 0$ is a constant from the inequality $|u|_{\infty,\Omega} \le c_4 \|u\|_{1+\alpha,\Omega}$.

Using the mean value theorem we have

$$\frac{1}{c_v(\eta_m,\theta_m)}=\frac{1}{c_v(\varrho_0,\theta_0)}+\left(\frac{1}{c_v}\right)_\eta(\varrho_0+s(\eta_m-\varrho_0))(\eta_m-\varrho_0)$$
$$+\left(\frac{1}{c_v}\right)_\theta(\theta_0+s(\theta_m-\theta_0))(\theta_m-\theta_0),$$

where $0<s<1$.

Hence, by (3.22) and by the interpolation inequality we have

$$\left|\frac{1}{c_v(\eta_m,\theta_m)}\right|_{\infty,\Omega^t}\le\sigma^*+B_1\bigg[\varepsilon^{1-\varkappa}(\sup_{0\le\tau\le t}\|\eta_m\|_{1+\alpha,\Omega}+\|\varrho_0\|_{1+\alpha,\Omega})$$
$$+c\varepsilon^{-\varkappa}\sup_{0\le\tau\le t}\left\|\int_0^\tau\eta_{mt}\,dt'\right\|_{0,\Omega}\bigg]$$
$$+B_2\bigg[\varepsilon^{1-\varkappa}(\sup_{0\le\tau\le t}\|\theta_m\|_{1+\alpha,\Omega}+\|\theta_0\|_{1+\alpha,\Omega})$$
$$+c\varepsilon^{-\varkappa}\sup_{0\le\tau\le t}\left\|\int_0^\tau\theta_{mt'}\,dt'\right\|_{0,\Omega}\bigg],$$

where $B_1,B_2>0$ are constants depending on $\bar{B}$. Therefore putting $\varepsilon=t^{1/2}$ we get

$$\left|\frac{1}{c_v(\eta_m,\theta_m)}\right|_{\infty,\Omega^t}\le\sigma^*+c(B_1+B_2)t^{1/2(1-\varkappa)}[\alpha_m(t)\tag{3.23}$$
$$+\|\varrho_0\|_{1+\alpha,\Omega}+\|\theta_0\|_{1+\alpha,\Omega}].$$

Next, from (3.6) it follows

$$\left|\frac{1}{\eta_m}\right|_{\infty,\Omega^t}\le|\varrho_0|_{\infty,\Omega}\exp\left[t^{1/2}\left(\int_0^t|\nabla_{u_m}\cdot u_m|^2dt'\right)^{1/2}\right]$$
$$\le|\varrho_0|_{\infty,\Omega}\exp(t^{1/2}\alpha_m(t)).\tag{3.24}$$

In the same way we estimate $\|\eta_m\|_{C^\beta(\overline{\Omega}^t)}$ and $\|\theta_m\|_{C^\beta(\overline{\Omega}^t)}$.

Now, taking into account estimate (3.12), Lemma 2.1, (3.23)–(3.24) and the estimates of l_i, $i=1,\dots,10$, (i.e. estimate (3.20) and the others) we get

$$\alpha_{m+1}(t)\le G(t,t^a\alpha_m(t),H_0)\quad\text{for } t\le T\tag{3.25}$$

if $T\le T_*$ and T_* is sufficiently small, $a>0$ is a constant. In (3.25) G is a positive increasing continuous function of its arguments,

$$H_0=\sigma^*+\frac{1}{\varrho^*}+\|\varrho_0\|_{1+\alpha,\Omega}+\|v_0\|_{1+\alpha,\Omega}+\|\vartheta_0\|_{1+\alpha,\Omega}+\|H(\xi,0)\|_{1/2+\alpha,S}\le\widetilde{H}_0,$$

and $\widetilde{H}_0$ is a constant.

Assume that the constant A is so large that

$$G(0,0,\widetilde{H}_0) < A. \tag{3.26}$$

Then by (3.11) and (3.25)–(3.26) if T_* is so small that

$$G(T_*, T_*^a A, \widetilde{H}_0) < A$$

we obtain $\alpha_{m+1}(t) \le A$ for $t \le T \le T_*$.

