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ON THE TOPOLOGICAL ENTROPY OF GREEN INTERVAL MAPS

J. BOBOK

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Abstract. We investigate the topological entropy of a green interval map. Defining the complexity we estimate from above the topological entropy of a green interval map with a given complexity. The main result of the paper — stated in Theorem 0.2 — should be regarded as a completion of results of [4].

0. Introduction and main result

The purpose of this paper is to evaluate the topological entropy of green interval maps. The topological entropy provides a numerical measure for the complexity of such one-dimensional dynamical systems and our aim is to describe this complexity by means of combinatorics. A particular case of a system given by a P-monotone map f_P for a green cycle P was investigated in [4]. Since a green interval map is not a uniform limit of such special green (Markov) maps, our Theorem 0.2 completes the result of [4].

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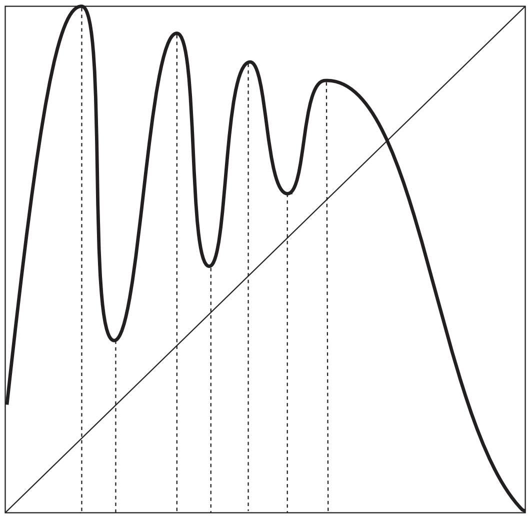

Figure 1: The green map f; $l = 3$

Green map ([2]). A continuous piecewise monotone map $f : [0,1] \to [0,1]$ is said to be *green* if $f(0) > 0$, $f(1) = 0$ and it has an odd number $2l+1$ of turning points $a_1 < a_2 < \cdots < a_{2l+1}$ in $(0,1)$ and a unique fixed point $b > a_{2l+1}$ such that (see Figure 1)

$$f(0) < f(a_2) < \cdots < f(a_{2l}) < b < f(a_{2l+1}) < \cdots < f(a_1) = 1. \quad (1)$$

Complexity. Let f be a green map satisfying (1). If $f^2(a_{2l+1}) < a_1$, the complexity $C(f)$ of f is equal to $2l+2$, otherwise $C(f)$ is defined as a minimal number $2k \in \{2, 4, \dots, 2l\}$ such that $f^2(a_{2i+1}) \geq a_{2i+1-2k}$ for each $i = k, k+1, \dots, l$.

Remark 0.1. The definition of complexity presented here slightly differs from the one given in [4].

By $\alpha(k)$ we denote the positive root of the polynomial equation

$$(\alpha+1)^k(1+\sqrt{1+k^2})^k + \alpha^2(\alpha-1)^k k^k(k-\sqrt{1+k^2}) = 0. \quad (2)$$

The following theorem generalizes a result from [4].

Theorem 0.2. *Let f be a green map such that $C(f) \leq 2k$. Then* $\mathrm{ent}(f) < \log\alpha(k)$.

Remark 0.3. (i) As it was shown in [4], the bound $\log \alpha(k)$ is the least possible even for special green Markov maps with the complexity less than or equal to $2k$, i.e. for any $\alpha < \alpha(k)$ there is a green Markov map of complexity less than or equal to $2k$ with the entropy from $(\alpha, \alpha(k))$. It is stated there that each $\alpha(k)$ is irrational greater than 1,

$$\lim_{k\to\infty} \frac{\alpha(k)}{k} = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{\alpha(k)}{\sqrt{k}} = \infty.$$

(ii) Using a method analogous to the one from [3] we can prove the inequality

$$\mathrm{ent}(f) \geq \frac{1}{2} \log(C(f) - 2).$$

1. Lemmas and proofs

We will use the symbolic dynamics [7]. As usually, for $m \in \mathbb{N}$ let us consider $N_m = \{1, \ldots, m\}$ as a finite space with the discrete topology, denote by Ω_m the infinite product space $\prod_{i=0}^{\infty} X_i$, where $X_i = N_m$ for all i. The shift map $\sigma : \Omega_m \to \Omega_m$ is defined by

$$(\sigma(\omega))_i = \omega_{i+1},\ i \in \mathbb{N} \cup \{0\}.$$

A subset Ω of Ω_m is σ-invariant if $\sigma(\Omega) \subset \Omega$, the pair (Ω, σ) is transitive if there is a point $\omega \in \Omega$ such that $\overline{\{\sigma^i(\omega) :\ i \in \mathbb{N}\}} = \Omega$. We write $\Omega(k) = \{(\omega_0, \ldots, \omega_{k-1}) :\ \omega \in \Omega\}$. The following proposition presents well-known results about topological entropy (for its definition see [6], [7]).

