Journal of Applied Analysis
Vol. 7, No. 2 (2001), pp. 113–130

ON THE CONTINUOUS DEPENDENCE ON PARAMETERS OF SOLUTIONS OF THE FOURTH ORDER PERIODIC PROBLEM

M. BARTKIEWICZ

Received September 14, 1999 and, in revised form, March 23, 2000

Abstract. We consider the fourth order periodic problem with a functional parameter. Some sufficient conditions under which solutions of this problem continuously depend on parameters are given. Proofs of theorems are based on variational methods.

Introduction

This paper is devoted to the continuous dependence on functional parameters of solutions of the fourth order periodic problem. Sufficient conditions for the existence of solutions of this problem and their continuous dependence on parameters are presented.

The question of the existence and uniqueness of solutions for the periodic problem was considered in many monographs and papers [3], [6], [2].

The problem of the continuous dependence on parameters for scalar equations of second order, was investigated in papers [4], [5]. In the case of the

1991 *Mathematics Subject Classification.* 34C25, 34B15.
Key words and phrases. Periodic problem, variational methods, continuous dependence on parameters.

ISSN 1425-6908

functional parameter from L^∞, sufficient conditions for the existence of solutions of the second order differential equations with Dirichlet-type boundary conditions and their continuous dependence on parameters, are given in paper [7]. The continuous dependence on functional parameters of solutions of the fourth order periodic problem has not been investigated so far.

In our paper we consider a problem of the form

$$\begin{gathered}\frac{d}{dt}\left(\frac{d}{dt}f_{\ddot u}\left(t,u,\dot u,\ddot u,\omega\right)-f_{\dot u}\left(t,u,\dot u,\ddot u,\omega\right)\right)+f_u\left(t,u,\dot u,\ddot u,\omega\right)=0\\ u(0)-u(T)=\dot u(0)-\dot u(T)=0 \qquad (0.1)\\ v(0)-v(T)=\dot v(0)-\dot v(T)=0\end{gathered}$$

where $v\left(\cdot\right)=f_{\ddot u}\left(\cdot,u\left(\cdot\right),\dot u\left(\cdot\right),\ddot u\left(\cdot\right),\omega\left(\cdot\right)\right)$, ω is a functional parameter from L^∞ and we look for $u\in H_T^2$. Under some suitable assumptions, we prove that the set $\tilde V_k$ of weak solutions of (0.1) is not empty, for any $\omega_k\in L^\infty$ and $\tilde V_{\omega_k}$ tends to $\tilde V_{\omega_0}$ in the sense of Painlevé–Kuratowski, as ω_k tends to ω_0 in the strong topology of L^∞. In many situations it is more natural to consider the normal form of (0.1):

$$\begin{gathered}u^{(4)}=\frac{d}{dt}F_{\dot u}\left(t,u,\dot u,\omega\right)-F_u\left(t,u,\dot u,\omega\right) \qquad (0.2)\\ u(0)-u(T)=\dot u(0)-\dot u(T)=\ddot u(0)-\ddot u(T)=u^{(3)}(0)-u^{(3)}(T)=0.\end{gathered}$$

We give sufficient conditions under which (0.2) depends continuously on the parameter ω. We are interested in cases when ∇F is bounded and F is convex.

1. Formulation of the fourth order problem

By H_T^2 we shall denote the space of functions $u:[0,T]\to\mathbb{R}^n$ such that u, $\dot u$ are absolutely continuous and $\ddot u\in L^2\left([0,T],\mathbb{R}^n\right)$, and $u(0)-u(T)=\dot u(0)-\dot u(T)=0$ where $T>0$.

In the space H_T^2 the norm is given by formula

$$||u||=\left(\int_0^T|u(t)|^2dt+\int_0^T|\dot u(t)|^2dt+\int_0^T|\ddot u(t)|^2dt\right)^{1/2}.$$

It is easy to check that $H_T^2\subset H_T^1$ where H_T^1 is the space of functions $u:[0,T]\to\mathbb{R}^n$ such that u is absolutely continuous and $\dot u\in L^2\left([0,T],\mathbb{R}^n\right)$, and $u(0)-u(T)=0$ where $T>0$. Moreover $\dot u\in H_T^1$ and $\int_0^T\dot u(t)dt=0$, so we can apply the Wirtinger inequality for $\dot{u}$

$$\int_0^T |\dot{u}(t)|^2 dt \leq \frac{T^2}{4\pi^2} \int_0^T |\ddot{u}(t)|^2 dt. \tag{1.1}$$

It is easy to calculate that

Lemma 1.1. *In the space H_T^2 the following norms are equivalent:*

1. $||u||_1 = \left(\int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt + \int_0^T |\ddot{u}(t)|^2 dt \right)^{1/2}$,
2. $||u||_2 = \left(\int_0^T |u(t)|^2 dt + \int_0^T |\ddot{u}(t)|^2 dt \right)^{1/2}$,
3. $||u||_3 = \left(\int_0^T |u(t)|^2 dt \right)^{1/2} + \left(\int_0^T |\ddot{u}(t)|^2 dt \right)^{1/2}$,
4. $||u||_4 = |u_0| + \left(\int_0^T |\ddot{u}(t)|^2 dt \right)^{1/2}$, $u_0 = u(0)$,
5. $||u||_5 = |\overline{u}| + \left(\int_0^T |\ddot{u}(t)|^2 dt \right)^{1/2}$, $\overline{u} = \frac{1}{T} \int_0^T u(s) ds$.

