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A TOPOLOGICAL APPROACH TO HEMIVARIATIONAL INEQUALITIES WITH UNILATERAL GROWTH CONDITION

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Abstract. The present paper is devoted to the study of the existence solution problem for a hemivariational inequality on vector-valued function space in the case when the nonlinear nonconvex part satisfies the unilateral growth condition. The critical point theory combined with the Galerkin approximation method have been used to establish the result.

1. Introduction

The theory of hemivariational inequalities begun in the early eighties with the works of P. D. Panagiotopoulos [22], [23], and a main reason for its birth was the need for description of important problems in physics and engineering, where nonmonotone, multivalued boundary or interface conditions occur, or where some nonmonotone, multivalued relations between stress and strain, or reaction and displacement have to be taken into account. The theory of hemivariational inequalities (as the generalization of variational inequalities (cf. [5]) has been proved to be very useful in

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understanding of many problems in mechanics and engineering involving nonconvex, nonsmooth energy functionals.

The aim of this paper is to give some existence results for hemivariational inequalities in the case of the unilateral growth conditions [19] imposed on the "nonlinearities". The approach presented here is based on the critical point theory [1], [27] suitably adopted to the nonsmooth case [2], [16]. See also [11], [7], [10], [15], [14], [13], [12], [28] for the study of topological methods concerning nonsmooth functionals.

For the general mathematical study of hemivariational inequalities and their applications the reader is referred to [24], [21], [16], [25], [17] and, additionally, to [8], [9] for their numerical treatment. Some results related to variational-hemivariational inequalities can be found in [4], [26], [18].

We pass now to the formulation of our main problem and, subsequently, of the imposed assumptions.

Let Ω be a bounded domain in $\mathbb{R}^m$ with sufficiently smooth boundary and V be a Banach space compactly imbedded into $L^p(\Omega;\mathbb{R}^N)$, $p > 2$. Moreover, assume that $g \in V^*$ and $a : V \times V \to \mathbb{R}$ is a continuous, symmetric, bilinear form which is coercive in the sense that there is a constant $\alpha > 0$ satisfying

$$a(v,v) \geq \alpha \|v\|_V^2, \quad \forall v \in V. \tag{1}$$

Suppose that $j : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function verifying the assumptions:

(H_1) $j(x,y)$ is Lipschitz continuous on the bounded subsets of $\mathbb{R}^N$ uniformly with respect to $x \in \Omega$, i.e., $\forall R > 0\ \exists K_R > 0$ such that

$$|j(x,y_1) - j(x,y_2)| \leq K_R |y_1 - y_2|, \quad \forall x \in \Omega,\ \forall |y_1|, |y_2| \leq R;$$

(H_2) there exist constants $\mu > 2$, $C_1 > 0$, $C_2 > 0$, $\sigma \in [1,2)$ such that

$$j(x,y) - \frac{1}{\mu} j^0(x,y;y) \geq -C_1 - C_2 |y|^\sigma, \quad \forall x \in \Omega,\ \forall y \in \mathbb{R}^N;$$

(H_3) $\int_\Omega j(x,0)\, dx \leq 0$ and

$$\liminf_{y \to 0} \frac{j(x,y)}{|y|^2} \geq 0 \text{ uniformly with respect to } x \in \Omega;$$

(H_4) for some $2 < q < p$ the unilateral growth condition holds (Naniewicz [19]):

$$j^0(x,\xi;\eta - \xi) \leq \widetilde{\alpha}(r)(1 + |\xi|^q) \quad \forall \xi, \eta \in \mathbb{R}^N,\ |\eta| \leq r,\ r \geq 0, \text{ a.e. in } \Omega,$$

where $\widetilde{\alpha} : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function from $\mathbb{R}^+$ into $\mathbb{R}^+$;

(H_5) there exists $v_0 \in V \cap L^\infty(\Omega; \mathbb{R}^N)$ such that

$$\liminf_{s\to\infty} s^{-\sigma} \int_\Omega j(x, sv_0)\, dx < \frac{\mu}{\sigma-\mu} C_2 \|v_0\|^\sigma_{L^\sigma},$$

with the positive constants C_2, μ, σ entering (H_2).

Here, for a.e. $x \in \Omega$, $j^0(x, \cdot\,; \cdot)$ stands for the Clarke's generalized directional derivative given by [3]:

$$j^0(x, \xi; \eta) = \limsup_{\substack{h\to 0\\ \lambda\to 0_+}} \frac{j(x, \xi + h + \lambda\eta) - j(x, \xi + h)}{\lambda},$$

and where

$$\partial_y j(x, \xi) = \{\eta \in \mathbb{R}^N : j^0(x, \xi; \mu) \geq \eta \cdot \mu \quad \forall \mu \in \mathbb{R}^N\}, \qquad \text{a.e. in } \Omega,$$

is the Clarke's generalized gradient of $j(x, \cdot)$ in $\xi \in \mathbb{R}^N$.

Remark 1. If $j(x, y)$ satisfies the unilateral growth condition (H_4) then the inequality below holds (see Naniewicz [20], Lemma 2.1):

(H_4') $$j(x, y) \geq -a_1 - a_2|y|^q, \quad \forall x \in \Omega,\ \forall y \in \mathbb{R}^N,$$

with constants $a_1 > 0$ and $a_2 > 0$.

Remark 2. Notice that our statement of hypothesis (H_3) is a weaker form of a celebrated condition of the form: $\beta : \Omega \times \mathbb{R} \to \mathbb{R}$ and

$(\star)$ $$\beta(x, z) = \mathrm{o}(|z|) \quad \text{at } z = 0 \text{ uniformely in } x \in \Omega,$$

where $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is related to β by

$$j(x, z) = \int_0^z \beta(x, t)\, dt,$$

used frequently in the variational theory of various nonlinear boundary value problems, for instance in the study of semilinear elliptic differential equations (cf. [1], [27], [2]). It must be also emphasised that in our paper we deal with vector-valued function space $(N \geq 1)$ while the hypothesis $(\star)$ is referred to the scalar case $(N = 1)$.

