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SOME REMARKS ON NONUNIQUENESS OF MINIMIZERS FOR DISCRETE MINIMIZATION PROBLEMS

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Abstract. We investigate the issue of uniqueness and nonuniqueness of minimizers for the approximation of variational problems. We show that when the continuous problem does not admit a minimizer its approximation by finite elements may lead to several discrete minimizers.

1. Introduction

Let Ω be a bounded polyhedral domain of $\mathbb{R}^n$ of boundary Γ and a function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ bounded from below. If $W_0^{1,\infty}(\Omega)$ denotes the space

$$W_0^{1,\infty}(\Omega) = \big\{ v \in W^{1,\infty}(\Omega) \ \text{ such that } v = 0 \text{ on } \Gamma \big\}, \tag{1.1}$$

we consider the following problem

$$I = \inf_{v \in W_0^{1,\infty}(\Omega)} \int_\Omega \varphi(\nabla v(x))\, dx. \tag{1.2}$$

It is well known that

$$I = |\Omega| \varphi^{**}(0) \tag{1.3}$$

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where φ^{**} is the convex envelope of φ that is to say the biggest convex function located below φ (see [10]) and $|\Omega|$ is the Lebesgue measure of Ω. The gap between nonexistence and existence of minimizers of the problem (1.2) is closed. Indeed, the integrands for which attainment occurs are completely determined and the minimizers are massively nonunique and they can be explicitly constructed (see [3], [12]). In this paper, we would like to address the question of uniqueness or nonuniqueness of minimizers for the approximation of (1.2) when this one does not admit minimizers. Let us denote by V_0^h a finite dimensional subspace of $W_0^{1,\infty}(\Omega)$. Then, we consider the following approximate version of (1.2)

$$I_h = \inf_{v \in V_0^h} \int_\Omega \varphi(\nabla v(x))\, dx. \tag{1.4}$$

We will consider the case of P_1-Lagrange finite elements in dimension 1 and 2 but our results can be extended to higher dimensions and one can consider other types of finite elements. When the continuous problem (1.2) does not attain its infimum, the approximate problem (1.4) may admit a minimizer if one assumes for instance that $\lim_{|t|\to+\infty} \varphi(t) = +\infty$ (see [3] and [5]). Nevertheless, it can happen that both the continuous problem and its approximation do not admit any minimizer. Indeed, let us assume that φ is such that

$$\varphi(t) > 0 \quad \forall t \in \mathbb{R}^n \tag{1.5}$$

$$\lim_{|t|\to\infty} \varphi(t) = 0, \tag{1.6}$$

then $\varphi^{**}(t) = 0$. Indeed, since 0 is a convex function located below φ one has first

$$0 \le \varphi^{**}. \tag{1.7}$$

Moreover, for any $t \neq 0$, $t \in \mathbb{R}^n$ and any $x \in \mathbb{R}^n$ it holds

$$\begin{aligned} \varphi^{**}(x) &= \varphi^{**}\left(\frac{1}{2}(x+kt) + \frac{1}{2}(x-kt)\right) \\ &\le \frac{1}{2}\varphi^{**}(x+kt) + \frac{1}{2}\varphi^{**}(x-kt) \\ &\le \frac{1}{2}\varphi(x+kt) + \frac{1}{2}\varphi(x-kt) \end{aligned} \tag{1.8}$$

for any $k \in \mathbb{N}$. Letting k go to infinity, by (1.6) it follows that

$$\varphi^{**} \le 0 \tag{1.9}$$

which leads, by (1.7), to $\varphi^{**} = 0$. We deduce then that the value of I is 0. Since φ is positive the infimum is not attained. Let us prove that the value of the discrete infimum is also equal to 0 provided that V_0^h contains a function u such that

$$\nabla u(x) \neq 0 \text{ for a.e. } x \in \Omega. \tag{1.10}$$

Indeed, since V_0^h is a vector space we have $k \cdot u \in V_0^h$ where k is an integer. Then we have

$$0 \leq I_h \leq \int_\Omega \varphi(k\nabla u(x))\, dx. \tag{1.11}$$

By (1.6) and (1.10) we have

$$\varphi(k\nabla u(x)) \longrightarrow 0 \text{ a.e.} \tag{1.12}$$

when k goes to infinity. Then we use the dominated convergence theorem (assuming φ bounded) to get $I_h = 0$ and the infimum is obviously not attained. Note here that we deduce that $I = 0$ without using the relaxation property (1.3).