Lemma 1.2. *If the sequence u_k converges weakly to u_0 in H_T^2, then $\ddot{u}_k$ converges weakly to $\ddot{u}_0$ in $L^2([0,T],\mathbb{R}^n)$ and $\dot{u}_k \to \dot{u}_0$, $u_k \to u_0$ uniformly on $[0,T]$.*

Proof. Let $\{u_k\} \subset H_T^2$ tends weakly to u_0 in H_T^2. Of course $\{\dot{u}_k\} \subset H_T^1$, so $\ddot{u}$ tends weakly to $\ddot{u}_0$ in $L^2([0,T],\mathbb{R}^n)$ and $\dot{u}_k$ tends uniformly to $\dot{u}_0$ on the interval $[0,T]$. Moreover $\{u_k\} \subset H_T^1$, so u_k tends uniformly to u_0 on the interval $[0,T]$. □

Let M be a convex and bounded subset of $\mathbb{R}^r$. Let us put

$$W = \{\omega \in L^\infty([0,T],\mathbb{R}^n) : \omega(t) \in M\}.$$

The set W will be referred to as a set of parameters.

Let $f = f(t,p_0,p_1,p_2,w)$ be any real function defined on the set $[0,T] \times (\mathbb{R}^n)^3 \times M$, satisfying the following assumptions:

(1-a) the functions $f, f_{p_0}, f_{p_1}, f_{p_2}, f_w$ are measurable with respect to $t \in [0,T]$ for any $(p_0,p_1,p_2,w) \in (\mathbb{R}^n)^3 \times M$ and continuous with respect to $(p_0,p_1,p_2,w) \in (\mathbb{R}^n)^3 \times M$ for a.e. $t \in [0,T]$, (1-b) $f(t, p_0, p_1, \cdot, w)$ is convex for a.e. $t \in [0, T]$ and any $(p_0, p_1, w) \in (\mathbb{R}^n)^2 \times M$,

(1-c) there exist some functions $a(\cdot) \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b(\cdot) \in L^1([0,T],\mathbb{R})$ and $c(\cdot) \in L^2([0,T],\mathbb{R})$ such that
$|f(t, p_0, p_1, p_2, w)| \le a(|(p_0, p_1)|) \left(b(t) + |p_2|^2\right)$,
$|f_{p_i}(t, p_0, p_1, p_2, w)| \le a(|(p_0, p_1)|) \left(b(t) + |p_2|^2\right)$, $i = 0, 1$,
$|f_{p_2}(t, p_0, p_1, p_2, w)| \le a(|(p_0, p_1)|) \left(c(t) + |p_2|\right)$,
$|f_w(t, p_0, p_1, p_2, w)| \le a(|(p_0, p_1)|) \left(b(t) + |p_2|^2\right)$,
for all $(p_0, p_1, p_2, w) \in (\mathbb{R}^n)^3 \times M$, and for a.e. $t \in [0, T]$.

(The abbreviation "a.e." means "almost every" in the sense of Lebesgue measure.)

Now let us consider a functional

$$\varphi_\omega(u) = \int_0^T f\left(t, u(t), \dot{u}(t), \ddot{u}(t), \omega(t)\right) dt. \tag{1.2}$$

Using the same method as for Theorem 1.4 in [3], we can prove

Theorem 1.3. *If a function f satisfies assumptions* (1-a)–(1-c)*, then the functional given by* (1.2) *is continuously differentiable on H_T^2 for all $\omega \in W$ and*

$$\left\langle \varphi_\omega'(u), h \right\rangle = \int_0^T \sum_{i=0}^{i=2} \left(f_{p_i}\left(t, u(t), \dot{u}(t), \ddot{u}(t), \omega(t)\right), h^{(i)}(t) \right) dt$$

for all u, $h \in H_T^2$.

Let us consider a boundary value problem with a parameter $\omega \in W$ of the form:

$$\begin{aligned} \frac{d}{dt}\left(\frac{d}{dt} f_{\ddot{u}}\left(t, u, \dot{u}, \ddot{u}, \omega\right) - f_{\dot{u}}\left(t, u, \dot{u}, \ddot{u}, \omega\right)\right) + f_u\left(t, u, \dot{u}, \ddot{u}, \omega\right) = 0 \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \\ v(0) - v(T) = \dot{v}(0) - \dot{v}(T) = 0 \qquad t \in [0, T] \text{ a.e.} \end{aligned} \tag{1.3}$$

where $v(\cdot) = f_{\ddot{u}}\left(\cdot, u(\cdot), \dot{u}(\cdot), \ddot{u}(\cdot), \omega(\cdot)\right)$ and $u \in H_T^2$.

For this problem, the corresponding functional is given by (1.2). It is easy to see that, under assumptions (1-a)–(1-c), the system defined by (1.3) is a system of Euler equations for functional (1.2).

Definition 1.1. We say that the functional $\varphi_\omega(\cdot)$ defined by (1.2) is uniformly coercive with respect to ω when

$$\forall K > 0 \quad \exists R \quad \forall |x| > R \quad \forall \omega \in W \quad \varphi_\omega(x) > K.$$

2. Principal lemma

Let $\varphi_k(\cdot) = \varphi_{\omega_k}(\cdot)$ $k = 0, 1, 2, \dots$ be a sequence of functionals defined by (1.2) with $\omega = \omega_k$, i.e.

$$\varphi_k(u) = \int_0^T f\left(t, u(t), \dot{u}(t), \ddot{u}(t), \omega_k(t)\right) dt$$

where $\{\omega_k\}$ is a sequence of admissible parameters. Denote by V_k the set of all minimizers of the functional φ_k, i.e.

$$V_k = \left\{u \in H_T^2 : \varphi_k(u) = \min\left\{\varphi_k(x) : x \in H_T^2\right\}\right\}. \tag{2.1}$$

Definition 2.1. We say that a sequence of sets V_k defined by (2.1) tends to V_0 in the weak topology of H_T^2 if any sequence $\{x_k\}$, $x_k \in V_k$, $k = 1, 2, \dots$ possesses cluster points (in the sense of the weak topology of H_T^2) in the set V_0 only.

The set of all cluster points of sequence $\{x_k\}$ is often referred to as the upper limit (in the sense of Painlevé–Kuratowski) of sets V_k and denoted by $\limsup V_k$.