We consider the problem of finding $u \in V$ such as to satisfy a hemivariational inequality of the form

(P) $$a(u, v - u) + \int_\Omega j^0(x, u; v - u)\, dx \geq \langle g, v - u\rangle_V, \quad \forall\, v \in V,$$

where the integral above is assumed to take $+\infty$ as its value whenever $j^0(x, u; v - u) \notin L^1(\Omega)$.

The main result of this paper concerning problem (P) is formulated in Theorem 6. To prove the main result, the critical point theory combined with the Galerkin approximation method will be applied. In this respect the basic fact is the variational interpretation of problem (P) in terms of a critical point existence problem for an associated nonsmooth functional.

The novelty of our approach consists mainly in using the nonsmooth version of Mountain Pass Theorem in Chang [2] on an appropriate family of finite dimensional subspaces and then proving a priori estimates for the finite dimensional approximate solutions on the basis of their minimax characterizations. Finally, a passing to limit process is developed. In comparison with our paper [11], where the subquadratic and superquadratic cases in the growth condition of the nonlinear term $j(x, y)$ has been discussed separately and treated by means of different methods, we present here an approach that works for all cases and improves substantially the previous results. In the present paper the unilateral growth condition of Naniewicz [19] is employed in all situations, independently of the growth rate for the nonlinear term $j(x, y)$, and shows its whole applicability. On the hand the nonsmooth critical point arguments and the Galerkin approximation technique are used in a nontrivial way relying essentially on the Mountain-Pass topological type of solutions which we construct on the finite dimensional spaces of the Galerkin basis.

The rest of the paper is organized as follows. Section 2 is devoted to some technical results of nonsmooth critical point theory that are needed in the sequel. Section 3 contains the exposition of the finite dimensional approximation in solving problem (P). Section 4 presents the main result of the paper and its complete proof pointing out the closed connection with the theory of hemivariational inequalities.

2. Some preliminaries

Lemma 1. *Assume that condition (H_1) holds. Then the functional $J : L^\infty(\Omega; \mathbb{R}^N) \to \mathbb{R}$ defined by*

$$J(v) = \int_\Omega j(x, v(x))\, dx, \quad \forall v \in L^\infty(\Omega; \mathbb{R}^N),$$

is Lipschitz continuous on the bounded subsets of $L^\infty(\Omega; \mathbb{R}^N)$. Moreover, J has the property

$$\partial J(v) \subset \int_\Omega \partial_y j(x, v(x))\, dx, \quad \forall v \in L^\infty(\Omega; \mathbb{R}^N), \tag{2}$$

in the sense that for each $z \in \partial J(v)$ there is a corresponding element $\tilde{z} \in L^1(\Omega;\mathbb{R}^N)$, which will be identified with z, such that

$$\langle z, w\rangle = \int_\Omega \tilde{z}(x)\cdot w(x)\,dx \equiv \int_\Omega z(x)\cdot w(x)\,dx, \tag{3}$$

for all $w \in L^\infty(\Omega;\mathbb{R}^N)$, and

$$\tilde{z}(x) \equiv z(x) \in \partial_y j(x, v(x)) \text{ for a.e. } x \in \Omega. \tag{4}$$

Proof. Let $R > 0$. If $v_1, v_2 \in L^\infty(\Omega;\mathbb{R}^N)$ satisfy $\|v_1\|_{L^\infty},\ \|v_2\|_{L^\infty} \le R$, according to assumption (H_1) we can write

$$\begin{aligned}|J(v_1) - J(v_2)| \le &\int_\Omega |j(x, v_1(x)) - j(x, v_2(x))|\,dx\\ \le &K_R\int_\Omega |v_1(x) - v_2(x)|\,dx \le K_R|\Omega|\|v_1 - v_2\|_{L^\infty}.\end{aligned}$$

Hence J is Lipschitz continuous on the closed ball $\|v\|_{L^\infty} \le R$. The representation formula (2) for the generalized gradient ∂J of J in the sense of relations (3) and (4) is proved in Clarke ([3] p. 76) under a hypothesis which is more general than (H_1). □

Throughout the rest of the paper we denote by Λ the family of all finite dimensional subspaces F of $V \cap L^\infty(\Omega;\mathbb{R}^N)$ such that $F \in \Lambda$ iff

$$v_0 \in F, \tag{5}$$

where v_0 is described in (H_5).

For every subspace $F \in \Lambda$ we introduce the functional $I_F : F \to \mathbb{R}$ as follows

$$I_F(v) = \frac{1}{2}a(v, v) - \langle g, v\rangle_V + J(v), \quad \forall v \in F. \tag{6}$$

From (6) it is clear that, assuming (H_1), Lemma 1 ensures that the functional I is locally Lipschitz and its generalized gradient is expressed by

$$\partial I_F(v) = i_F^* A i_F - i_F^* g + \bar{i}_F^* \partial J(v)\bar{i}_F, \quad \forall v \in F, \tag{7}$$

where $i_F : F \to V$ and $\bar{i}_F : F \to L^\infty(\Omega;\mathbb{R}^N)$ are the inclusion maps, while $A : V \to V^*$ stands for the continuous linear operator which corresponds to the bilinear form $a : V \times V \to \mathbb{R}$, that is

$$\langle Au, v\rangle_V = a(u, v), \quad \forall u, v \in V.$$

Lemma 2. *Assume that conditions (H_1) and (H_2) hold. Then for each $F \in \Lambda$ the functional $I_F : F \to \mathbb{R}$ satisfies the Palais-Smale condition in the sense of Chang* [2].