We have the following convergence theorem:

Theorem 1.1. *Assume that φ is bounded on bounded subsets of $\mathbb{R}^n$ then we have*

$$\lim_{h\to 0} I_h = I. \tag{1.13}$$

We refer to [1] or [2] for a proof. Remark that if $\varphi^{**}(0) = \varphi(0)$, the problems (1.2), (1.4) attain their infima at 0. In the sequel, we will assume that

$$\varphi^{**}(0) < \varphi(0). \tag{1.14}$$

Note that the continuous problem (1.2) may admit a minimizer even if (1.14) is verified (see Example 2.2 below).

Roughly speaking, our main result is the following: when the approximate problem (1.4) admits a minimizer then this minimizer is in general not unique. Remark that this is trivial if φ is an even function. Indeed, if u is a minimizer of (1.4) then for h small it is not equal to 0 thanks to (1.3), (1.14) and theorem 1.1. Moreover its opposite $-u$ is also a minimizer.

Example 1.1. (The two-well antiplane shear problem) Take $n = 2$, and consider the energy density $\varphi(x, y) = (x^2-1)^2+y^2$. This energy arises in the model studied by R. V. Kohn and S. Müller to analyse branching of twins near austenite/twinned martensite interface (see [13]). The value of the continuous problem (1.2) is 0. Moreover, the infimum is not attained. When h is small, the discrete problem admits at least two opposite minimizers.

Remark 1.1. Let $a \in \mathbb{R}^n$ and consider the following problem and its discrete version

$$J = \inf_{v \in W_a^{1,\infty}(\Omega)} \int_\Omega \varphi(\nabla v(x)) \, dx. \tag{1.15}$$

$$J_h = \inf_{v \in V_a^h} \int_\Omega \varphi(\nabla v(x)) \, dx. \tag{1.16}$$

where

$$\begin{aligned} W_a^{1,\infty}(\Omega) &= \{v + a \cdot x : v \in W_0^{1,\infty}(\Omega)\}, \\ V_a^h &= \{v + a \cdot x : v \in V_0^h\}. \end{aligned} \tag{1.17}$$

($a \cdot x$ denotes either the scalar product between a and x or the function $x \to a \cdot x$). Then the problems (1.15) and (1.16) are equivalent to the following ones

$$I = \inf_{v \in W_0^{1,\infty}(\Omega)} \int_\Omega \tilde{\varphi}(\nabla v(x)) \, dx \tag{1.18}$$

$$I_h = \inf_{v \in V_0^h} \int_\Omega \tilde{\varphi}(\nabla v(x)) \, dx \tag{1.19}$$

where

$$\tilde{\varphi}(w) = \varphi(w + a), \tag{1.20}$$

so that the results obtained in this paper are also valid if one imposes linear nonhomogeneous boundary conditions.

2. A one-dimensional case

In this section we assume that $\Omega = (a, b)$ and we consider the following subdivision of Ω:

$$a_i = a + i \frac{b-a}{2^j}; \quad i = 0, 1, \ldots, 2^j; \quad j \in \mathbb{N}, \tag{2.1}$$

then we set

$$\begin{aligned} V_0^j(a,b) = V_0^j = \{&v \colon \Omega \to \mathbb{R},\ v \text{ is continuous on } \Omega, \\ &v(a) = v(b) = 0,\ v \text{ is affine on each } (a_{i-1}, a_i),\ i = 1, \ldots, 2^j\}. \end{aligned} \tag{2.2}$$

We denote by I^j the infimum of the following discrete problem:

$$I^j(a,b) = I^j = \inf_{v \in V_0^j} \int_a^b \varphi(u'(t))dt. \tag{2.3}$$

We are going to prove that if I^j admits a unique minimizer then it is necessary equal to zero. We will deduce that I^j cannot admit a unique minimizer using Theorem 1.1. For this purpose, we will need some lemmas. First, we have the following lemma which is in fact true if one considers subdivisions of the interval (a,b) such that their corresponding nodes are symmetric with respect to the midpoint $c=(a+b)/2$ of (a,b).