This completes the proof. $\square$

Now we prove the convergence of the sequence $(\eta_m, u_m, \vartheta_m)$. To do this we consider the differences

$$U_{m+1} = u_{m+1} - u_m, \qquad \Theta_{m+1} = \vartheta_{m+1} - \vartheta_m, \qquad H_m = \eta_m - \eta_{m-1}.$$

The above differences satisfy the problems

$$\begin{aligned}
&\eta_m \partial_t U_{m+1} - \mu \nabla_{u_m}^2 U_{m+1} - \nu \nabla_{u_m} \nabla_{u_m} \cdot U_{m+1} \\
&= -H_m \partial_t u_m - \mu (\nabla_{u_m}^2 - \nabla_{u_{m-1}}^2) u_m \\
&- \nu (\nabla_{u_m} \nabla_{u_m} \cdot - \nabla_{u_{m-1}} \nabla_{u_{m-1}} \cdot) u_m + \nabla_{u_m} p(\eta_m, \vartheta_m) \\
&- \nabla_{u_{m-1}} p(\eta_{m-1}, \vartheta_{m-1}) + H_m g_m + \eta_{m-1}(g_m - g_{m-1}) \\
&\equiv \sum_{i=1}^{7} L_i, \\
&\Pi_0 \Pi_{u_m} \mathbb{D}_{u_m}(U_{m+1}) \bar{n}_{u_m} \\
&= \Pi_0 (\Pi_{u_{m-1}} \mathbb{D}_{u_{m-1}}(u_m) \bar{n}_{u_{m-1}} - \Pi_{u_m} \mathbb{D}_{u_m}(u_m) \bar{n}_{u_m}) \equiv L_8 + L_9, \\
&\bar{n}_0 \cdot \mathbb{D}_{u_m}(U_{m+1}) \bar{n}_{u_m} - \sigma \bar{n}_0 \cdot \Delta_{u_m}(t) \int_0^t U_{m+1}(\tau) d\tau \\
&= \bar{n}_0 \cdot [\mathbb{D}_{u_{m-1}}(u_m) \bar{n}_{u_{m-1}} - \mathbb{D}_{u_m}(u_m) \bar{n}_{u_m}] \\
&+ \bar{n}_0 \cdot [p(\eta_m, \vartheta_m) \bar{n}_{u_m} - p(\eta_{m-1}, \vartheta_{m-1}) \bar{n}_{u_{m-1}}] \\
&- p_0 \bar{n}_0 \cdot (\bar{n}_{u_m} - \bar{n}_{u_{m-1}}) - \sigma \bar{n}_0 \cdot (\Delta_{u_m}(t) - \Delta_{u_{m-1}}(t)) \xi \\
&+ \sigma \int_0^t \bar{n}_0 \cdot (\dot{\Delta}_{u_m}(\tau) - \dot{\Delta}_{u_{m-1}}(\tau)) \int_0^\tau u_m(t') dt' d\tau \\
&+ \sigma \int_0^t \bar{n}_0 \cdot (\Delta_{u_m}(\tau) - \Delta_{u_{m-1}}(\tau)) u_m(\tau) d\tau \\
&\equiv \sum_{i=10}^{13} L_i + \sum_{i=14}^{15} \int_0^t L_i d\tau,
\end{aligned}$$