Proof. Let $\{u_n\} \subset F$ and $\{w_n\} \subset F^*$ be sequences such that

$$|I_F(u_n)| \leq M, \quad \forall n \geq 1, \tag{8}$$

for a constant $M > 0$, and

$$w_n \in \partial I_F(u_n), \quad w_n \to 0 \text{ in } F^* \text{ as } n \to \infty. \tag{9}$$

From (7) we see that w_n in (9) can be written as follows

$$w_n = i_F^* A i_F - i_F^* g + \bar{i}_F^* z_n \bar{i}_F, \quad \text{with } z_n \in \partial J(u_n). \tag{10}$$

Using (8), (9) and (10), in conjunction with (2), (3), we get that for n sufficiently large one has

$$M + \|u_n\|_V \geq I_F(u_n) - \frac{1}{\mu}\langle w_n, u_n\rangle_F$$
$$= \left(\frac{1}{2} - \frac{1}{\mu}\right) a(u_n, u_n) + \left(\frac{1}{\mu} - 1\right)\langle g, u_n\rangle_V + \int_\Omega \left[j(x, u_n) - \frac{1}{\mu} z_n \cdot u_n\right] dx.$$

Then on the basis of relation (1) and hypothesis (H_2) one obtains that

$$M + \|u_n\|_V \geq \alpha\left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_V^2 + \left(\frac{1}{\mu} - 1\right)\|g\|_{V^*}\|u_n\|_V - C_1|\Omega| - C_2\|u_n\|_{L^\sigma}^\sigma$$
$$\geq \alpha\left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_V^2 + \left(\frac{1}{\mu} - 1\right)\|g\|_{V^*}\|u_n\|_V - C_1|\Omega| - \overline{C}_2\|u_n\|_V^\sigma,$$

for a new constant $\overline{C}_2$ determined by the imbedding $F \subset V \subset L^\sigma(\Omega; \mathbb{R}^N)$, where $|\Omega|$ stands for the Lebesgue measure of Ω.

Since $\mu > 2$ and $\sigma < 2$, the estimate above yields the boundedness of $\{u_n\}$ in V, so in F. Taking into account that F is finite dimensional, $\{u_n\}$ contains a convergent subsequence in F. This completes the proof. □

Lemma 3. *Assume that conditions* (H_1), (H_2) *and* (H_5) *hold. Then, for each finite dimensional linear subspace* F *of* X *satisfying* (5), *one has*

$$\liminf_{t \to \infty} I_F(tv_0) = -\infty. \tag{11}$$

Proof. For all $\tau \neq 0$, $x \in \Omega$ and $y \in R^N$ it is seen that the formula below for the generalized gradient with respect to τ is valid

$$\partial_\tau(\tau^{-\mu} j(x, \tau y)) = \tau^{-\mu-1}[-\mu j(x, \tau y) + \partial_y j(x, \tau y)(\tau y)]. \tag{12}$$

Since the function $\tau \mapsto \tau^{-\mu} j(x, \tau y)$ is differentiable a.e. on $\mathbb{R}$, equality (12) and a classical property of Clarke's generalized directional derivative imply that

$$t^{-\mu}j(x,ty)-j(x,y)=\int_1^t \frac{d}{d\tau}(\tau^{-\mu}j(x,\tau y))d\tau$$
$$\leq \int_1^t \tau^{-\mu-1}[-\mu j(x,\tau y)+j_y^0(x,\tau y;\tau y)]d\tau, \quad \forall t>1,\ \text{a.e. } x\in\Omega,\ y\in\mathbb{R}^N.$$

Making now use of assumption (H_2) we infer that

$$\begin{aligned} t^{-\mu}j(x,ty)-j(x,y) &\leq \mu\int_1^t \tau^{-\mu-1}[C_1+C_2\tau^{\sigma}|y|^{\sigma}]d\tau \\ &= \mu\left[C_1\left(-\frac{1}{\mu}t^{-\mu}+\frac{1}{\mu}\right)+C_2|y|^{\sigma}\left(\frac{1}{\sigma-\mu}t^{\sigma-\mu}-\frac{1}{\sigma-\mu}\right)\right] \\ &\leq C_1+\mu(\mu-\sigma)^{-1}C_2|y|^{\sigma}, \quad \forall t>1,\ \text{a.e. } x\in\Omega,\ y\in\mathbb{R}^N. \end{aligned} \tag{13}$$

Set $y=sv_0(x)$, with $x\in\Omega$ and $s>0$, in (13). We find the following estimate

$$j(x,tsv_0(x))\leq t^{\mu}[j(x,sv_0(x))+C_1+\mu(\mu-\sigma)^{-1}C_2s^{\sigma}|v_0(x)|^{\sigma}],$$
$$\forall t>1,\ s>0,\ \text{a.e. } x\in\Omega. \tag{14}$$

Combining (6) and (14) we infer that

$$\begin{aligned} I_F(tsv_0) \leq\ & \frac{1}{2}t^2s^2a(v_0,v_0)-ts\langle g,v_0\rangle_V \\ & + t^{\mu}s^{\sigma}[s^{-\sigma}\int_\Omega j(x,sv_0(x))\,dx+s^{-\sigma}C_1|\Omega| \\ & + C_2\mu(\mu-\sigma)^{-1}\|v_0\|_{L^\sigma}^{\sigma}], \quad \forall t>1,\ s>0. \end{aligned} \tag{15}$$

Assumption (H_5) allows to fix some number $s>0$ such that

$$s^{-\sigma}\int_\Omega j(x,sv_0(x))\,dx+s^{-\sigma}C_1|\Omega|+C_2\mu(\mu-\sigma)^{-1}\|v_0\|_{L^\sigma}^{\sigma}<0. \tag{16}$$

With $s>0$ fixed as in (16) we pass to the limit in (15) for $t\to\infty$. This leads to the conclusion that

$$\lim_{t\to\infty} I_F(tsv_0)=-\infty.$$

Therefore assertion (11) is obtained and the proof of Lemma 3 is complete. □

3. Finite dimensional approximation

We state the main results concerning the finite dimensional approximation of (P).

Theorem 4. *Assume that conditions $(H_1) - (H_5)$ hold. Then there exists a constant $B > 0$ such that, whenever $\|g\|_{V^*} \le B$ and $F \in \Lambda$, the problem:*

$(P_{F,g})$ *Find $u \in F$ and $\xi \in L^1(\Omega;\mathbb{R}^N)$ such that*

$$a(u, v-u) + \int_\Omega \xi(x) \cdot \big(v(x) - u(x)\big)\, dx = \langle g, v-u\rangle_V, \ \forall v \in F, \tag{17}$$

$$\xi(x) \in \partial_y j(x, u(x)) \text{ for a.e. } x \in \Omega, \tag{18}$$

admits a solution $(u_{F,g}, \xi_{F,g}) \in F \times L^1(\Omega;\mathbb{R}^N)$. Moreover, there exists a constant $b > 0$ depending only on B such that

$$\|u_{F,g}\|_V \le b. \tag{19}$$

In addition, we have the uniform energy estimate

$$\frac{1}{2} a(u_{F,g}, u_{F,g}) - \langle g, u_{F,g}\rangle_V + \int_\Omega j\big(x, u_{F,g}(x)\big)\, dx \ge \beta \tag{20}$$

for all F, g as required above, with a positive constant β depending only on B.