In the case when sets V_k are singletons, i.e. $V_k = \{x_k\}$ $k = 0, 1, 2, \dots,$ then the convergence of the sets is identical with the convergence of points in the weak topology of H_T^2 (see Lemma 1.2).

Now we prove a lemma

Lemma 2.1. *If the sequence $\{\omega_k\} \subset W$ $k = 1, 2, \dots$ tends to $\omega_0 \in W$ in the strong topology of L^∞ then the sequence $\varphi_k(\cdot)$ tends to $\varphi_0(\cdot)$ uniformly on the ball $B(0, R) \subset H_T^2$ for any fixed $R > 0$.*

Proof. By the *mean-value* theorem and assumption (1-c) we obtain

$$\begin{aligned} |\varphi_k(u) - \varphi_0(u)| &\le \int_0^T \left|f_w\left(t, u(t), \dot{u}(t), \ddot{u}(t), \tilde{\omega}_k(t)\right)\right| |\omega_k(t) - \omega_0(t)|\, dt \\ &\le \int_0^T a\left(\left|\left(u(t), \dot{u}(t)\right)\right|\right)\left(b(t) + |\ddot{u}(t)|^2\right) |\omega_k(t) - \omega_0(t)|\, dt \end{aligned}$$

where $\tilde{\omega}_k(t) = \omega_0(t) + \Theta(t)(\omega_k(t) - \omega_0(t))$ and $0 \le \Theta(t) \le 1$. Since $||u|| \le R$, there exists a constant $C > 0$ such that $a\left(\left|\left(u(t), \dot{u}(t)\right)\right|\right) \le C$ and $\int_0^T |\ddot{u}(t)|^2 dt \le ||u||^2 \le R^2$ for any $u \in B(0,R)$.

Let us put $\varepsilon > 0$. Since $\omega_k \to \omega_0$ in L^∞, there exists some K, such that for any $k > K$ and $t \in [0,T]$ we have

$$|\omega_k(t) - \omega_0(t)| < \varepsilon.$$

Therefore we obtain

$$|\varphi_k(u) - \varphi_0(u)| \le C\varepsilon \int_0^T b(t)\,dt + C\varepsilon \int_0^T |\ddot{u}(t)|^2 dt \le C_1\varepsilon + C\varepsilon R^2 = C_2\varepsilon$$

for positive constants C_1, C_2. This ends the proof. □

We shall prove

Lemma 2.2. *If*

1. *the function f satisfies assumptions* (1-a)–(1-c)*,*
2. *$\varphi_k(\cdot)$ are weakly lower semicontinuous and uniformly coercive with respect to ω_k,*

then

a) *for any admissable parameter ω_k the set V_k of minimizers of functional $\varphi_k(\cdot)$ is not empty,*

b) *there exists a ball $B(0,R) \subset H_T^2$ such that $V_k \subset B(0,R)$ for $k = 0, 1, 2, \ldots$.*

Proof. Since by assumption 2, there exists at least one minimizer u_k of $\varphi_k(\cdot)$, thus V_k, $k = 0, 1, 2, \ldots$ is a nonempty set. Hence $\varphi_k(u_k) \le \varphi_k(0)$ for $\omega_k \in W$.

Let us put $P = \sup_{\omega_k \in W} \varphi_k(0) < \infty$. So there exists $R > 0$ such that for all $\omega_k \in W$ we have

$$u_k \in V_k \subset A_k = \left\{u \in H_T^2 : \varphi_k(u) \le P\right\} \subset B(0,R). \tag{2.2}$$

Indeed suppose that the second inclusion in (2.2) does not hold. Then for all $R > 0$, for instance $R = k$, $k = 1, 2, \ldots$, there exists a parameter $\omega_k \in W$ such that $A_k \subsetneq B(0,R)$. Thus there exists a sequence $\{u_k\}$, where $u_k \in A_k$, such that $||u_k|| > R = k$, $k = 1, 2, \ldots$. Because $\varphi_k(\cdot)$ is uniformly coercive, so for $||u_k|| \to \infty$, $k \to \infty$ we have $\varphi_k(u_k) \to \infty$. Hence $u_k \notin A_k$, for $k > P$ and we have got a contradiction. It means that $A_k \subset B(0,R)$ for some $R > 0$ and $k = 0, 1, 2, \ldots$. □

3. Continuous dependence on parameters theorem for the fourth order equation

Let $\{\omega_k\} \subset W$ be an arbitrary sequence, and by $\tilde{V}_k \subset H_T^2$ denote the set of solutions of the periodic problem

$$\begin{gathered}\frac{d}{dt}\left(\frac{d}{dt}f_{\ddot{u}}\left(t,u(t),\dot{u}(t),\ddot{u}(t),\omega_k(t)\right)-f_{\dot{u}}\left(t,u(t),\dot{u}(t),\ddot{u}(t),\omega_k(t)\right)\right)\\ +f_u\left(t,u(t),\dot{u}(t),\ddot{u}(t),\omega_k(t)\right)=0\\ u(0)-u(T)=\dot{u}(0)-\dot{u}(T)=0\\ v(0)-v(T)=\dot{v}(0)-\dot{v}(T)=0\\ \text{for a.e. } t\in[0,T] \text{ and } k=0,1,2,\dots,\end{gathered} \tag{3.1}$$

and denote by

$$V_k=\left\{u\in H_T^2:\varphi_{\omega_k}(u)=\min\left\{\varphi_{\omega_k}(x):x\in H_T^2\right\}\right\}$$

the set of minimizers of functional $\varphi_{\omega_k}(\cdot)$ for $k=0,1,2,\dots$.

Theorem 3.1. *If*

1. f *satisfies assumptions* (1-a)–(1-c),
2. *for any admissable parameter* ω_k*, the set* V_k *of minimizers of functional* $\varphi_{\omega_k}(\cdot)$*, is not empty,*
3. *there exist a ball* $B(0,R)\subset H_T^2$ *such that* $V_k\subset B(0,R)$ *for* $k=0,1,2,\dots,$
4. $\varphi_{\omega_k}(\cdot)$ *is convex for any* $\omega_k\in W$*, for* $k=0,1,2,\dots,$
5. *the sequence* $\{\omega_k\}\subset W$ *tends to* $\omega_0\in W$ *in the strong topology of* L^∞*,*

then $\limsup \tilde{V}_k$ *is a nonempty set and* $\limsup \tilde{V}_k\subset \tilde{V}_0$.