$$
\begin{aligned}
&U_{m+1}|_{t=0}=0,\\
&\eta_m c_v(\eta_m,\vartheta_m)\partial_t\Theta_{m+1}-\varkappa\nabla^2_{u_m}\Theta_{m+1}\\
&=-H_m c_v(\eta_m,\vartheta_m)\partial_t\vartheta_m\\
&+\eta_{m-1}\partial_t\vartheta_m[c_v(\eta_{m-1},\vartheta_{m-1})-c_v(\eta_m,\vartheta_m)]\\
&-\varkappa(\nabla^2_{u_m}-\nabla^2_{u_{m-1}})\vartheta_m\\
&+\frac{\mu}{2}\sum_{i,j=1}^{3}[(\xi_{x_i}(u_m)\cdot\nabla_\xi u_{m_j}+\xi_{x_j}(u_m)\cdot\nabla_\xi u_{m_i})^2\\
&-(\xi_{x_i}(u_{m-1})\cdot\nabla_\xi u_{m-1,j}+\xi_{x_j}(u_{m-1})\cdot\nabla_\xi u_{m-1,i})^2]\\
&+(\nu-\mu)[(\nabla_{u_m}\cdot u_m)^2-(\nabla_{u_{m-1}}\cdot u_{m-1})^2]\\
&+[\vartheta_{m-1}p_\vartheta(\eta_{m-1},\vartheta_{m-1})\nabla_{u_{m-1}}\cdot u_{m-1}\\
&-\vartheta_m p_\vartheta(\eta_m,\vartheta_m)\nabla_{u_m}\cdot u_m]\\
&+H_m k_m+\eta_{m-1}(k_m-k_{m-1})\equiv\sum_{i=16}^{23}L_i,\\
&\bar n_{u_m}\cdot\nabla_{u_m}\Theta_{m+1}\\
&=-(\bar n_{u_m}\cdot\nabla_{u_m}\vartheta_m-\bar n_{u_m}\cdot\nabla_{u_{m-1}}\vartheta_m)+(\bar\vartheta_m-\bar\vartheta_{m-1})\\
&\equiv\sum_{i=24}^{25}L_i,\\
&\Theta_{m+1}|_{t=0}=0,\\
&\partial_t H_m+H_m\operatorname{div}_{u_m}u_m \qquad\qquad (3.27)\\
&=-\eta_{m-1}(\operatorname{div}_{u_m}u_m-\operatorname{div}_{u_{m-1}}u_{m-1}),\\
&H_m|_{t=0}=0.
\end{aligned}
$$

Now, we obtain the following local existence theorem.

Theorem 3.4. *Let the assumptions of Lemma* 3.3 *be satisfied and let* $\alpha\in[3/4,1)$. *Then there exists* $T>0$ *such that there exists a unique solution* $(u,\vartheta,\eta)\in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)\times W_2^{2+\alpha,1+\alpha/2}(\Omega^T)\times C(0,T;W_2^{1+\alpha}(\Omega))\cap W_2^{1+\alpha,1/2+\alpha/2}(\Omega^T)$ *of problem* (1.4) *and*

$$\|u\|^{(\alpha+2,\alpha/2+1)}_{\Omega^T}+\|\vartheta\|^{(\alpha+2,\alpha/2+1)}_{\Omega^T}\le A \tag{3.28}$$

and

$$\|\eta\|_{1+\alpha,\Omega^T}+\sup_t\|\eta\|_{1+\alpha,\Omega}\le\varphi(A), \tag{3.29}$$

where $A>0$ *is given by* (3.9) *and* (3.26), φ *is a positive continuous function of* A.

Proof. Similarly as in the proof of Lemma 3.3 using Lemmas 3.1 and 3.2 we get

$$
\begin{aligned}
&\|U_{m+1}\|_{\Omega^T}^{(\alpha+2,\alpha/2+1)} + \|\Theta_{m+1}\|_{\Omega^T}^{(\alpha+2,\alpha/2+1)} \qquad (3.30)\\
&\le \varphi_3\Big(T, \|\eta_m\|_{C^\beta(\overline{\Omega}^T)}, \|\vartheta_m\|_{C^\beta(\overline{\Omega}^T)}, \left|\frac{1}{\eta_m}\right|_{\infty,\Omega^T}, \left|\frac{1}{c_v(\eta_m,\vartheta_m)}\right|_{\infty,\Omega^T}\Big)\\
&\quad \cdot\Big(\sum_{i=1}^{7}\|L_i\|_{\Omega^T}^{(\alpha,\alpha/2)} + \sum_{i=16}^{23}\|L_i\|_{\Omega^T}^{(\alpha,\alpha/2)}\\
&\quad + \sum_{i=8}^{13}\|L_i\|_{\alpha+1/2,S^T} + \sum_{i=24}^{25}\|L_i\|_{\alpha+1/2,S^T} + \sum_{i=14}^{15}\|L_i\|_{S^T}^{(\alpha-1/2,\alpha/2-1/4)}\Big).
\end{aligned}
$$