Proof. For each $F \in \Lambda$ consider the locally Lipschitz functional $I_F : F \to \mathbb{R}$ defined as in (6). We apply to each functional I_F Chang's variant of Mountain Pass Theorem for locally Lipschitz functionals (see Chang [2]). Towards this we note that by (H_3) we have

$$I_F(0) \le 0. \tag{21}$$

We show that there exist constants $B > 0$, $\beta > 0$ and $\rho > 0$ such that

$$I_F(v) \ge \beta, \quad \forall v \in F \text{ with } \|v\|_V = \rho, \tag{22}$$

whenever $g \in V^*$ entering I_F satisfies $\|g\|_{V^*} \le B$ and for all subspaces $F \in \Lambda$.

Indeed, from (H_3) we know that for each $\varepsilon > 0$ one finds $\delta > 0$ such that

$$j(x, y) \ge -\varepsilon |y|^2, \quad \forall x \in \Omega \text{ and } |y| \le \delta.$$

Taking into account (H_4') in Section 1 we derive that

$$j(x, y) \ge -\varepsilon |y|^2 - (a_1 \delta^{-q} + a_2)|y|^q, \quad \forall x \in \Omega, \ \forall y \in \mathbb{R}^N. \tag{23}$$

Then, with new constants $c_0, b_1, b_2 > 0$, one obtains from (1), (6), (23), the continuous imbedding $V \subset L^q(\Omega;\mathbb{R}^N) \subset L^2(\Omega;\mathbb{R}^N)$ and Young's inequality, the estimate

$$\begin{aligned} I_F(v) &\geq \frac{1}{2}\alpha\|v\|_V^2 - \|g\|_{V^*}\|v\|_V - \varepsilon\|v\|_{L^2}^2 - (a_1\delta^{-q} + a_2)\|v\|_{L^q}^q \\ &\geq \left[\frac{1}{2}\alpha - c_0\varepsilon - \left(\frac{b_1}{\delta^q} + b_2 + \frac{1}{q}\right)\|v\|_V^{q-2}\right]\|v\|_V^2 - \frac{q-1}{q}\|g\|_{V^*}^{q/(q-1)}, \\ &\qquad \forall v \in F,\ \forall F \in \Lambda. \end{aligned} \tag{24}$$

Let us take $\rho > 0$ and $\varepsilon > 0$ sufficiently small to have

$$E := \frac{1}{2}\alpha - c_0\varepsilon - \left(\frac{b_1}{\delta^q} + b_2 + \frac{1}{q}\right)\rho^{q-2} > 0. \tag{25}$$

Then, in view of (25), inequality (24) becomes

$$I_F(v) \geq E\rho^2 - \frac{q-1}{q}\|g\|_{V^*}^{q/(q-1)}, \tag{26}$$

for any $v \in F$ with $\|v\|_V = \rho$, and $\forall F \in \Lambda$. Relation (26) ensures that it suffices to choose $B > 0$ such that

$$\beta := E\rho^2 - \frac{q-1}{q}B^{q/(q-1)} > 0 \tag{27}$$

to guarantee that property (22) is valid.

In virtue of Lemma 3 we can choose some $t_0 > 0$ sufficiently large so that $e := t_0 v_0$ has the properties

$$\|e\|_V > \rho \tag{28}$$

and

$$I(e) \leq 0. \tag{29}$$

Furthermore, we know that

$$e \in F, \quad \forall F \in \Lambda. \tag{30}$$

Lemma 2 and the assertions (21), (22) (with (27)), (28) and (29) permit to apply to I_F the Mountain Pass Theorem for locally Lipschitz functionals (see Chang [2]), for each $F \in \Lambda$ and $g \in V^*$ provided $\|g\|_{V^*} \leq B$. This provides a critical point $u_{F,g} \in F$ of I_F, that is

$$0 \in \partial I_F(u_{F,g}). \tag{31}$$

In addition, $I_F(u_{F,g})$ has the following minimax characterization

$$I_F(u_{F,g}) = \inf_{\gamma \in \Gamma_F} \max_{t \in [0,1]} I_F(\gamma(t)), \tag{32}$$

where

$$\Gamma_F = \{\gamma \in C([0,1], F) :\ \gamma(0) = 0,\ \gamma(1) = e\}. \tag{33}$$

Writing explicitly (31) by means of (7), and taking into account (2), (3), (4), we see that for $u_F \in F$ there is $\xi_F \in L^1(\Omega;\mathbb{R}^N)$ solving problem $(P_{F,g})$ with $u = u_{F,g}$ and $\xi = \xi_{F,g}$.

Let us establish the boundedness stated in (19). To this end we note that formula (31) can be expressed in the following way

$$a(u_{F,g}, v) - \langle g, v\rangle_V + \int_\Omega \xi_{F,g}(x) \cdot v(x)\,dx = 0, \quad \forall v \in F. \tag{34}$$

On the other hand we notice that $e \in F$ for all subspaces $F \in \Lambda$ (cf. (30)). Consequently, the segment $[0, e]$, viewed as a path in V, is contained in all subspaces $F \in \Lambda$, thus it belongs to every family Γ_F given by (33). Therefore we can take

$$\begin{aligned}\bar{b} &:= \max_{\|g\|_{V^*}\le B} \max_{t\in[0,1]} I_F(te)\\ &= \max_{\|g\|_{V^*}\le B} \max_{t\in[0,1]} \left[\frac{1}{2}t^2 a(e,e) - t\langle g, e\rangle_V + J(te)\right]\end{aligned} \tag{35}$$

which is independent of F and g and depends on B only.