Proof. Let $\{u_k\}\subset H_T^2$ be a sequence such that $u_k\in V_k$ for $k=1,2,\dots$. Because $V_k\subset B(0,R)$, $k=1,2,\dots$ with some $R>0$, we may assume that u_k tends weakly to u_0 in H_T^2. Denote

$$m_k=\varphi_k(u_k)=\inf\left\{\varphi_k(x):x\in H_T^2\right\}=\inf\left\{\varphi_k(x):x\in B(0,R)\right\}.$$

Since by assumption 5 and Lemma 2.1, $\varphi_k(\cdot)$ tends to $\varphi_0(\cdot)$ uniformly on the ball $B(0,R)$, we have

$$m_k\to m_0. \tag{3.2}$$

Suppose that u_0 does not belong to V_0. The set is not empty, therefore there exists $x\in V_0$ such that $u_0\neq x$. We have

$$\begin{aligned}m_k-m_0=\varphi_k(u_k)-\varphi_0(x)&=[\varphi_k(u_k)-\varphi_0(u_k)]\\ &+[\varphi_0(u_k)-\varphi_0(u_0)]+[\varphi_0(u_0)-\varphi_0(x)].\end{aligned} \tag{3.3}$$

It is easy to check that $\varphi_0(u_0) - \varphi_0(x) > 0$. So passing with k to ∞ in (3.3) we get a contradiction with (3.2). Hence $\limsup V_k \subset V_0$.

Moreover the functionals $\varphi_{\omega_k}(\cdot)$ are convex and differentiable in the sense of Gateaux, therefore $\tilde{V}_k = V_k\,, \quad k = 0, 1, 2, \ldots$. This ends the proof. □

Corollary 3.2. *If the assumptions of Theorem* 3.1 *are satisfied and the functionals $\varphi_{\omega_k}(\cdot)$ are strictly convex then problem* (3.1) *possesses a unique solution, i.e. the set $\tilde{V}_k = \{u_k\}$, is singleton for $k = 0, 1, 2, \ldots$ and u_k tends to u_0 in the weak topology of H^2_T.*

Theorem 3.3. *If*

1. *f satisfies assumptions* (1-a)–(1-c),
2. *there exist some constants $a_2, a_0 > 0$, $a_1 \geq 0$ and $a_2 > a_1 T^2/4\pi^2$, $b_i \geq 0$ $(i = 0, 1, 2)$ and a function $c_0 \in L^1([0,T],\mathbb{R})$ such that*
$$f(t, p_0, p_1, p_2, w) \geq \sum_{i=0}^{2} (-1)^i \left(a_i \left|p_i\right|^2 - b_i \left|p_i\right| \right) - c_0,$$
3. *$\varphi_{\omega_k}(\cdot)$ is convex for any $\omega_k \in W$,*
4. *the sequence $\{\omega_k\} \subset W$ tends to $\omega_0 \in W$ in the strong topology of L^∞,*

then $\limsup \tilde{V}_k$ *is a nonempty set and* $\limsup \tilde{V}_k \subset \tilde{V}_0$.

Proof. To prove this theorem we must show that $\varphi_{\omega_k}(\cdot)$ is uniformly coercive with respect to $\omega_k \in W$, $k = 0, 1, 2, \ldots$ and weakly lower semicontinuous. By assumption 2 we have

$$f\left(t, u(t), \dot{u}(t), \ddot{u}(t), \omega_k(t)\right) \geq a_2 \left|\ddot{u}(t)\right|^2 - b_2 \left|\ddot{u}(t)\right| - a_1 \left|\dot{u}(t)\right|^2 - b_1 \left|\dot{u}(t)\right| + a_0 \left|u(t)\right|^2 - b_0 \left|u(t)\right| - c_0 \tag{3.4}$$

for any $\omega_k \in W$. Using the Wirtinger inequality (1.1), we get

$$\varphi_{\omega_k}(u) \geq a_{\min} ||u||^2 - \sum_{i=0}^{2} b_i \sqrt{T} ||u|| - \overline{c},$$

where $||u|| = \left(\int_0^T |u(t)|^2 dt + \int_0^T |\ddot{u}(t)|^2 dt \right)^{1/2}$, $a_{\min} = \min\{a_2 - a_1 T^2/4\pi^2, a_0\}$, $\overline{c} = \int_0^T c_0(t) dt$. Because $a_2 - a_1 T^2/4\pi^2 > 0$ and $a_0 > 0$, the functional $\varphi_{\omega_k}(\cdot)$ is uniformly coercive with respect to ω_k.

Our next step is to prove that the functional $\varphi_{\omega_k}(\cdot)$ is weakly lower semicontinuous for any $u_0 \in H^2_T$. Denote

$$Z = \left\{ (t, p_0, p_1, p_2) \in [0,T] \times (\mathbb{R}^n)^3, |p_1| \leq \rho_0 \right\}$$

where $\rho_0 = \max_{t \in [0,T]} \left|\dot{u}_0(t)\right|$.

By assumption 2 we have

$$f(t, x, p, w) \geq -\psi(t)$$

for some positive and integrable function ψ and for any $(t, p_0, p_1, p_2) \in Z$. So the fact that $\varphi_{\omega_k}(\cdot)$ is weakly lower semicontinuous in $u_0 \in H_T^2$ is obtained from Theorem 10.8.i in [1].