We estimate the terms on the right-hand side of (3.30). Consider for example L_{17}. Using Lemma 3.3 we get

$$
\begin{aligned}
&[L_{17}]^2_{\alpha,\Omega^T,\xi} \le \bar\psi_1(A) \qquad (3.31)\\
&\cdot\int_0^T\int_\Omega\int_\Omega\Big[\frac{|\eta_{m-1}(\xi)-\eta_{m-1}(\xi')|^2}{|\xi-\xi'|^{3+2\alpha}}|\partial_t\vartheta_m|^2|H_m|^2\\
&+\frac{|\eta_{m-1}(\xi)-\eta_{m-1}(\xi')|^2}{|\xi-\xi'|^{3+2\alpha}}|\partial_t\vartheta_m|^2|\Theta_m|^2\\
&+|\eta_{m-1}|^2\frac{|\partial_t\vartheta_m(\xi)-\partial_t\vartheta_m(\xi')|^2}{|\xi-\xi'|^{3+2\alpha}}|H_m|^2\\
&+|\eta_{m-1}|^2\frac{|\partial_t\vartheta_m(\xi)-\partial_t\vartheta_m(\xi')|^2}{|\xi-\xi'|^{3+2\alpha}}|\Theta_m|^2\\
&+|\eta_{m-1}|^2|\partial_t\vartheta_m|^2\frac{|H_m(\xi)-H_m(\xi')|^2}{|\xi-\xi'|^{3+2\alpha}}\\
&+|\eta_{m-1}|^2|\partial_t\vartheta_m|^2\frac{|\Theta_m(\xi)-\Theta_m(\xi')|^2}{|\xi-\xi'|^{3+2\alpha}}\Big]dtd\xi d\xi'\\
&\equiv\sum_{i=1}^{6}\mathcal{I}_i,
\end{aligned}
$$

where $\bar\psi_1$ and all functions $\bar\psi_i$ below are positive increasing continuous functions, A is given by (3.9) and (3.26). First, we have

$$
\begin{aligned}
\mathcal{I}_1 &\le \bar\psi_2(A)\sup_{t\le T}|H_m|^2_{\infty,\Omega}\sup_{t\le T}\bigg(\int_\Omega\int_\Omega\frac{|\eta_{m-1}(\xi)-\eta_{m-1}(\xi')|^{2p_1}}{|\xi-\xi'|^{3+2p_1(3\mu_1/2-3/(2p_1)+\alpha)}}d\xi d\xi'\bigg)^{1/p_1}\\
&\quad\cdot\int_0^T\bigg(\int_\Omega\int_\Omega\frac{|\vartheta_{mt}|^{2p_2}}{|\xi-\xi'|^{3\mu_2p_2}}\bigg)^{1/p_2},
\end{aligned}
$$

where $1/p_1+1/p_2=1$, $\mu_1+\mu_2=1$ and $\mu_2p_2<1$. Using the imbeddings $W_2^{1+\alpha}(\Omega)\subset W_{2p_1}^{\alpha+3\mu_1/p_1-3/(2p_1)}(\Omega)$ and $W_2^\alpha(\Omega)\subset L_{2p_2}(\Omega)$ which hold simultaneously for any p_1 and p_2 such that $3/2-3/(2p_1)+\alpha+(3/2)\mu_1-3/(2p_1)\le 1+\alpha$ and $3/2-3/(2p_2)\le\alpha$ if $\alpha>1/2$, we get

$$\begin{aligned}\mathcal{I}_1\le&\bar\psi_3(A)\sup_{t\le T}\|H_m\|_{1+\alpha,\Omega}^2\\&\cdot\sup_{t\le T}\|\eta_{m-1}\|_{1+\alpha,\Omega}^2\|\vartheta_{mt}\|_{\alpha,\Omega^T}^2.\end{aligned}\tag{3.32}$$

Now, since

$$\sup_{t\le T}\|H_m\|_{1+\alpha,\Omega}^2\le\bar\psi_4(A,T)T^{a_1}\|U_m\|_{2+\alpha,\Omega^T}^2.\tag{3.33}$$

(where $a_1>0$ is a constant) estimate (3.32) implies

$$\mathcal{I}_1\le\bar\psi_5(A,T)T^{a_1}\|U_m\|_{2+\alpha,\Omega^T}^2,\tag{3.34}$$

where we have also used (3.10).

