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QUOTIENTS OF QUASI-CONTINUOUS FUNCTIONS

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Abstract. The main goal of this paper is to characterize both the quotients of quasi-continuous and the quotients of Darboux quasi-continuous functions. We prove also theorems concerning common divisor for the families of the quotients of quasi-continuous (Darboux quasi-continuous) functions with respect to quasi-continuity (Darboux property and quasi-continuity, respectively).

1. Introduction

The letters $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ denote the real line, the set of rationals, the set of integers, and the set of positive integers, respectively. The word *function* denotes a mapping from $\mathbb{R}$ to $\mathbb{R}$ unless otherwise explicitly stated. For each set A we use the symbols $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{bd} A$, χ_A, and $\operatorname{card} A$ to denote the interior, the closure, the boundary, the characteristic function, and the cardinality of A, respectively. We say that a set $A \subset \mathbb{R}$ is *semi-open* [8], if $A \subset \operatorname{cl} \operatorname{int} A$.

Let $f: \mathbb{R} \to \mathbb{R}$. The symbol $C(f)$ denotes the set of points of continuity of f. For each $y \in \mathbb{R}$ let $[f < y] = \{x \in \mathbb{R}: f(x) < y\}$. Similarly we define

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the symbols $[f > y]$, $[f = y]$, etc. If $A \subset \mathbb{R}$ is nonempty, then let $\operatorname{osc}(f, A)$ be the *oscillation of f on A*, i.e., $\operatorname{osc}(f, A) = \sup\bigl\{|f(x) - f(t)| : x, t \in A\bigr\}$.

The following denote classes of functions:

- $\mathcal{D}$: consists of all *Darboux* functions, i.e., $f \in \mathcal{D}$ iff it has the intermediate value property;
- $\mathcal{Q}$: consists of all *quasi-continuous* functions in the sense of Kempisty [7]; recall that $f \in \mathcal{Q}$ iff for each $x \in \mathbb{R}$ there is a sequence $(x_n) \subset C(f)$ such that $x_n \to x$ and $f(x_n) \to f(x)$ (see, e.g., [5] or [6, Lemma 2]);
- $\mathcal{DQ}$: denotes the family of Darboux quasi-continuous functions;
- $\acute{\mathcal{S}}_s$: consists of all *strong Świątkowski* functions [9], i.e., $f \in \acute{\mathcal{S}}_s$ iff for all $a < b$ and y between $f(a)$ and $f(b)$, there is an $x \in (a, b) \cap C(f)$ with $f(x) = y$; one can easily see that $\acute{\mathcal{S}}_s \subset \mathcal{DQ}$;
- $\mathcal{C}_q$: consists of all *cliquish* functions [15]; recall that $f \in \mathcal{C}_q$ iff $\operatorname{cl} C(f) = \mathbb{R}$ (see, e.g., [14]).

There are several papers concerning theorems on a common summand [4], [3], or factor [12]. In this paper we deal with theorems on a common divisor. In particular we characterize the cardinal

$$\mathrm{q}(\mathcal{A}) \stackrel{\mathrm{df}}{=} \min\Bigl(\bigl\{\operatorname{card} \mathcal{F} : \mathcal{F} \subset {}^{\mathcal{A}}\!/_{\mathcal{A}} \ \&\ \neg\bigl(\exists_g \forall_{f \in \mathcal{F}}\ f/g \in \mathcal{A}\bigr)\bigr\} \cup \bigl\{(\operatorname{card} {}^{\mathcal{A}}\!/_{\mathcal{A}})^+\bigr\}\Bigr)$$

for the families $\mathcal{Q}$ and $\mathcal{DQ}$, where

$${}^{\mathcal{A}}\!/_{\mathcal{A}} \stackrel{\mathrm{df}}{=} \bigl\{f/g : f, g \in \mathcal{A}\ \&\ g(x) \neq 0 \text{ for each } x \in \mathbb{R}\bigr\}.$$

In the above definition it is quite natural to restrict ourselves to subfamilies of ${}^{\mathcal{A}}\!/_{\mathcal{A}}$ only. Indeed, if there is a function g such that both f/g and $1/g$ are in $\mathcal{A}$, then $f \in {}^{\mathcal{A}}\!/_{\mathcal{A}}$. Therefore, before we can examine the value of $\mathrm{q}(\mathcal{A})$, we should know what the family ${}^{\mathcal{A}}\!/_{\mathcal{A}}$ is.

2. Quasi-continuous functions

Denote by $\mathcal{B}$ the family of all cliquish functions f such that the set $[f \neq 0]$ is semi-open. We start with a simple proposition.