By Lemma 2.2 the set V_k of minimizers of functional $\varphi_{\omega_k}(\cdot)$ for $k = 0, 1, 2, \dots$ is not empty and there exist a ball $B(0, R) \subset H_T^2$ such that $V_k \subset B(0, R)$ for $k = 0, 1, 2, \dots$. Now we can apply Theorem 3.1 and we obtain the assertion. □

4. The normal form of the fourth order equation

Let $W = \{w \in L^\infty([0, T], \mathbb{R}^r) : w(t) \in M\}$ where M is any convex and bounded subset of $\mathbb{R}^r$ and let $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times M \to \mathbb{R}$ satisfy the following assumptions:

(2-a) $F(t, p_0, p_1, w)$ is measurable with respect to $t \in [0, T]$ for any $(p_0, p_1, w) \in \mathbb{R}^n \times \mathbb{R}^n \times M$ and continuously differentiable in (p_0, p_1) for a.e. $t \in [0, T]$,

(2-b) there exist functions $a(\cdot) \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b(\cdot) \in L^1([0, T], \mathbb{R})$ such that
$|F(t, p_0, p_1, w)| \leq a(|(p_0, p_1)|)b(t)$,
$|F_{p_i}(t, p_0, p_1, w)| \leq a(|(p_0, p_1)|)b(t)$, $i = 0, 1$,
for all $(p_0, p_1, w) \in \mathbb{R}^n \times \mathbb{R}^n \times M$, and a.e. $t \in [0, T]$.

Let us consider the functional Φ_ω given by

$$\Phi_\omega(u) = \int_0^T \left(\frac{1}{2} |\ddot{u}(t)|^2 + F\left(t, u(t), \dot{u}(t), \omega(t)\right) \right) dt. \tag{4.1}$$

The functional (4.1) is continuously differentiable on H_T^2 and

$$\begin{aligned} \langle \Phi'_\omega(u), h \rangle = & \int_0^T \left(F_u\left(t, u(t), \dot{u}(t), \omega(t)\right), h(t) \right) dt \\ & + \int_0^T \left(F_{\dot{u}}\left(t, u(t), \dot{u}(t), \omega(t)\right), \dot{h}(t) \right) dt + \int_0^T \left(\ddot{u}(t), \ddot{h}(t) \right) dt. \end{aligned}$$

We will use some generalization of the Du-Bois-Reymond lemma (see [2]).

Lemma 4.1. *If* $v \in L^2([0,T],\mathbb{R}^n)$, $w \in L^1([0,T],\mathbb{R}^n)$ *and*

$$\int_0^T \left(v(t), \ddot{h}(t)\right) dt = \int_0^T (w(t), h(t))\, dt$$

for any $h \in H_T^2$, *then there exist constants* c_0, $c_1 \in \mathbb{R}$ *such that*

$$v(t) = \int_0^t \int_0^{t_1} w(s) ds dt_1 + c_1 t + c_0$$

for a.e. $t \in [0,T]$ *and* $v(0) = v(T)$, $\dot{v}(0) = \dot{v}(T)$.

Theorem 4.2. *Let* F *satisfy assumptions* (2-a) *and* (2-b). *If* $u \in H_T^2$ *is a solution of the corresponding Euler equation for functional* (4.1), *then* $\ddot{u}$ *has weak derivatives* $u^{(3)}$ *and* $u^{(4)}$, *and*

$$u^{(4)}(t) = \frac{d}{dt} F_{\dot{u}}\left(t, u(t), \dot{u}(t), \omega(t)\right) - F_u\left(t, u(t), \dot{u}(t), \omega(t)\right) \tag{4.2}$$

$$u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = \ddot{u}(0) - \ddot{u}(T) = u^{(3)}(0) - u^{(3)}(T) = 0$$

for a.e. $t \in [0,T]$.

Proof. By assumptions we have

$$\langle \Phi'_\omega(u), h\rangle = 0$$

for all $h \in H_T^2$. So

$$\int_0^T \left(\ddot{u}(t), \ddot{h}(t)\right) dt = -\int_0^T \left(F_{\dot{u}}\left(t, u(t), \dot{u}(t), \omega(t)\right), \dot{h}(t)\right) dt - \int_0^T \left(F_u\left(t, u(t), \dot{u}(t), \omega(t)\right), h(t)\right) dt. \tag{4.3}$$

Integrating by parts the second integral in (4.3) and using the boundary conditions we obtain :

$$\int_0^T \left(\ddot{u}(t), \ddot{h}(t)\right) dt = \int_0^T \left(\left(\frac{d}{dt} F_{\dot{u}}\left(t, u(t), \dot{u}(t), \omega(t)\right) - F_u\left(t, u(t), \dot{u}(t), \omega(t)\right)\right), h(t)\right) dt.$$

Applying the fundamental Lemma 4.1, we get assertion of the theorem. □

Lemma 4.3. *The functional $\Phi_\omega(\cdot)$ given by* (4.1) *is weakly lower semicontinuous in H_T^2.*

Proof. The following functional:

$$H_T^2 \ni u \longmapsto \int_0^T \frac{1}{2}|\ddot{u}(t)|^2 dt$$

is convex and continuous so is weakly l.s.c., and the functional

$$H_T^2 \ni u \longmapsto \int_0^T F\left(t, u(t), \dot{u}(t), \omega(t)\right) dt$$

is weakly continuous (see Lemma 1.2). Therefore the functional $\Phi_\omega(\cdot)$ as the sum of weakly lower semicontinuous functionals is weakly lower semicontinuous in H_T^2. □

For later considerations, in the space H_T^2 we will consider the norm given by

$$||u|| = |\overline{u}| + \left(\int_0^T |\ddot{u}(t)|^2 dt\right)^{1/2}, \quad \overline{u} = \frac{1}{T}\int_0^T u(s)ds.$$

Let us denote by $\tilde{V}_k \subset H_T^2$ the set of solutions of the periodic problem of the form:

$$u^{(4)}(t) = \frac{d}{dt}F_{\dot{u}}\left(t, u(t), \dot{u}(t), \omega_k(t)\right) - F_u\left(t, u(t), \dot{u}(t), \omega_k(t)\right)$$

for a.e. $t \in [0, T]$,

$$u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = \ddot{u}(0) - \ddot{u}(T) = u^{(3)}(0) - u^{(3)}(T) = 0$$

where $\omega_k \in W,\ k = 0, 1, 2, \dots$,

and by

$$V_k = \left\{u \in H_T^2 : \Phi_{\omega_k}(u) = \min\left\{\Phi_{\omega_k}(v) : v \in H_T^2\right\}\right\}$$

— the set of minimizers of functional $\Phi_{\omega_k}(\cdot)$ for $k = 0, 1, 2, \dots$.