Next using (3.10), we have

$$\begin{aligned}\mathcal{I}_2&\le\bar\psi_6(A)\sup_{t\le T}|\Theta_m|_{\infty,\Omega}^2\sup_{t\le T}\|\eta_{m-1}\|_{1+\alpha,\Omega}^2\|\vartheta_{mt}\|_{\alpha,\Omega^T}^2\\&\le\bar\psi_7(A)(\varepsilon^{1-\varkappa}\sup_{t\le T}\|\Theta_m\|_{1+\alpha,\Omega}^2+\varepsilon^{-\varkappa}\sup_{t\le T}|\Theta_m|_{2,\Omega}^2).\end{aligned}\tag{3.35}$$

Choosing $\varepsilon=T$ we obtain

$$\begin{aligned}\mathcal{I}_2&\le\bar\psi_8(A)T^{1-\varkappa}\Big(\|\Theta_m\|_{2+\alpha,\Omega^T}^2+T^{-\varkappa}\sup_{t\le T}\Big|\int_0^t\Theta_{m\tau}d\tau\Big|_{2,\Omega}^2\Big)\\&\le\bar\psi_8(A)T^{1-\varkappa}\Big(\|\Theta_m\|_{2+\alpha,\Omega^T}^2+\int_0^T|\Theta_{m\tau}|_{2,\Omega}^2d\tau\Big)\\&\le\bar\psi_9(A)T^{1-\varkappa}\|\Theta_m\|_{2+\alpha,\Omega^T}^2.\end{aligned}\tag{3.36}$$

Now, we estimate

$$\begin{aligned}\mathcal{I}_3+\mathcal{I}_4&\le\bar\psi_{10}(A)\sup_{t\le T}|\eta_{m-1}|_{\infty,\Omega}^2\|\vartheta_{mt}\|_{\alpha,\Omega^T}^2\\&\quad\cdot(\sup_{t\le T}|H_m|_{\alpha,\Omega}^2+\sup_{t\le T}|\Theta_m|_{\infty,\Omega}^2)\\&\le\bar\psi_{11}(A,T)T^{a_2}(\|U_m\|_{2+\alpha,\Omega^T}^2+\|\Theta_m\|_{2+\alpha,\Omega^T}^2),\end{aligned}\tag{3.37}$$

where we have used (3.33), (3.10) and the interpolation inequality from (3.35), and $a_2>0$ is a constant.

Next, we have

$$\begin{aligned}\mathcal{I}_6 \le &\bar{\psi}_{12}(A)\sup_{t\le T}|\eta_{m-1}|^2_{\infty,\Omega}\sup_{t\le T}\left(\int_\Omega\int_\Omega\frac{|\Theta_m(\xi)-\Theta_m(\xi')|^{2p_1}}{|\xi-\xi'|^{3+2p_1(3\mu_1/2-3/(2p_1)+\alpha)}}d\xi d\xi'\right)^{1/p_1}\\ &\cdot\int_0^T\left(\int_\Omega\int_\Omega\frac{|\vartheta_{mt}|^{2p_2}}{|\xi-\xi'|^{3\mu_2p_2}}d\xi\,d\xi'\right)^{1/p_2},\end{aligned}$$

where as before $1/p_1+1/p_2=1$, $\mu_1+\mu_2=1$ and $\mu_2p_2<1$. Using the interpolation inequality

$$\|\Theta_m\|^2_{W^{\alpha+(3/2)\mu_1-3/(2p_1)}_{2p_1}(\Omega)}\le\varepsilon^{1-\varkappa}\|\Theta_m\|^2_{1+\alpha,\Omega}+c\varepsilon^{-\varkappa}|\Theta_m|^2_{2,\Omega}$$