Proposition 2.1. ${}^{\mathcal{Q}}\!/_{\mathcal{Q}} \subset \mathcal{B}$.

Proof. Let $f = g/h$, where $g, h \in \mathcal{Q}$. Then the cliquishness of f is obvious. By [14], the set $[g \neq 0]$ is semi-open. Clearly $[f \neq 0] = [g \neq 0]$. □

Our next goal is to show that ${}^{\mathcal{Q}}\!/_{\mathcal{Q}} = \mathcal{B}$.

Lemma 2.2. *Let $I = [a, b]$ and $m < M$. Suppose that $f_1, \dots, f_l \in \mathcal{C}_q$, and $\max\{\operatorname{osc}(f_i, I) : i \in \{1, \dots, l\}\} < 1$. There is a Baire one function g such that $g = 0$ on $\operatorname{bd} I$, and for each i, $(f_i + g)[I] \supset [m, M]$ and $(f_i + g)\restriction I$ is strong Świątkowski.*

Proof. Put

$$\widetilde{m} = m - \max\{\sup |f_i|[I] : i \in \{1, \dots, l\}\} - 1,$$
$$\widetilde{M} = M + \max\{\sup |f_i|[I] : i \in \{1, \dots, l\}\} + 1.$$

Let φ be a continuous function such that $\varphi[I] \supset [\widetilde{m}, \widetilde{M}]$ and $\varphi = 0$ on $\operatorname{bd} I$. For each i define $\widetilde{f}_i = (f_i + \varphi)\chi_I + f_i(a)\chi_{(-\infty,a)} + f_i(b)\chi_{(b,\infty)}$. By [10, Theorem 4], there is a Baire one function $\tilde{g}$ such that $\widetilde{f}_i + \tilde{g} \in \acute{\mathcal{S}}_s$ for each i, and $\sup |\tilde{g}|[\mathbb{R}] < 1$; by its proof, we can conclude that $\tilde{g} = 0$ on $\operatorname{bd} I$.

Put $g = \varphi + \tilde{g}$. Then for each i, $(f_i + g)\restriction I$ is strong Świątkowski and

$$\begin{aligned}(f_i + g)[I] &\supset \big(\inf(f_i + g)[I], \sup(f_i + g)[I]\big)\\ &\supset \big[\sup f_i[I] + \widetilde{m} + 1, \inf f_i[I] + \widetilde{M} - 1\big] \supset [m, M].\end{aligned}$$

Clearly g is Baire one and $g = 0$ on $\operatorname{bd} I$. □

Lemma 2.3. *Let $f_1, \dots, f_l \in \mathcal{C}_q$, and assume that each f_i is either positive or negative on (a, b). There is a Baire one function $g\colon (a, b) \to \mathbb{R} \setminus \{0\}$ such that for each i, $(f_i/g)[(a, c)] = (f_i/g)[(c, b)] = \mathbb{R} \setminus \{0\}$ for each $c \in (a, b)$, and f_i/g is quasi-continuous.*

Proof. For each i define

$$\widetilde{f}_i(x) = \begin{cases} \ln |f_i|(x) & \text{if } x \in (a, b),\\ 0 & \text{otherwise.}\end{cases}$$

Clearly each $\widetilde{f}_i$ is cliquish. So by [10, Theorem 4], there is a Baire one function $\bar{g}$ such that $\widetilde{f}_i - \bar{g} \in \acute{\mathcal{S}}_s$ for each i. Let $\{a_z : z \in \mathbb{Z}\} \subset \bigcap_{i=1}^{l} C(\widetilde{f}_i - \bar{g})$ be an increasing sequence with limit points a and b. Fix $z \in \mathbb{Z}$. Choose $b_z \in (a_z, a_{z+1})$ such that $\max\{\operatorname{osc}(\widetilde{f}_i - \bar{g}, [a_z, b_z]) : i \in \{1, \dots, l\}\} < 1$. Use Lemma 2.2 to construct a Baire one function g_z such that $g_z = 0$ on $\{a_z, b_z\}$, and for each i, $(\widetilde{f}_i - \bar{g} - g_z)\restriction[a_z, b_z]$ is strong Świątkowski and $(\widetilde{f}_i - \bar{g} - g_z)\big[[a_z, b_z]\big] \supset [-|z|, |z|]$. Define

$$\tilde{g}(x) = \begin{cases} \bar{g}(x) + g_z(x) & \text{if } x \in [a_z, b_z],\ z \in \mathbb{Z},\\ \bar{g}(x) & \text{otherwise,}\end{cases}$$

and let $g(x) = (-1)^z \exp(\tilde{g}(x))$ if $x \in [a_z, a_{z+1})$, $z \in \mathbb{Z}$. We will prove that g has all required properties. Clearly g is Baire one.