Under some conditions about F we can prove the following theorems:

Theorem 4.4. *If*

1. *F satisfies assumptions* (2-a)–(2-b)*,*
2. *there exist $g \in L^1\left([0, T], \mathbb{R}^+\right)$ such that*

$$|F_{p_i}(t, x, y, \omega_k)| \le g(t) \quad \forall x, y \in \mathbb{R}^n \text{ and } \omega_k \in W,$$

for $i = 0, 1$ and $k = 0, 1, 2, \dots$,

3. $\int_0^T F(t,x,0,\omega_k(t))\,dt \to \infty$ *uniformly with respect to* ω_k, *when* $|x| \to \infty$ *for* $k = 0,1,2,\ldots$,
4. $\Phi_{\omega_k}(\cdot)$ *is convex for any* $\omega_k \in W$ *with* $k = 0,1,2,\ldots$,
5. *the sequence* $\{\omega_k\} \subset W$ *tends to* $\omega_0 \in W$ *in the strong topology of* L^∞,

then $\limsup \tilde{V}_k$ *is a nonempty set and* $\limsup \tilde{V}_k \subset \tilde{V}_0$.

Proof. We have to prove that $\Phi_{\omega_k}(\cdot)$ is uniformly coercive with respect to ω_k, for $k = 0,1,2,\ldots$. Let $\omega_k \in W$, $k = 0,1,2,\ldots$. For $u \in H_T^2$ we have $u = \overline{u} + \tilde{u}$ where $\overline{u} = \int_0^T u(s)ds$ and $\dot{u} = \dot{\tilde{u}}$. So

$$\Phi_{\omega_k}(u) = \int_0^T \left(\frac{1}{2}|\ddot{u}(t)|^2 + F(t,\overline{u},0,\omega_k(t))\right) dt$$
$$+ \int_0^T \left(F\left(t,u(t),\dot{u}(t),\omega_k(t)\right) - F\left(t,\overline{u},\dot{u}(t),\omega_k(t)\right)\right) dt$$
$$+ \int_0^T \left(F\left(t,\bar{u},\dot{u}(t),\omega_k(t)\right) - F\left(t,\overline{u},0,\omega_k(t)\right)\right) dt.$$

Hence we have that

$$\Phi_{\omega_k}(u) = \int_0^T \left(\frac{1}{2}|\ddot{u}(t)|^2 + F(t,\overline{u},0,\omega_k(t))\right) dt$$
$$+ \int_0^T \int_0^1 \left(F_u\left(t,\overline{u}+s\tilde{u}(t),\dot{u}(t),\omega_k(t)\right),\tilde{u}(t)\right) dsdt$$
$$+ \int_0^T \int_0^1 \left(F_{\dot{u}}\left(t,\bar{u},s\dot{\tilde{u}},\omega_k(t)\right),\dot{\tilde{u}}(t)\right) dsdt.$$

By the Sobolev inequality and the Wirtinger inequality we obtain

$$\Phi_{\omega_k}(u) \geq \frac{1}{2}\int_0^T |\ddot{u}(t)|^2 dt - \int_0^T g(t)dt||\tilde{u}||_\infty - \int_0^T g(t)dt||\dot{\tilde{u}}||_\infty$$
$$+ \int_0^T F(t,\overline{u},0,\omega_k(t))\,dt$$

$$\geq \frac{1}{2}\int_0^T |\ddot{u}(t)|^2 dt - C_1\left(\int_0^T |\ddot{u}(t)|^2 dt\right)^{1/2} - C_2\left(\int_0^T |\ddot{u}(t)|^2 dt\right)^{1/2}$$
$$+\int_0^T F\left(t,\overline{u},0,\omega_k(t)\right)dt$$
$$=\frac{1}{2}||\ddot{u}||_{L^2}^2 - C||\ddot{u}||_{L^2} + \int_0^T F\left(t,\overline{u},0,\omega_k(t)\right)dt$$

where C, C_1, C_2 are constants and $k=0,1,2,\ldots$. So if $||u||\to\infty$, then $\Phi_{\omega_k}(u)\to\infty$ uniformly with respect to ω_k, for $k=0,1,2,\ldots$. By Lemma 4.3 and Lemma 2.2 the set V_k of minimizers of functional $\Phi_{\omega_k}(\cdot)$ for $k=0,1,2,\ldots$ is not empty and there exists a ball $B(0,R)\subset H_T^2$ such that $V_k\subset B(0,R)$ for $k=0,1,2,\ldots$. Applying Theorem 3.1 we get the thesis. □

Theorem 4.5. *If*

1. *F satisfies assumptions* (2-a)–(2-b),
2. *$F(t,\cdot,\cdot,\omega_k(t))$ is convex for $t\in[0,T]$ a.e., for all $\omega_k\in W$ with $k=0,1,2,\ldots$,*
3. *$F(t,x,y,\omega_k)\geq\alpha|x|-\beta+\psi(|y|)$ for all $x,y\in\mathbb{R}^n$ and $\omega_k\in W$, $k=0,1,2,\ldots$ where $\alpha>0$ and $\beta\geq 0$ are some constants and $\phi\in L^1([0,T],\mathbb{R}^n)$,*
4. *the sequence $\{\omega_k\}\subset W$ tends to $\omega_0\in W$ in the strong topology of L^∞,*

then $\limsup\tilde{V}_k$ *is a nonempty set and* $\limsup\tilde{V}_k\subset\tilde{V}_0$.