(where $\varkappa=(1/1+\alpha)(3/2-3/(2p_1)+\alpha+(3/2)\mu_1-3/(2p_1)))$ and the imbedding $W^\alpha_2(\Omega)\subset L_{2p_2}(\Omega)$ which hold simultaneously for any p_1 and p_2 such that $3/2-3/(2p_1)+\alpha+(3/2)\mu_1-3/(2p_1)<1+\alpha$ and $3/2-3/(2p_2)\le\alpha$ we get choosing $\varepsilon=T$

$$\mathcal{I}_6\le\bar{\psi}_{13}(A,T)T^{a_3}\|\Theta_m\|^2_{2+\alpha,\Omega^T},\tag{3.38}$$

where $a_3=1-\varkappa$.

In a similar way we obtain

$$\mathcal{I}_5\le\bar{\psi}_{14}(A,T)T^{a_4}\|U_m\|^2_{2+\alpha,\Omega^T},\tag{3.39}$$

where $a_4>0$ is a constant.

Taking into account estimates (3.31), (3.34), and (3.36)–(3.39) we have

$$\begin{aligned}[L_{17}]^2_{\alpha,\Omega^T,\xi}\le&\bar{\psi}_{15}(A,T)T^{a_5}(\|U_m\|^2_{2+\alpha,\Omega^T}\\ &+\|\Theta_m\|^2_{2+\alpha,\Omega^T}),\end{aligned}\tag{3.40}$$

where $a_5>0$ is a constant.

Now, we estimate

$$\begin{aligned}&[L_{17}]^2_{\alpha/2,\Omega^T,t}\le\bar{\psi}_{16}(A)\\ &\cdot\int_\Omega\int_0^T\int_0^T\Big[\frac{|\eta_{m-1}(t)-\eta_{m-1}(t')|^2}{|t-t'|^{1+\alpha}}|\partial_t\vartheta_m|^2|H_m|^2\\ &+\frac{|\eta_{m-1}(t)-\eta_{m-1}(t')|^2}{|t-t'|^{1+\alpha}}|\partial_t\vartheta_m|^2|\Theta_m|^2\\ &+|\eta_{m-1}|^2\frac{|\partial_t\vartheta_m(t)-\partial_{t'}\vartheta_m(t')|^2}{|t-t'|^{1+\alpha}}|H_m|^2\\ &+|\eta_{m-1}|^2\frac{|\partial_t\vartheta_m(t)-\partial_{t'}\vartheta_m(t')|^2}{|t-t'|^{1+\alpha}}|\Theta_m|^2\end{aligned}\tag{3.41}$$

$$
\begin{aligned}
&+|\eta_{m-1}|^2|\partial_t\vartheta_m|^2\frac{|H_m(t)-H_m(t')|^2}{|t-t'|^{1+\alpha}}\\
&+|\eta_{m-1}|^2|\partial_t\vartheta_m|^2\frac{|\Theta_m(t)-\Theta_m(t')|^2}{|t-t'|^{1+\alpha}}\bigg]dtdt'd\xi\\
&\equiv\sum_{i=7}^{12}\mathcal{I}_i.
\end{aligned}
$$

First, in the same way as $\mathcal{I}_3$ and $\mathcal{I}_4$ we estimate

$$
\begin{aligned}
\mathcal{I}_9+\mathcal{I}_{10}&\le\bar\psi_{17}(A)\sup_{t\le T}|\eta_{m-1}|^2_{\infty,\Omega}\|\vartheta_{mt}\|^2_{\alpha,\Omega^T}\\
&\quad\cdot(\sup_{t\le T}|\Theta_m|^2_{\infty,\Omega}+\sup_{t\le T}|H_m|^2_{\infty,\Omega})\\
&\le\bar\psi_{18}(A)T^{a_2}(\|U_m\|^2_{2+\alpha,\Omega^T}+\|\Theta_m\|^2_{2+\alpha,\Omega^T}).
\end{aligned}
\tag{3.42}
$$