Using (32)–(35) we conclude that for each $F \in \Lambda$,

$$\bar{b} \ge \max_{t\in[0,1]} I_F(te) \ge I_F(u_{F,g}),$$

provided $\|g\|_{V^*} \le B$. By means of the inequality above combined with (6), (7) and (34), we arrive at

$$\begin{aligned}\bar{b} &\ge I_F(u_{F,g}) - \frac{1}{\mu}\Big[a(u_{F,g}, u_{F,g}) - \langle g, u_{F,g}\rangle_V + \int_\Omega \xi_{F,g}(x) \cdot u_{F,g}(x)\,dx\Big]\\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right) a(u_{F,g}, u_{F,g}) + \left(\frac{1}{\mu} - 1\right)\langle g, u_{F,g}\rangle_V\\ &+ \int_\Omega \left(j(x, u_{F,g}(x)) - \frac{1}{\mu}\xi_{F,g}(x)\cdot u_{F,g}(x)\right) dx,\end{aligned} \tag{36}$$

for an arbitrary subspace $F \in \Lambda$ and each $g \in V^*$ with $\|g\|_{V^*} \le B$.

If now we make use of relations (1), (4) (with $z = \xi_{F,g}$) and assumption (H_2), inequality (36) yields

$$\begin{aligned}\bar{b} &\ge \left(\frac{1}{2} - \frac{1}{\mu}\right)\alpha\|u_{F,g}\|_V^2 + \left(\frac{1}{\mu} - 1\right)\|g\|_{V^*}\|u_{F,g}\|_V\\ &- c_1 - c_2\|u_{F,g}\|_V^\sigma,\end{aligned} \tag{37}$$

with F and g as in (36) and constants $c_1 > 0$, $c_2 > 0$. Estimate (37) shows that the claim in (19) is checked.

The energy estimate in (20) is a consequence of the minimax characterization (32), (33) in conjunction with the boundedness from below on the sphere $\|v\|_V = \rho$, as shown in (26), by the constant $\beta > 0$ indicated in (27). This completes the proof of Theorem 4. □

Theorem 5. *Assume all the hypotheses in Theorem 4. If* $(u_{F,g}, \xi_{F,g}) \in F \times L^1(\Omega; \mathbb{R}^N)$ *is a solution of* $(P_{F,g})$, $F \in \Lambda$, *then the set* $\{\xi_{F,g}\}_{F\in\Lambda}$ *is weakly precompact in* $L^1(\Omega; \mathbb{R}^N)$.

Proof. According to the Dunford-Pettis theorem (see, e.g., [6]) it suffices to show that for each $\varepsilon > 0$ a $\delta > 0$ can be determined such that for any $\omega \subset \Omega$ with $|\omega| < \delta$,

$$\int_\omega |\xi_{F,g}|\, dx < \varepsilon, \quad \forall F \in \Lambda. \tag{38}$$

Fix $r > 0$ and let $\eta \in \mathbb{R}^N$ be such that $|\eta| \le r$. Then one has $\xi_{F,g}\cdot(\eta - u_{F,g}) \le j^0(x, u_{F,g}; \eta - u_{F,g})$, from which, by virtue of (H_4), it results that

$$\xi_{F,g} \cdot \eta \le \xi_{F,g} \cdot u_{F,g} + \widetilde{\alpha}(r)(1 + |u_{F,g}|^q) \tag{39}$$

a.e. in Ω. Let us set $\eta \equiv (r/\sqrt{N})(\operatorname{sgn}\chi_{R_1}, \dots, \operatorname{sgn}\chi_{R_N})$, where χ_{R_i}, $i = 1, 2, \dots, N$, are the components of $\xi_{F,g}$ and where $\operatorname{sgn} y = 1$ if $y > 0$, $\operatorname{sgn} y = 0$ if $y = 0$, and $\operatorname{sgn} y = -1$ if $y < 0$. It is not difficult to verify that $|\eta| \le r$ for almost all $x \in \Omega$ and that $\xi_{F,g} \cdot \eta \ge (r/\sqrt{N})|\xi_{F,g}|$. Therefore by virtue of (39) one is led to the estimate

$$\frac{r}{\sqrt{N}}|\xi_{F,g}| \le \xi_{F,g} \cdot u_{F,g} + \widetilde{\alpha}(r)(1 + |u_{F,g}|^q).$$

Integrating this inequality over $\omega \subset \Omega$ yields

$$\begin{aligned} \int_\omega |\xi_{F,g}|\, dx \le{}& \frac{\sqrt{N}}{r}\int_\omega \xi_{F,g} \cdot u_{F,g}\, dx \\ &+ \frac{\sqrt{N}}{r}\widetilde{\alpha}(r)|\omega| + \frac{\sqrt{N}}{r}\widetilde{\alpha}(r)|\omega|^{(p-q)/p}||u_{F,g}||^q_{L^p(\Omega)}. \end{aligned} \tag{40}$$

Thus, from (19) one obtains

$$\begin{aligned} \int_\omega |\xi_{F,g}|\, dx &\le \frac{\sqrt{N}}{r}\int_\omega \xi_{F,g} \cdot u_{F,g}\, dx + \frac{\sqrt{N}}{r}\widetilde{\alpha}(r)|\omega| + \frac{\sqrt{N}}{r}\widetilde{\alpha}(r)|\omega|^{(p-q)/p}\gamma^q||u_{F,g}||^q_V \\ &\le \frac{\sqrt{N}}{r}\int_\omega \xi_{F,g} \cdot u_{F,g}\, dx + \frac{\sqrt{N}}{r}\widetilde{\alpha}(r)|\omega| + \frac{\sqrt{N}}{r}\widetilde{\alpha}(r)|\omega|^{(p-q)/p}\gamma^q b^q \end{aligned} \tag{41}$$

(where $||\cdot||_{L^p(\Omega;\mathbb{R}^N)} \le \gamma||\cdot||_V$). The next claim is that

$$\int_\omega \xi_{F,g} \cdot u_{F,g}\, dx \le C \tag{42}$$

for some positive constant C not depending on $\omega \subset \Omega$ and $F \in \Lambda$. Indeed, from (H_4) one can easily deduce that

$$\xi_{F,g} \cdot u_{F,g} + \widetilde{\alpha}(0)\big(|u_{F,g}|^q + 1\big) \geq 0 \ \text{ a.e. in } \ \Omega.$$