Proof. By assumptions we conclude that the real function

$$g_k:\mathbb{R}^n\ni x\to\int_0^T F\left(t,x,0,\omega_k(t)\right)dt,\quad k=0,1,2,\ldots,$$

has a minimum at some point $\overline{x}_{\omega_k}$ for which

$$\int_0^T F_{p_0}\left(t,\overline{x}_{\omega_k},0,\omega_k(t)\right)dt=0,\quad k=0,1,2,\ldots \tag{4.4}$$

and

$$\int_0^T F\left(t,x,0,\omega_k(t)\right)dt\to\infty \text{ when } |x|\to\infty,\ k=0,1,2,\ldots. \tag{4.5}$$

Let us fix for a moment k and let $\{u_n\} \subset H^1_T$ be a minimizing sequence for $\Phi_{\omega_k}(\cdot)$. By the assumption of convexity we obtain

$$\Phi_{\omega_k}(u_n) \geq \frac{1}{2}\int_0^T |\ddot{u}_n(t)|^2 dt + \int_0^T F\left(t, \overline{x}_{\omega_k}, 0, \omega_k(t)\right) dt$$
$$+ \int_0^T \left(F_{p_0}\left(t, \overline{x}_{\omega_k}, 0, \omega_k(t)\right), u_n(t) - \overline{x}_{\omega_k}\right) dt$$
$$+ \int_0^T \left(F_{p_1}\left(t, \overline{x}_{\omega_k}, 0, \omega_k(t)\right), \dot{u}_n(t)\right) dt$$

and by (4.4) we have

$$\Phi_{\omega_k}(u_n) \geq \frac{1}{2}\int_0^T |\ddot{u}_n(t)|^2 dt + \int_0^T F\left(t, \overline{x}_{\omega_k}, 0, \omega_k(t)\right) dt$$
$$- \int_0^T \left|F_{p_0}\left(t, \overline{x}_{\omega_k}, 0, \omega_k(t)\right)\right| dt ||\tilde{u}_n||_\infty$$
$$- \int_0^T \left|F_{p_1}\left(t, \overline{x}_{\omega_k}, 0, \omega_k(t)\right)\right| dt ||\dot{\tilde{u}}||_\infty.$$

Using assumption (2-b) and the Sobolev inequality we obtain

$$\Phi_{\omega_k}(u_n) \geq \frac{1}{2}\int_0^T |\ddot{u}_n(t)|^2 dt + \int_0^T F\left(t, \overline{x}_{\omega_k}, 0, \omega_k(t)\right) dt$$
$$- \left|\int_0^T F_{p_0}\left(t, \overline{x}_{\omega_k}, 0, \omega_k(t)\right) dt\right| ||\tilde{u}_n||_\infty$$
$$- \int_0^T a\left(|(\overline{x}_{\omega_k}, 0)|\right) b\left(t\right) dt ||\dot{\tilde{u}}||_\infty$$
$$\geq \frac{1}{2}||\ddot{u}_n||^2_{L^2} - C_1 - C_2||\ddot{u}_n||_{L^2}$$

where C_1, $C_2 > 0$, $u_n = \overline{u}_n + \tilde{u}_n$, $\dot{u}_n = \dot{\tilde{u}}$.

Hence there exists a constant $D_1 > 0$, such that

$$||\ddot{u}_n||_{L^2} \leq D_1$$

and inequality (1.1) and the Sobolev inequality imply that

$$||\dot{\tilde{u}}||_\infty \leq C_3 \text{ and } ||\tilde{u}_n||_\infty \leq C_4$$

where $C_3, C_4 > 0$ are constants.

From the convexity of F we obtain:

$$F\left(t, \frac{1}{2}\overline{u}_n, 0, \omega_k(t)\right) = F\left(t, \frac{1}{2}\left(u_n(t) - \tilde{u}_n(t)\right), \frac{1}{2}\left(\dot{u}_n(t) - \dot{\tilde{u}}_n(t)\right), \omega_k(t)\right)$$
$$\leq \frac{1}{2}F\left(t, u_n(t), \dot{u}_n(t), \omega_k(t)\right) + \frac{1}{2}F\left(t, -\tilde{u}_n(t), -\dot{\tilde{u}}_n(t), \omega_k(t)\right)$$

for a.e. $t \in [0,T]$. Hence

$$\begin{aligned}\Phi_{\omega_k}(u_n) &\geq \frac{1}{2}\int_0^T |\ddot{u}_n(t)|^2 dt + 2\int_0^T F\left(t, \frac{1}{2}\overline{u}_n, 0, \omega_k(t)\right) dt \\ &\quad - \int_0^T F\left(t, -\tilde{u}_n(t), -\dot{\tilde{u}}_n(t), \omega_k(t)\right) dt \\ &\geq 2\int_0^T F\left(t, \frac{1}{2}\overline{u}_n, 0, \omega_k(t)\right) dt - C_5\end{aligned}$$

for some $C_5 > 0$. Thus there exists a constant $D_2 > 0$, such that $|\overline{u}_n| \leq D_2$.

Therefore $\Phi_{\omega_k}(\cdot)$ for $k = 0, 1, 2, \ldots$ has a bounded minimizing sequence and the set V_k is not empty for $k = 0, 1, 2, \ldots$.

Now we have to prove that there exists a ball $B(0,R) \subset H_T^2$ such that $V_k \subset B(0,R)$ for $k = 0, 1, 2, \ldots$.

First, let us notice that

$$g_k(x) = \int_0^T F(t, x, 0, \omega_k(t))\, dt \geq \int_0^T (\alpha|x| - \beta + \psi(|0|)) \geq \alpha_0|x| - \beta_0$$

where $\alpha_0 > 0$.