Fix $i \in \{1,\ldots,l\}$. Notice that for each z, $f_i/g = (-1)^z \exp\circ\big(\widetilde{f_i} - \tilde{g}\big)\operatorname{sgn}\circ f_i$ on $[a_z, a_{z+1})$ and $\big(\widetilde{f_i} - \tilde{g}\big)\restriction[a_z, a_{z+1})$ is quasi-continuous. It follows that f_i/g is quasi-continuous on $(a,b) = \bigcup_{z\in\mathbb{Z}}[a_z, a_{z+1})$.

Finally let $c \in (a,b)$ and $y \neq 0$. Choose a $z \in \mathbb{Z}$ with $\big|\ln|y|\big| \le |z|$ such that $[a_z, b_z] \subset (a,c)$ and $(-1)^z = \operatorname{sgn} y \cdot \operatorname{sgn}\circ f_i$ on (a,b). Since $\big(\widetilde{f_i} - \tilde{g}\big)\big[[a_z, b_z]\big] \supset [-|z|, |z|]$, there is an $x \in [a_z, b_z]$ such that $\ln|y| = \ln|f_i|(x) - \tilde{g}(x)$. Thus

$$y = |y|\operatorname{sgn} y = \big(|f_i|/\exp\circ\tilde{g}\big)(x)(-1)^z \operatorname{sgn} f_i(x) = (f_i/g)(x).$$

Similarly we can show that $y = (f_i/g)(x')$ for some $x' \in (c,b)$. □

Remark. We say that a set $A \subset \mathbb{R}$ is *simply open* [1] if $\operatorname{bd} A$ is nowhere dense. It is easy to show that each semi-open set is simply open. So by [2], if $f \in \mathcal{B}$, then the set $\operatorname{int}[f=0] \cup \operatorname{int}[f>0] \cup \operatorname{int}[f<0]$ is dense in $\mathbb{R}$.

Theorem 2.4. *Let $f_1,\ldots f_k \in \mathcal{B}$. There is a Baire one function $g\colon \mathbb{R} \to \mathbb{R}\setminus\{0\}$ such that $f_i/g \in \mathcal{Q}$ for each i.*

Proof. Let $\big\{(a_n, b_n) : n < r\big\}$, where $r \le \omega$, be a sequence of all components of the set

$$U = \bigcap_{i=1}^{k}\big(\operatorname{int}[f_i = 0] \cup \operatorname{int}[f_i > 0] \cup \operatorname{int}[f_i < 0]\big). \tag{1}$$

By the remark preceding this theorem, U is dense in $\mathbb{R}$. Clearly $\operatorname{sgn}\circ f_i$ is constant on (a_n, b_n) for each i and n. By Lemma 2.3, for each n there is a Baire one function $g_n\colon (a_n, b_n) \to \mathbb{R}\setminus\{0\}$ such that for each i, f_i/g_n is quasi-continuous on (a_n, b_n) and if f_i is nonzero on (a_n, b_n), then

$$\begin{aligned}&(f_i/g_n)[(a_n, c)]\\ &= (f_i/g_n)[(c, b_n)] = \mathbb{R}\setminus\{0\} \quad \text{for each } c \in (a_n, b_n).\end{aligned} \tag{2}$$

Define

$$g(x) = \begin{cases} g_n(x) & \text{if } x \in (a_n, b_n),\ n < r,\\ 1 & \text{otherwise.}\end{cases}$$

Fix $i \in \{1,\ldots,k\}$. Clearly f_i/g is quasi-continuous on U. So, let $x \in \mathbb{R}\setminus U$. We consider two cases.

If $x \notin \operatorname{cl}\operatorname{int}[f_i \neq 0]$, then since $[f_i \neq 0]$ is semi-open, $x \notin \operatorname{cl}[f_i \neq 0]$. Hence $x \in \operatorname{int}[f_i = 0] = \operatorname{int}[(f_i/g) = 0]$, and f_i/g is continuous at x.