Thus it follows

$$\int_\omega \big(\xi_{F,g} \cdot u_{F,g} + \widetilde{\alpha}(0)\big(|u_{F,g}|^q + 1\big)\big)\, dx \leq \int_\Omega \big(\xi_{F,g} \cdot u_{F,g} + \widetilde{\alpha}(0)\big(|u_{F,g}|^q + 1\big)\big)\, dx,$$

and consequently

$$\int_\omega \xi_{F,g} \cdot u_{F,g}\, dx \leq \int_\Omega \xi_{F,g} \cdot u_{F,g}\, dx + \overline{k}_1\big(||u_{F,g}||_V^q + |\Omega|\big),$$

where $\overline{k}_1 > 0$ is a constant. But A maps bounded sets into bounded sets. Therefore, by means of (17) and (19),

$$\begin{aligned}\int_\Omega \xi_{F,g} \cdot u_{F,g}\ dx &= -\langle Au_{F,g} - g, u_{F,g}\rangle_V \\ &\leq ||Au_{F,g} - g||_{V^\star} ||u_{F,g}||_V \leq C_0, \quad C_0 = \text{const},\end{aligned}$$

and consequently, (42) easily follows. Further, from (41) and (42), for $r > 0$,

$$\int_\omega |\xi_{F,g}| dx \leq \frac{\sqrt{N}}{r} C + \frac{\sqrt{N}}{r} \widetilde{\alpha}(r)|\omega| + \frac{\sqrt{N}}{r} \widetilde{\alpha}(r)|\omega|^{(p-q)/p} \gamma^q b^q. \tag{43}$$

This estimate is crucial for (38) to be obtained. Namely, let $\varepsilon > 0$. Fix $r > 0$ with

$$\frac{\sqrt{N}}{r} C < \frac{\varepsilon}{2} \tag{44}$$

and determine $\delta > 0$ small enough to fulfill

$$\frac{\sqrt{N}}{r} \widetilde{\alpha}(r)|\omega| + \frac{\sqrt{N}}{r} \widetilde{\alpha}(r)|\omega|^{(p-q)/p} \gamma^q b^q \leq \frac{\varepsilon}{2},$$

provided that $|\omega| < \delta$. Thus from (43) it follows that for any $\omega \subset \Omega$,

$$\int_\omega |\xi_{F,g}|\, dx \leq \varepsilon, \quad \forall F \in \Lambda, \tag{45}$$

whenever $|\omega| < \delta$. Finally, $\{\xi_{F,g}\}_{F \in \Lambda}$ is equi-integrable and its precompactness in $L^1(\Omega; \mathbb{R}^N)$ has been proved (see [6]). □

4. Hemivariational inequality

Now we are ready to formulate the main result of the paper.

Theorem 6. *Assume that conditions (H_1)–(H_5) hold. Then there exists a constant $B > 0$ such that if $\|g\|_{V^*} \le B$, the problem:*

(P) *Find $u \in V$ and $\xi \in L^1(\Omega; \mathbb{R}^N)$ such that*

$$a(u, v-u) + \int_\Omega \xi(x) \cdot \big(v(x) - u(x)\big)\, dx = \langle g, v-u\rangle_V, \qquad \forall v \in V \cap L^\infty(\Omega; \mathbb{R}^N), \quad (46)$$

$$\xi(x) \in \partial_y j(x, u(x)) \quad \textit{for a.e. } x \in \Omega, \quad (47)$$

$$\xi \cdot u \in L^1(\Omega) \quad (48)$$

admits at least a solution. Moreover, $u \in V$ satisfies the hemivariational inequality:

$$a(u, v-u) + \int_\Omega j^0\big(x, u(x), v(x) - u(x)\big)\, dx \ge \langle g, v-u\rangle_V, \qquad \forall v \in V, \quad (49)$$

where the integral above takes $+\infty$ as its value whenever $j^0(x, u, v-u) \notin L^1(\Omega; \mathbb{R}^N)$.

Proof. The proof is divided into a sequence of steps.

Step 1. First we show that there exist $u \in V$ and $\xi \in L^1(\Omega; \mathbb{R}^N)$ such that

$$a(u, v) + \int_\Omega \xi(x) \cdot v(x) dx = \langle g, v\rangle_V, \ \forall v \in V \cap L^\infty(\Omega; \mathbb{R}^N). \quad (50)$$

For any $F \in \Lambda$ define

$$U_F = \big\{(u_{F,g}, \xi_{F,g}) \colon (u_{F,g}, \xi_{F,g}) \text{ satisfies } (P_{F,g})\big\}$$

and let

$$W_F = \bigcup_{\substack{F' \in \Lambda \\ F' \supset F}} U_{F'}.$$

By Theorem 4, W_F is not empty. We use the symbol weakcl (W_F) to denote the closure of W_F in the weak topology of $V \times L^1(\Omega; R^N)$. From Theorem 4 and Theorem 5 it follows that weakcl (W_F) is weakly compact in $V \times L^1(\Omega; \mathbb{R}^N)$. Moreover, this family has the finite intersection property because for any $F_1, \dots, F_k \in \Lambda$ it follows $W_{F_1} \cap \ldots \cap W_{F_k} \supset W_F$, with

$F = F_1 + \ldots + F_k$. Accordingly, by a classical argument we conclude that the intersection

$$\bigcap_{F\in\Lambda} \operatorname{weakcl}(W_F)$$

is not empty. Let (u,ξ) belong to this intersection, i.e.

$$(u,\xi) \in \bigcap_{F\in\Lambda} \operatorname{weakcl}(W_F).$$

To show that (50) holds let us fix $v \in V \cap L^\infty(\Omega;\mathbb{R}^N)$ arbitrarily and choose $F \in \Lambda$ with $v \in F$. There exists a sequence $(u_{F_n,g}, \xi_{F_n,g}) \in W_F$ (for simplicity of the notations denoted by (u_n, ξ_n)) such that

$$u_n \to u \quad \text{weakly in } V,$$
$$\xi_n \to \xi \quad \text{weakly in } L^1(\Omega;\mathbb{R}^N)$$

and, by (17),

$$a(u_n, w) + \int_\Omega \xi_n(x)\cdot w(x)\,dx = \langle g, w\rangle_V, \quad \forall w \in F_n.$$

Since for any n, $v \in F_n$,

$$a(u_n, v) + \int_\Omega \xi_n(x)\cdot v(x)\,dx = \langle g, v\rangle_V,$$

so by letting $n \to \infty$ we get (50), as required.