Lemma 1.1. [7] *Let* $\Omega, \Gamma \subset \Omega_m$ *be closed and* σ*-invariant,* (Γ, σ) *transitive,* $\Omega \subset \Gamma$ *and* $\Omega \neq \Gamma$*. Then* $\mathrm{ent}(\sigma|\Omega) < \mathrm{ent}(\sigma|\Gamma)$.

For $I = [0, 1]$ let $C(I)$ be a space of all continuous maps which map I into itself. By $\mathcal{Q}$ we denote the set

$$\{Q = \{q_k\}_{k=0}^{n} :\ 0 = q_0 < q_1 < \cdots < q_{n-1} < q_n = 1,\ n \in \mathbb{N}\}.$$

A map $g \in C(I)$ is piecewise monotone if there is a $Q \in \mathcal{Q}$ such that the map g is monotone (not necessarily strictly) on each interval $[q_{k-1}, q_k]$ given by Q. Such a Q with the minimal cardinality is denoted by T_g. In the sequel the following notation will be useful. For $g \in C(I)$ and $Q = \{q_k\}_{k=0}^{n} \in \mathcal{Q}$, $Q(g) = \{x \in I :\ \forall i \in \mathbb{N} \cup \{0\}\ g^i(x) \notin Q\}$ and

$$\Omega(Q(g)) = \{\omega \in \Omega_n :\ \exists x \in Q(g)\ \forall i \in \mathbb{N} \cup \{0\}\ g^i(x) \in (q_{\omega_i - 1}, q_{\omega_i})\}.$$

The following lemma shows the basic facts about relation of an entropy of a map $g \in C(I)$ and the symbolic dynamics given by g and $Q \in \mathcal{Q}$. The references needed for the proofs are given.

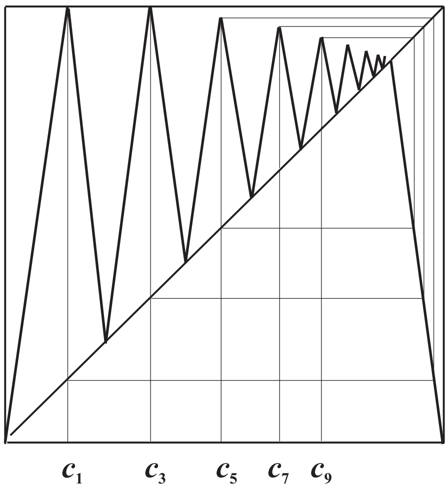

Figure 2: The map F_2

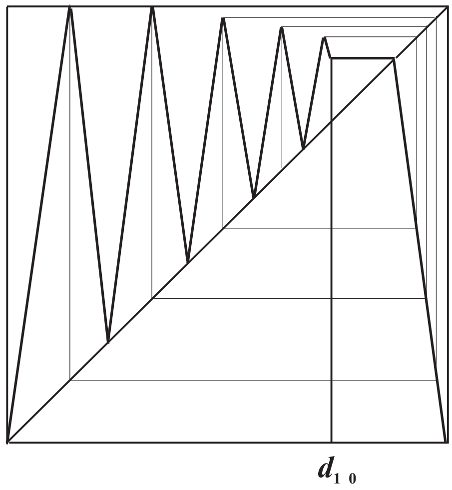

Figure 3: The map $G_{2,10}$

Lemma 1.2.

(i) [1], [5] *For any* $g \in C(I)$ *and a* g*-invariant* $Q \in \mathcal{Q}$,

$$\operatorname{ent}(g) \geq \lim_k \frac{1}{k} \log \operatorname{card} \Omega(Q(g))(k).$$

(ii) [8], [1] *Let* $g \in C(I)$ *be piecewise monotone. Then*

$$\operatorname{ent}(g) = \lim_k \frac{1}{k} \log \operatorname{card} \Omega(T_g(g))(k).$$

In order to prove our Theorem we use a special map constructed in [4] (see Figure 2).