Let us denote $A = a(0)\int_0^T b(t)\,dt$ and $B(0,\rho) = \{x \in \mathbb{R}^n : \alpha_0|x| - \beta_0 \leq A\}$. By the assumption (2-b) we have

$$g_k(\overline{x}_{\omega_k}) \leq g_k(0) \leq A \quad k = 0, 1, 2, \ldots.$$

Let us notice that all minimizers $\overline{x}_{\omega_k}$ are in $B(0,\rho)$, $k = 0, 1, 2, \ldots$. Indeed

$$\overline{x}_{\omega_k} \in \{x \in \mathbb{R}^n : g_k(x) \leq A\} \subset \{x \in \mathbb{R}^n : \alpha_0|x| - \beta_0 \leq A\} = B(0,\rho).$$

Now let $u_k = \tilde{u}_k + \bar{u}_k$ be a minimizer of $\Phi_{\omega_k}(\cdot)$ for $k = 0, 1, 2, \ldots$. In the analogous way as previously we can show that $||u_k|| \leq R$. By Theorem 3.1 we get the thesis. □

Example 4.1. Let $f:[0,T]\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times M\to\mathbb{R}$ be a function defined by the formula

$$\begin{aligned} f\left(t,p_0,p_1,p_2,w\right) = \left|p_2\right|^2 + p_2^2 - w\left|p_1\right|^2 + e^{-wp_1^2} \\ + \sin p_0^1\left|p_1\right|^2 + \left|p_0\right|^2 + twp_0^2 \end{aligned} \tag{4.6}$$

where $p_i=\left(p_i^1,p_i^2\right)$, $i=0,1,2$, and $M=[-1,1]$. Let us notice that

$$f\left(t,p_0,p_1,p_2,w\right) \geq \left|p_2\right|^2 - \left|p_2\right| - 2\left|p_1\right|^2 + \left|p_0\right|^2 - T\left|p_0\right|.$$

Consider the functional

$$\varphi_\omega(u) = \int_0^T f\left(t,u(t),\dot{u}(t),\ddot{u}(t),\omega(t)\right)dt$$

where f is given by (4.6). One can show that φ_ω is strictly convex for any $w\in W=\{\omega\in L^\infty([0,T],\mathbb{R}^r):\omega(t)\in M\}$. Let $\{\omega_k\}\subset W$ be any sequence strongly converging to $\omega_0\in W$. Consider a periodic problem with parameters ω_k, $k=0,1,2,\ldots$

$$\begin{aligned} &\frac{d}{dt}\left(\frac{d}{dt}2\ddot{u}^1 + 2\omega_k\dot{u}^1 - 2\dot{u}^1\sin u^1\right) + 2u^1 + \left|\dot{u}\right|^2\cos u^1 = 0 \\ &\frac{d}{dt}\left(\frac{d}{dt}\left(2\ddot{u}^2+1\right)+2\omega_k\dot{u}^2-2\dot{u}^1\sin u^1-\omega_k e^{-\omega_k\dot{u}^2}\right)+2u^2+t\omega_k=0 \\ &u(0)-u(T)=\dot{u}(0)-\dot{u}(T)=\ddot{u}(0)-\ddot{u}(T)=u^{(3)}(0)-u^{(3)}(T)=0. \end{aligned} \tag{4.7}$$

From Theorem 3.3 and Corollary 3.2 it follows that for each ω_k, $k=0,1,2,\ldots$, problem (4.7) possesses a uniquely defined solution $u_k\in H_T^2$ and that the sequence $\{u_k\}$ tends to u_0 in the weak topology of H_T^2.

Example 4.2. Let $W=\{\omega\in L^\infty([0,T],\mathbb{R}):0\leq\omega(t)\leq 1\}$ and let $\{\omega_k\}\subset W$, $k=1,2,\ldots$ be any sequence strongly converging to ω_0. For $k=0,1,2,\ldots$ consider the scalar problem

$$\begin{aligned} u^{(4)} &= \frac{d}{dt}\left(\omega_k\sin\left(u-\dot{u}\right) - \omega_k\cos\dot{u} + e\left(t\right)\right) \\ &\quad - \omega_k\operatorname{sgn}u + \omega_k\sin\left(u-\dot{u}\right), \\ u(0)-u(T)&=\dot{u}(0)-\dot{u}(T)=\ddot{u}(0)-\ddot{u}(T)=u^{(3)}(0)-u^{(3)}(T)=0 \end{aligned} \tag{4.8}$$

where function $e\in L^1\left([0,T],\mathbb{R}^+\right)$. In this case, F is of the form

$$F\left(t,p_0,p_1,w\right) = w\left(p_0^2+p_1^2\right)^{1/2} + w\cos\left(p_0-p_1\right) - w\sin p_1 + e\left(t\right)p_1$$

and hence

$$\int_0^T F(t,x,0,w)\,dt = (|x| + \cos x)\int_0^T w(t)\,dt \to \infty$$

if $|x| \to \infty$. It is easy to show that

$$|F_{p_i}(t,x,y,\omega_k)| \le g(t) \quad \forall x, y \in \mathbb{R}^n \text{ and } \omega_k \in W, \quad i = 0,1,\ k = 0,1,2,\dots$$

where $g(t) = \max(2, 2 + e(t))$ for $t \in [0,T]$. Moreover the functional

$$\Phi_\omega(u) = \int_0^T \left(\frac{1}{2}|\ddot{u}(t)|^2 + F\big(t, u(t), \dot{u}(t), \omega(t)\big)\right) dt$$

is strictly convex, so from Theorem 4.4 and Corollary 3.2 it follows that for each ω_k, $k = 0,1,2,\dots$, problem (4.7) possesses a uniquely defined solution $u_k \in H_T^2$, and that the sequence $\{u_k\}$ tends to u_0 in the weak topology of H_T^2.

Acknowledgment. The author expresses his gratitude to unknown referees for their valuable remarks and improvements.

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Monika Bartkiewicz
Faculty of Mathematics
University of Łódź
S. Banacha 22
90–238 Łódź, Poland
email: bartkiew@imul.uni.lodz.pl