Next, we consider

$$
\begin{aligned}
\mathcal{I}_{11}&\le\bar\psi_{19}(A)\sup_{t\le T}|\eta_{m-1}|^2_{\infty,\Omega}\\
&\quad\cdot\int_0^T\int_0^T\frac{|\int_{t'}^t H_{m\tau}\,d\tau|^2_{\infty,\Omega}}{|t-t'|^{1+\alpha}}|\vartheta_{mt}|^2_{2,\Omega}dtdt'.
\end{aligned}
$$

Hence by equation (3.27) and estimate (3.33) we have

$$
\begin{aligned}
\mathcal{I}_{11}&\le\bar\psi_{20}(A)T^{1-\alpha}\int_0^T|H_{mt}|^2_{\infty,\Omega}dt\int_0^T|\vartheta_{mt}|^2_{2,\Omega}dt\\
&\le\bar\psi_{21}(A)T^{1-\alpha}\bigg(\sup_{t\le T}|H_m|^2_{\infty,\Omega}\int_0^T|u_{m\xi}|^2_{\infty,\Omega}dt\\
&\quad+T\sup_{t\le T}|\eta_{m-1}|^2_{\infty,\Omega}\int_0^T|U_{m\xi}|^2_{\infty,\Omega}dt\int_0^T|u_{m\xi}|^2_{\infty,\Omega}dt\\
&\quad+\sup_{t\le T}|\eta_{m-1}|^2_{\infty,\Omega}\int_0^T|U_{m\xi}|^2_{\infty,\Omega}dt\bigg)\int_0^T|\vartheta_{mt}|^2_{2,\Omega}dt\\
&\le\bar\psi_{22}(A)T^{1-\alpha}(\sup_{t\le T}\|H_m\|^2_{1+\alpha,\Omega}\|u_m\|^2_{2+\alpha,\Omega^T}\\
&\quad+T\sup_{t\le T}\|\eta_{m-1}\|^2_{1+\alpha,\Omega}\|U_m\|^2_{2+\alpha,\Omega^T}\|u_m\|^2_{2+\alpha,\Omega^T}\\
&\quad+\sup_{t\le T}\|\eta_{m-1}\|^2_{1+\alpha,\Omega}\|U_m\|^2_{2+\alpha,\Omega^T})\|\vartheta_{mt}\|^2_{0,\Omega^T}\\
&\le\bar\psi_{23}(A,T)T^{a_6}\|U_m\|^2_{2+\alpha,\Omega^T},
\end{aligned}
\tag{3.43}
$$

where $a_6>0$ is a constant.

In the same way we obtain the estimate

$$
\mathcal{I}_7+\mathcal{I}_8\le\bar\psi_{24}(A,T)T^{a_7}\|U_m\|^2_{2+\alpha,\Omega^T},
\tag{3.44}
$$

where $a_7 > 0$ is a constant.

Finally, we estimate $\mathcal{I}_{12}$. We get

$$\begin{aligned}\mathcal{I}_{12} &\le \bar\psi_{25}(A)\sup_{t\le T}|\eta_{m-1}|^2_{\infty,\Omega} \\ &\quad\cdot\int_0^T\int_0^T\frac{|\int_{t'}^t\Theta_{m\tau}d\tau|^2_{4,\Omega}}{|t-t'|^{1+\alpha}}|\partial_t\vartheta_m|^2_{4,\Omega}dtdt' \\ &\le \bar\psi_{26}(A)T^{1-\alpha}\int_0^T|\Theta_{mt}|^2_{4,\Omega}dt\int_0^T|\vartheta_{mt}|^2_{4,\Omega}dt \\ &\le \bar\psi_{27}(A)T^{1-\alpha}\|\Theta_{mt}\|^2_{\alpha,\Omega^T}\|\vartheta_{mt}\|^2_{\alpha,\Omega^T} \\ &\le \bar\psi_{28}(A)T^{1-\alpha}\|\Theta_{mt}\|^2_{\alpha,\Omega^T}\end{aligned} \tag{3.45}$$

which holds for $\alpha \ge 3/4$.