Step 2. Now we prove that $\xi \in \partial_y j(x,u)$ a.e in Ω, i.e. the condition (47) is fulfilled. Since V is compactly imbedded into $L^p(\Omega;\mathbb{R}^N)$ one may suppose that

$$u_n \to u \quad \text{strongly in } L^p(\Omega;\mathbb{R}^N).$$

This implies that for a subsequence of $\{u_n\}$ (again denoted by the same symbol) one gets $u_n \to u$ a.e. in Ω (by passing to a subsequence, if necessary). Egoroff's theorem asserts that for any $\varepsilon > 0$ a subset $\omega \subset \Omega$ with $|\omega| < \varepsilon$ can be determined such that $u_n \to u$ uniformly in $\Omega \setminus \omega$ with $u \in L^\infty(\Omega\setminus\omega;\mathbb{R}^N)$. Let $v \in L^\infty(\Omega\setminus\omega;\mathbb{R}^N)$ be an arbitrary function. From the inequality

$$\int_{\Omega\setminus\omega} \xi_n \cdot v\,dx \le \int_{\Omega\setminus\omega} j^0(x,u_n;v)\,dx$$

and the upper semicontinuity of

$$L^\infty(\Omega\setminus\omega;\mathbb{R}^N) \ni u_n \longmapsto \int_{\Omega\setminus\omega} j^0(x,u_n;v)\,dx$$

it follows

$$\int_{\Omega\setminus\omega} \xi\cdot v\,dx \le \int_{\Omega\setminus\omega} j^0(x,u;v)\,dx, \quad \forall\, v \in L^\infty(\Omega\setminus\omega;\mathbb{R}^N).$$

But the last inequality amounts to saying that $\xi \in \partial_y j(x,u)$ a.e. in $\Omega \setminus \omega$. Since $|\omega| < \varepsilon$ and ε was chosen arbitrarily,

$$\xi \in \partial_y j(x,u) \quad \text{a.e. in } \Omega,$$

as claimed.

Step 3. Now we show that $\xi \cdot u \in L^1(\Omega)$, i.e. (48) holds. For this purpose we shall need the following truncation result for vector-valued Sobolev spaces.

Theorem 7 ([20])**.** *For each $v \in H^1(\Omega;\mathbb{R}^N)$ there exists a sequence of functions $\{\varepsilon_n\} \subset L^\infty(\Omega)$ with $0 \le \varepsilon_n \le 1$ such that*

$$\begin{aligned} &\{(1-\varepsilon_n)v\} \subset H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N) \\ &(1-\varepsilon_n)v \to v \quad \textit{strongly in } H^1(\Omega;\mathbb{R}^N). \end{aligned} \tag{51}$$

In a similar way to the aforementioned theorem, for $u \in V$ one can find a sequence $\{\varepsilon_k\} \subset L^\infty(\Omega)$ with $0 \le \varepsilon_k \le 1$ such that $\tilde{u}_k := (1-\varepsilon_k)u \in V \cap L^\infty(\Omega;\mathbb{R}^N)$ and $\tilde{u}_k \to u$ in V as $k \to \infty$. Without loss of generality it can be assumed that $\tilde{u}_k \to u$ a.e. in Ω. Since it is already known that $\xi \in \partial_y j(x,u)$, one can apply (H_4) to obtain $\xi \cdot (-u) \le j^0(x,u;-u) \le \widetilde{\alpha}(0)(1+|u^q|)$. Hence

$$\xi \cdot \tilde{u}_k = (1-\varepsilon_k)\xi \cdot u \ge -\widetilde{\alpha}(0)(1+|u^q|). \tag{52}$$

This implies that the sequence $\{\xi \cdot \tilde{u}_k\}$ is bounded from below and $\xi \cdot \tilde{u}_k \to \xi \cdot u$ a.e. in Ω. On the other hand, due to (50) and (19) one gets

$$C \ge \langle g, \tilde{u}_k \rangle_V - a(u,\tilde{u}_k) = \int_\Omega \xi \cdot \tilde{u}_k \, dx$$

for a positive constant C. Now, letting $k \to \infty$, by Fatou's lemma we arrive at $\xi \cdot u \in L^1(\Omega)$, as required.

Step 4. Now the inequality

$$\liminf_{n\to\infty} \int_\Omega \xi_n \cdot u_n \, dx \ge \int_\Omega \xi \cdot u \, dx \tag{53}$$

will be established. It can be supposed that $u_n \to u$ a.e. in Ω, since $u_n \to u$ in $L^p(\Omega;\mathbb{R}^N)$. Fix $v \in L^\infty(\Omega;\mathbb{R}^N)$ arbitrarily. In view of $\xi_n \in \partial_y j(x,u_n)$,

$$\xi_n \cdot (v-u_n) \le j^0(x,u_n;v-u_n) \le \widetilde{\alpha}(||v||_{L^\infty(\Omega;\mathbb{R}^N)})(1+|u_n|^q). \tag{54}$$