In the opposite case, there is a sequence $(I_m) \subset \big\{(a_n, b_n) \cap [f_i \neq 0] : n < r\big\}$ such that $\varrho(x, I_m) \to 0$, where $\varrho(x, I_m) \overset{\text{df}}{=} \inf\big\{|x - t| : t \in I_m\big\}$. (We use the fact that U is dense in $\mathbb{R}$, and the definition of $\mathcal{B}$.) Notice that f_i/g is quasi-continuous on each I_m. So by (2), there is a sequence $(x_m) \subset C(f_i/g)$ such that $x_m \to x$ and $(f_i/g)(x_m) \to (f_i/g)(x)$. So, f_i/g is quasi-continuous at x. □

Corollary 2.5. ${}^{\mathcal{Q}}\!/_{\mathcal{Q}} = \mathcal{B}$.

Proof. Let $f \in \mathcal{B}$. By Theorem 2.4, there is a function g such that $f/g \in \mathcal{Q}$ and $1/g \in \mathcal{Q}$. Hence $f = (f/g)/(1/g) \in {}^{\mathcal{Q}}\!/_{\mathcal{Q}}$.

The opposite inclusion follows by Proposition 2.1. □

Proposition 2.6. *For each function* $g\colon \mathbb{R} \to \mathbb{R} \setminus \{0\}$ *there is a* $q \in \mathbb{Q}$ *such that* $(\exp \circ \chi_{\{q\}})/g \notin \mathcal{Q}$.

Proof. If $C(g) = \emptyset$, then $C\big((\exp \circ \chi_{\{0\}})/g\big) \subset \{0\}$, so $(\exp \circ \chi_{\{0\}})/g \notin \mathcal{Q}$.

Otherwise by [11, Proposition 3.3], $\chi_{\{q\}} - \ln|g| \notin \mathcal{Q}$ for some $q \in \mathbb{Q}$. Hence

$$\exp \circ (\chi_{\{q\}} - \ln|g|) = (\exp \circ \chi_{\{q\}})/|g| \notin \mathcal{Q},$$

and consequently, $(\exp \circ \chi_{\{q\}})/g \notin \mathcal{Q}$. □

Theorem 2.7. $\mathrm{q}(\mathcal{Q}) = \omega$.

Proof. The inequality $\mathrm{q}(\mathcal{Q}) \geq \omega$ follows by Theorem 2.4 and Corollary 2.5. The opposite inequality follows by Proposition 2.6. □

3. Darboux quasi-continuous functions

Now we turn to the quotients of Darboux quasi-continuous functions. Denote by $\mathcal{B}^\star$ the family of all cliquish functions f such that

a) both $[f > 0]$ and $[f < 0]$ are semi-open;
b) if $a < b$ and $f(a)f(b) < 0$, then $[f = 0] \cap (a, b) \neq \emptyset$;
c) both $[f > 0]$ and $[f < 0]$ are bilaterally dense in itself.

Proposition 3.1. ${}^{\mathcal{DQ}}\!/_{\mathcal{DQ}} \subset \mathcal{B}^\star$.

Proof. Let $f \in {}^{\mathcal{DQ}}\!/_{\mathcal{DQ}}$. Evidently $f \in {}^{\mathcal{D}}\!/_{\mathcal{D}} \cap {}^{\mathcal{Q}}\!/_{\mathcal{Q}}$. So by Proposition 2.1 and [13], f is cliquish and conditions b) and c) hold. Choose $g \in \mathcal{Q}$ and $h \in \mathcal{D}$ such that $f = g/h$. We may assume that $h > 0$. Then $[f > 0] = [g > 0]$ and $[f < 0] = [g < 0]$. Now condition a) follows by [14]. □

Lemma 3.2. *Let $f_1, \dots, f_l \in \mathcal{C}_q$, and assume that each f_i is either positive or negative on (a,b). There is a Baire one function $g\colon (a,b) \to (0,\infty)$ such that for each i, $(|f_i|/g)[(a,c)] = (|f_i|/g)[(c,b)] = (0,\infty)$ for each $c \in (a,b)$, and f_i/g is both Darboux and quasi-continuous.*

Proof. First we proceed as in the proof of Lemma 2.3 to construct the function $\tilde{g}\colon \mathbb{R} \to \mathbb{R}$. Define $g = \exp \circ \tilde{g}\restriction(a,b)$. One can easily see that then the requirements of the lemma are fulfilled. □

Theorem 3.3. *Let $f_1, \dots, f_k \in \mathcal{B}^\star$. There is a Baire one function $g\colon \mathbb{R} \to (0,\infty)$ such that $f_i/g \in \mathcal{DQ}$ for each i.*

Proof. Define U by (1), and let $\{(a_n,b_n)\colon n < r\}$ be a sequence of all its components. By Lemma 3.2, for each n there is a function $g_n\colon (a_n,b_n) \to (0,\infty)$ such that for each i, f_i/g_n is both Darboux and quasi-continuous on (a_n,b_n) and if f_i is nonzero on (a_n,b_n), then

$$(|f_i|/g_n)[(a_n,c)] = (|f_i|/g_n)[(c,b_n)] = (0,\infty) \quad \text{for each } c \in (a_n,b_n). \tag{3}$$