Now, estimates (3.41)–(3.45) yield

$$\begin{aligned}&[L_{17}]^2_{\alpha/2,\Omega^T,t} \\ &\quad\le \bar\psi_{25}(A,T)T^{a_8}(\|U_m\|^2_{2+\alpha,\Omega^T}+\|\Theta_m\|^2_{2+\alpha,\Omega^T}),\end{aligned} \tag{3.46}$$

where $a_8 > 0$ is a constant. By (3.40) and (3.46) we have

$$\|L_{17}\|^{(\alpha,\alpha/2)}_{\Omega^T} \le \bar\psi_{26}(A,T)T^{a_9}(\|U_m\|_{2+\alpha,\Omega^T}+\|\Theta_m\|_{2+\alpha,\Omega^T}),$$

where $a_9 > 0$ is a constant.

In the same way we estimate the other terms on the right-hand side of (3.30). Thus, summarizing the above considerations we get by using Lemma 3.3

$$\|U_{m+1}\|_{2+\alpha,\Omega^T}+\|\Theta_{m+1}\|_{2+\alpha,\Omega^T} \le \bar\psi(A,T)T^a(\|U_m\|_{2+\alpha,\Omega^T}+\|\Theta_m\|_{2+\alpha,\Omega^T}),$$

where $a > 0$ is a constant, $\bar\psi$ is a positive increasing continuous function of its arguments. Hence we obtain for $T \le T_*$ (where T_* is sufficiently small) the strong convergence of $(u_m, \vartheta_m, \eta_m)$ to a solution (u, ϑ, η) of problem (1.3).

The uniqueness follows for T_* sufficiently small from the inequality

$$\|U\|_{2+\alpha,\Omega^T}+\|\Theta\|_{2+\alpha,\Omega^T} \le \bar\psi(A,T)T^a(\|U\|_{2+\alpha,\Omega^T}+\|\Theta\|_{2+\alpha,\Omega^T}),$$

where $T \le T_*$, $U = u_1 - u_2$, $\Theta = \vartheta_1 - \vartheta_2$ and (u_1, ϑ_1), (u_2, ϑ_2) are two solutions of problem (1.3).

Estimate (3.28) follows from Lemma 3.3 and the convergence of (u_m, ϑ_m), and estimate (3.29) follows from (3.28) and Lemma 2.1.

This completes the proof of the theorem. □

References

[1] Besov, O. V., Ill'in, V. P., Nikol'skii S. M., *Integral representations of functions and imbedding theorems* (in Russian), Nauka, Moscow, 1975; English transl.: Washington D.C., Scripta Series in Mathematics Halsted Press Book, Winston, 1979.

[2] Burnat, M., Zajączkowski W. M., *On local motion of a compressible barotropic viscous fluid with the boundary slip condition*, Topol. Methods Nonlinear Anal. **10** (1997), 195–223.

[3] Landau, L., Lifschitz, E., *Hydrodynamics* (in Russian), Nauka, Moscow, 1986.

[4] Serrin, J., *Mathematical principles of classical fluid mechanics*, in "Handbuch der Physik", Bd. VIII//1, Berlin, Göttingen, Heidelberg, Springer, 1959.

[5] Solonnikov, V. A., *Solvability of a problem of evolution of viscous incompressible fluid bounded by a free boundary on finite time interval*, Algebra and Analysis **3** (1991), 222–257.

[6] Solonnikov, V. A., Tani, A., *Free boundary problem for a viscous compressible flow with a surface tension*, Constantin Caratheodory: An International Tribute, ed. by Th. M. Rassias, World Scientific Publ. Co., (1991), 1270–1303.

[7] Solonnikov, V. A., Tani A., *Evolution free boundary problem for equations of motion of viscous compressible barotropic liquid*, Paderborn University, preprint.

[8] Zadrzyńska, E., *On nonstationary motion of a fixed mass of a general viscous compressible heat conducting capillary fluid bounded by a free boundary*, Appl. Math. **25** (1999), 489–511.

[9] Zadrzyńska, E., Zajączkowski W. M., *On local motion of a general compressible viscous heat conducting fluid bounded by a free surface*, Ann. Polon. Math. **59** (1994), 133–170.

[10] Zajączkowski W. M., *On nonstationary motion of a compressible barotropic viscous capillary fluid bounded by a free surface*, SIAM J. Math. Anal. **25** (1994), 1–84.

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