From Egoroff's theorem it follows that for any $\varepsilon > 0$ a subset $\omega \subset \Omega$ with $|\omega| < \varepsilon$ can be determined such that $u_n \to u$ uniformly in $\Omega \setminus \omega$. One can also suppose that $|\omega|$ is small enough to fulfill $\int_\omega \widetilde{\alpha}(||v||_{L^\infty(\Omega;\mathbb{R}^N)})(1+|u_n|^q)\,dx \le \varepsilon$, $n \ge 1$, and $\int_\omega \widetilde{\alpha}(||v||_{L^\infty(\Omega;\mathbb{R}^N)})(1+|u|^q)\,dx \le \varepsilon$. Hence

$$\int_\Omega j^0(x,u_n;v-u_n)\,dx \le \int_{\Omega\setminus\omega} j^0(x,u_n;v-u_n)\,dx + \varepsilon$$

which by Fatou's lemma and the upper semicontinuity of $j^0(\cdot\,;\cdot)$ yields

$$\begin{aligned}\liminf_{n\to\infty}\int_\Omega -j^0(x,u_n;v-u_n)\,dx &\geq \int_{\Omega\setminus\omega} -j^0(x,u;v-u)\,dx-\varepsilon\\ &\geq \int_\Omega -j^0(x,u;v-u)\,dx-2\varepsilon,\end{aligned}$$

where it was admitted that $\int_\Omega j^0(x,u;v-u)\,dx=+\infty$ if $j^0(\cdot,u;v-u)\notin L^1(\Omega)$, and $\int_\omega j^0(x,u;v-u)\,dx>-\varepsilon$ if $j^0(\cdot,u;v-u)\in L^1(\Omega)$. By the arbitrariness of $\varepsilon>0$ and (54) one obtains

$$\liminf_{n\to\infty}\int_\Omega \xi_n\cdot u_n\,dx\geq\int_\Omega \xi\cdot v\,dx-\int_\Omega j^0(x,u;v-u)\,dx,\qquad \forall\, v\in V\cap L^\infty(\Omega;\mathbb{R}^N). \tag{55}$$

By substituting $v=\tilde{u}_k:=(1-\varepsilon_k)u$ (with $\tilde{u}_k$ as described in the truncation argument of Theorem 7) into the right hand side of (55) one gets

$$\begin{aligned}\liminf_{n\to\infty}\int_\Omega \xi_n\cdot u_n\,dx &\geq \liminf_{k\to\infty}\int_\Omega \xi\cdot\tilde{u}_k\,dx\\ &\quad-\limsup_{k\to\infty}\int_\Omega j^0(x,u;\tilde{u}_k-u)\,dx.\end{aligned} \tag{56}$$

Taking into account that $\tilde{u}_k\to u$ a.e. in Ω,

$$j^0(x,u;\tilde{u}_k-u)=\varepsilon_k j^0(x,u;-u)\leq\varepsilon_k\widetilde{\alpha}(0)(1+|u|^q)\leq\widetilde{\alpha}(0)(1+|u|^q)$$

and $|\xi\cdot u|\geq\xi\cdot\tilde{u}_k=(1-\varepsilon_k)\xi\cdot u\geq-\widetilde{\alpha}(0)(1+|u|^q)$, Fatou's lemma and the dominated convergence can be used to deduce

$$\limsup_{k\to\infty}\int_\Omega j^0(x,u;\tilde{u}_k-u)\,dx\leq 0,$$

and

$$\lim_{k\to\infty}\int_\Omega \xi\cdot\tilde{u}_k\,dx\geq\int_\Omega \xi\cdot u\,dx.$$

Finally, combining the last two inequalities with (56) yields (53), as required.

Step 5. The next claim is that

$$a(u,u)+\int_\Omega \xi(x)\cdot u(x)\,dx=\langle g,u\rangle_V. \tag{57}$$

Indeed, (50) implies

$$a(u,\tilde{u}_k)+\int_\Omega \xi(x)\cdot\tilde{u}_k(x)\,dx=\langle g,\tilde{u}_k\rangle_V, \tag{58}$$

with $\{\tilde{u}_k\}$ as in Step 3. Since $\xi\cdot u\in L^1(\Omega)$ and

$$-\widetilde{\alpha}(0)(1+|u|^q)\leq\xi\cdot\tilde{u}_k=(1-\varepsilon_k)\xi\cdot u\leq|\xi\cdot u|,$$

by the dominated convergence,

$$\int_\Omega \xi \cdot \tilde{u}_k \, dx \to \int_\Omega \xi \cdot u \, dx,$$

which means that (57) has to hold by passing to the limit as $k \to \infty$ in (58). Combining (50) and (57) yields (46). Accordingly, (u, ξ) is a solution of (P).

Step 6. In the final step of the proof it will be shown that (46) – (48) imply (49). For this purpose choose $v \in V \cap L^\infty(\Omega; \mathbb{R}^N)$ arbitrarily. From (47) one has $\xi \cdot (v - u) \le j^0(x, u; v - u) \le \widetilde{\alpha}(||v||_{L^\infty(\Omega;\mathbb{R}^N)})(1 + |u|^q)$ with $\xi \cdot (v - u) \in L^1(\Omega)$ and $\tilde{\alpha}(||v||_{L^\infty(\Omega;\mathbb{R}^N)})(1 + |u|^q) \in L^1(\Omega)$. Hence $j^0(x, u; v - u)$ is finite integrable and therefore (49) easily follows.

Now let us consider the case $j^0(x, u; v - u) \in L^1(\Omega)$ with $v \notin V \cap L^\infty(\Omega; \mathbb{R}^N)$. According to an analogous result to Theorem 7 there exists a sequence $\tilde{v}_k = (1 - \varepsilon_k)v$ such that $\{\tilde{v}_k\} \subset V \cap L^\infty(\Omega; \mathbb{R}^N)$ and $\tilde{v}_k \to v$ strongly in V. Since

$$a(u, \tilde{v}_k - u) + \int_\Omega \xi(x) \cdot \big(\tilde{v}_k(x) - u(x)\big) \, dx = \langle g, \tilde{v}_k - u \rangle_V,$$

so in order to establish (49) it remains to show that

$$\limsup_{k\to\infty} \int_\Omega j^0(x, u; \tilde{v}_k - u) \, dx \le \int_\Omega j^0(x, u; v - u) \, dx.$$

For this purpose let us observe that $\tilde{v}_k - u = (1 - \varepsilon_k)(v - u) + \varepsilon_k(-u)$ which combined with the convexity of $j^0(x, u; \cdot)$ yields the estimate

$$\begin{aligned} j^0(x, u; \tilde{v}_k - u) &\le (1 - \varepsilon_k) j^0(x, u; v - u) \ + \varepsilon_k j^0(x, u; -u) \\ &\le |j^0(x, u; v - u)| + \widetilde{\alpha}(0)(1 + |u|^q). \end{aligned}$$

Thus Fatou's lemma implies the assertion. The proof of Theorem 6 is complete. □

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