Define

$$g(x) = \begin{cases} g_n(x) & \text{if } x \in (a_n,b_n),\ n < r, \\ 1 & \text{otherwise.} \end{cases}$$

Fix $i \in \{1,\dots,k\}$. The proof of quasi-continuity of f_i/g is a repetition of the argument used in Theorem 2.4. The only difference is that if $x \in \operatorname{cl}\operatorname{int}[f_i \neq 0]$ and $f_i(x) \neq 0$, then we require that $\operatorname{sgn} \circ f_i = \operatorname{sgn} f_i(x)$ on $\bigcup_{m\in\mathbb{N}} I_m$. (We use conditions (3) and a)).

Finally we will show that $f_i/g \in \mathcal{D}$. Fix $a < b$ and let $I = [a,b]$. Clearly f_i/g is Darboux on each connected component of U, so we may assume that $I \setminus U \neq \emptyset$. Define $A = I \cap [f_i > 0]$ and $B = I \cap [f_i < 0]$. First we will show that

$$\text{if } A \neq \emptyset, \text{ then } (f_i/g)[I] \supset (0,\infty).$$

Indeed, if $A \neq \emptyset$, then by c), $(a,b) \cap [f_i > 0] \neq \emptyset$. Now since $[f_i > 0]$ is semi-open and U is dense in $\mathbb{R}$, there is an $n < r$ such that $J \overset{\text{df}}{=} (a,b) \cap \operatorname{int}[f_i > 0] \cap (a_n,b_n) \neq \emptyset$. Observe that $I \cap \{a_n,b_n\} \neq \emptyset$ (recall that $I \setminus U \neq \emptyset$) and $f_i/g = f_i/g_n$ on J. Finally by (3), $(f_i/g)[I] \supset (f_i/g)[J] = (0,\infty)$.

Similarly we can show that if $B \neq \emptyset$, then $(f_i/g)[I] \supset (-\infty,0)$. Now we consider four cases.

If $A = \emptyset = B$, then $(f_i/g)[I] = \{0\}$.

If $A \neq \emptyset = B$, then $(0,\infty) \subset (f_i/g)[I] \subset [0,\infty)$, so $(f_i/g)[I]$ is an interval. Analogously we proceed if $A = \emptyset \neq B$.

If $A \neq \emptyset \neq B$, then $(f_i/g)[I] \supset (-\infty, 0) \cup (0, \infty)$. But by b), $0 \in f_i[I]$. Consequently, $(f_i/g)[I] = \mathbb{R}$. □

The next corollary follows by Proposition 3.1 and Theorem 3.3. Its proof is analogous to that of Corollary 2.5.

Corollary 3.4. ${}^{\mathcal{DQ}}/_{\mathcal{DQ}} = \mathcal{B}^\star$.

Theorem 3.5. $\mathrm{q}(\mathcal{DQ}) = \omega$.

Proof. The inequality $\mathrm{q}(\mathcal{DQ}) \geq \omega$ follows by Theorem 3.3 and Corollary 3.4. The opposite inequality follows by Proposition 2.6. □

The next proposition shows that ${}^{\mathcal{DQ}}/_{\mathcal{DQ}}$ is a proper subset of ${}^{\mathcal{D}}/_{\mathcal{D}} \cap {}^{\mathcal{Q}}/_{\mathcal{Q}}$.

Proposition 3.6. ${}^{\mathcal{DQ}}/_{\mathcal{DQ}} \neq {}^{\mathcal{D}}/_{\mathcal{D}} \cap {}^{\mathcal{Q}}/_{\mathcal{Q}}$.

Proof. Let C be the Cantor ternary set and let C_0 denote the set of points of bilateral accumulation of C. Define

$$f(x) = \begin{cases} 1 & \text{if } x \in C_0, \\ 0 & \text{if } x \in C \setminus C_0, \\ -1 & \text{if } x \in \mathbb{R} \setminus C. \end{cases}$$

Then $f \in {}^{\mathcal{D}}/_{\mathcal{D}} \cap {}^{\mathcal{Q}}/_{\mathcal{Q}}$ (see [13] and Corollary 2.5). On the other hand, $[f > 0] = C_0$. So by Proposition 3.1, $f \notin {}^{\mathcal{DQ}}/_{\mathcal{DQ}}$. □

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