Lemma 1.3. *Let* $k \in \mathbb{N}$ *be fixed. The value* $\alpha = \alpha(k)$ *given by* (2) *is the least one for which there exists a transitive* α*-Lipschitz map* $F_k : [0,1] \to [0,1]$ *with the following properties* $(d = \alpha/(\alpha+1))$. *For the sequence* $\{c_n\}_{n=0}^{\infty}$ *such that* $0 = c_0 < c_1 < c_2 < c_3 < \cdots < d$, $\lim c_n = d$,

(i) F_k *has a constant slope* α *on each interval* $[c_{2n-2}, c_{2n-1}]$ *and a slope* $(-\alpha)$ *on each* $[c_{2n-1}, c_{2n}]$, $[d, 1]$,
(ii) $F_k(c_{2n-2}) = c_{2n-2}$ *for* $n \geq 1$, $F_k^2(c_{2n-1}) = c_{2(n-k)-1}$ *for* $n \geq k+1$,
(iii) $F_k(c_{2n-1}) = 1$ *for* $n \in \{1, \dots, k\}$,
(iv) $\operatorname{ent}(F_k) = \log \alpha$.

Let us denote $F_k^{-1}(\{d\}) = \{d_1 < d_2 < \cdots < d_n < \cdots\}$; for each $n \in \mathbb{N}$, $d_n \in (c_{n-1}, c_n)$. The continuous map $G_{k,m} : [0,1] \to [0,1]$ is defined by $G_{k,m}(x) = F_k(x)$, $x \in [0, d_m] \cup [d, 1]$ and $G_{k,m}(x) = d$, $x \in [d_m, d]$ (see Figure 3).

Let f be the green map satisfying (1), for the sake of simplicity denote $G = G_{k,2l+2}$ and $F = F_k$. We have $T_f = \{0,1\} \cup \{a_k\}_{k=1}^{2l+1}$, $T_G = \{0,1\} \cup \{c_k\}_{k=1}^{2l+1}$.

Put also $Q = \{0,1\} \cup \{c_k\}_{k=1}^{2l+3}$. The following lemma describes the properties of the subsets (of Ω_{2l+4})

$$\Omega(f) = \overline{\Omega}(T_f(f)),\ \Omega(G) = \overline{\Omega}(T_G(G)),\ \Omega(F) = \overline{\Omega}(Q(F)).$$

Note that all those subsets are closed and σ-invariant. By $\omega' + \Omega$ we denote the set $\{(\omega'_1 + \omega_1, \omega'_2 + \omega_2, \dots) : \ \omega \in \Omega\}$.

Lemma 1.4.

(i) $\Omega(f) \subset \Omega(G)$.
(ii) $(1,1,\dots) + \Omega(G) \subset \Omega(F)$, $(\Omega(F), \sigma)$ *is transitive and* $(1,1,\dots) + \Omega(G) \neq \Omega(F)$.

Proof. By our assumption on the complexity of f and the definition of G it holds $G((c_{k-1}, c_k)) \supset (c_{j-1}, c_j)$, whenever $f((a_{k-1}, a_k)) \cap (a_{j-1}, a_j) \neq \emptyset$ and $k, j \in \{1, \dots, 2l+1\}$. This gives (i). Similarly it can be verified that $(1,1,\dots) + \Omega(G) \subset \Omega(F)$; the transitivity of $(\Omega(F), \sigma)$ follows from that of F. The relation $(1,1,\dots) + \Omega(G) \neq \Omega(F)$ is clear. $\square$

Now we are ready to give the proof of our main theorem.

Proof of Theorem 0.2. As above, $G = G_{k,2l+2}$ and $F = F_k$. Put $P = Q \cup F(Q)$. Clearly, by Lemma 1.2(i) the entropy $\operatorname{ent}(F)$ is greater than or equal to $\lim_k (1/k) \log \operatorname{card} \Omega(P(F))(k)$. Since $P \supset Q$ we have also

$$\lim_k \frac{1}{k} \log \operatorname{card} \Omega(P(F))(k) \geq \lim_k \frac{1}{k} \log \operatorname{card} \Omega(Q(F))(k).$$

Now, it follows from Lemmas 1.4(ii), 1.1, 1.2(ii) and 1.4(i) that

$$\lim_k \frac{1}{k} \log \operatorname{card} \Omega(Q(F))(k) > \operatorname{ent}(G) \geq \operatorname{ent}(f).$$

Summarizing, $\operatorname{ent}(F) = \log \alpha(k) > \operatorname{ent}(f)$. This proves the theorem. $\square$

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Jozef Bobok
KM FSv ČVUT
Thákurova 7
166 29 Praha 6
Czech Republic
e-mail: erastus@mbox.cesnet.cz