Lemma 2.1. *If u is a unique minimizer of I^j then*

$$u(t) = -u(a+b-t). \tag{2.4}$$

In particular one has $u(c)=0$.

Proof. We consider the following function

$$v(t) = -u(a+b-t).$$

It is clear that $v \in V_0^j$ and one has

$$\begin{aligned}\int_a^b \varphi(v'(t))dt &= \int_a^b \varphi(u'(a+b-t))dt \\ &= -\int_b^a \varphi(u'(t))dt \\ &= \int_a^b \varphi(u'(t))dt.\end{aligned} \tag{2.5}$$

Since u is assumed to be the unique minimizer of I^j, we deduce that $v(t) = u(t)$. This completes the proof of the lemma. □

Remark 2.1. Let $u \in V_0^j$ verifying (2.4) then we have

$$\int_a^b \varphi(u'(t))dt = 2\int_a^c \varphi(u'(t))dt. \tag{2.6}$$

Lemma 2.2. *Let u be a minimizer of $I^j(a,b)$ such that*

$$u(t) = -u(a+b-t), \tag{2.7}$$

then $\tilde{u}$ the restriction of u to (a,c) is a minimizer of $I^{j-1}(a,c)$. Moreover if u is unique then $\tilde{u}$ is also unique.

Proof. Let $\tilde{v} \in V_0^{j-1}(a,c)$ with an obvious definition for $V_0^{j-1}(a,c)$, we consider the function v defined by

$$v(t) = \begin{cases} \tilde{v}(t) & \text{if } t \in (a,c) \\ -\tilde{v}(a+b-t) & \text{if } t \in (c,b). \end{cases} \tag{2.8}$$

It is easy to see that $v \in V_0^j(a,b)$ and

$$\int_a^b \varphi(v'(t))dt = 2\int_a^c \varphi(\tilde{v}'(t))dt. \tag{2.9}$$

Since u is a minimizer of $I^j(a,b)$, we have

$$\begin{aligned}\int_a^b \varphi(v'(t))dt &\geq \int_a^b \varphi(u'(t))dt \\ &= 2\int_a^c \varphi(\tilde{u}'(t))dt. \end{aligned} \tag{2.10}$$

Combining (2.9) and (2.10) we obtain

$$\int_a^c \varphi(\tilde{v}'(t))dt \geq \int_a^c \varphi(\tilde{u}'(t))dt \quad \forall \tilde{v} \in V_0^{j-1}(a,c). \tag{2.11}$$

This completes the proof of the first part of the Lemma. We assume now that $I^{j-1}(a,c)$ admits another minimizer $\tilde{v}$ and we denote by v its extension to (a,b) defined by (2.8). Then we have

$$\int_a^c \varphi(\tilde{v}'(t))dt = \int_a^c \varphi(\tilde{u}'(t))dt. \tag{2.12}$$

Using Remark 2.1, we conclude that

$$\int_a^b \varphi(v'(t))dt = \int_a^b \varphi(u'(t))dt. \tag{2.13}$$

Therefore, v is another minimizer of $I^j(a,b)$. □

We can now prove the following theorem:

Theorem 2.1. *If the problem I^j admits a unique minimizer u, then u is identically equal to zero.*

Proof. One proceeds by induction on j. Let u be a minimizer of I^1 then u is affine on (a,c) and (c,b) (recall that $c=(a+b)/2$). By the second part of Lemma 2.1 we have $u(a)=u(c)=u(b)=0$ which implies that u is the zero function. We assume now that the theorem is true for the energy I^{j-1} and let us prove that the theorem is still valid for the energy I^j. Let u be a minimizer of $I^j(a,b)$. We deduce by Lemma 2.2 that $\tilde{u}$ the restriction of u to (a,c) is the unique minimizer of $I^{j-1}(a,c)$. Applying the induction assumption to the energy $I^{j-1}(a,c)$. We deduce that $\tilde{u}=0$. But u verifies (2.4), it is therefore equal to the zero function. □

We can now prove our main result:

Theorem 2.2. *We assume that φ is bounded on bounded subsets of $\mathbb{R}$. Then for j sufficiently large I^j cannot admit a unique minimizer.*

Proof. The sequence I^j is nonincreasing and converges, by Theorem 1.1, to $(b-a)\varphi^{**}(0)$. Since we assumed

$$\varphi^{**}(0) < \varphi(0), \tag{2.14}$$

there exists $j_0 \in \mathbb{N}$ such that

$$I^j < (b-a)\varphi(0) \quad \forall j \geq j_0.$$

So I^j cannot attain its infimum at 0. Therefore, I^j cannot have a unique minimizer by Theorem 2.1. □

Now we illustrate our results by some examples

Example 2.1. (An energy density with a finite well and a well at infinity) Let $\varphi\colon \mathbb{R} \to \mathbb{R}_+$ be a continuous function convex on $(-\infty, w)$ where w is a negative real number such that

$$\varphi(w) = 0, \quad \varphi(t) > 0 \quad \forall t \neq w \tag{2.15}$$

$$\lim_{|t|\to+\infty} \varphi(t) = +\infty, \quad \lim_{t\to+\infty} \frac{\varphi(t)}{t} = 0. \tag{2.16}$$

Then

$$\varphi^{**}(t) = \begin{cases} \varphi(t) & \text{for } t \in (-\infty, w] \\ 0 & \text{for } t \in [w, +\infty). \end{cases} \tag{2.17}$$

One can take φ as follows

$$\varphi(t) = \begin{cases} (t+1)^2 & \text{for } t \leq -1, \\ \dfrac{t+1}{\ln(t+e)} & \text{for } t \geq -1. \end{cases} \tag{2.18}$$

The value of the continuous infimum is zero and it is not attained. The discrete infimum I^j admits at least two minimizers when j is large.

Example 2.2. (An energy density with two wells) Let $w_1, w_2 \in \mathbb{R}$ such be that

$$w_1 < 0 < w_2. \tag{2.19}$$

We consider an energy density φ which is a continuous function such that

$$\varphi \text{ is convex on } (-\infty, w_1] \text{ and } [w_2, +\infty),$$

$$\varphi(t) \geq \psi(t) = \frac{\varphi(w_2) - \varphi(w_1)}{w_2 - w_1}(t - w_1) + \varphi(w_1) \quad \text{on } \mathbb{R}, \tag{2.20}$$

$$\varphi(0) > \psi(0), \tag{2.21}$$

$$\lim_{|t|\to\infty} \varphi(t) = +\infty. \tag{2.22}$$

Then we have

$$\varphi^{**}(t) = \begin{cases} \varphi(t) & \text{if } t \in (-\infty, w_1] \cup [w_2, +\infty), \\ \psi(t) & \text{if } t \in [w_1, w_2]. \end{cases} \tag{2.23}$$

Using (2.19), (2.21) and (2.23), it is easy to see that

$$\varphi^{**}(0) < \varphi(0). \tag{2.24}$$

Using for example Theorem 2.12 in [12] we conclude that the continuous problem admits minimizers. Theorem 2.2 implies that its discrete approximation admits at least two minimizers. As we mentioned before the continuous minimizers can be constructed and the numerical approximation of it is not useful.

3. A two-dimensional case

In this section we assume that $\Omega = (0,1) \times (0,1)$. Recall that for h small I_h cannot attain its infimum at 0 if φ verifies (1.14) so that if φ is an even function i.e.

$$\varphi\big(-(x,y)\big) = \varphi\big((x,y)\big) \quad \forall (x,y) \in \mathbb{R}^2, \tag{3.1}$$

I_h cannot admit a unique minimizer. If now φ is even with respect to each of its variables i.e.

$$\varphi\big((-x,y)\big) = \varphi\big((x,y)\big) = \varphi\big((x,-y)\big) \quad \forall (x,y) \in \mathbb{R}^2, \tag{3.2}$$

then φ verifies (3.1) and I_h cannot have a unique minimizer.

In what follows we will exhibit a family of triangulation of Ω for which the nonuniqueness of minimizers holds if one assumes only that φ is even with respect to one of its variables. We assume for example that φ verifies

$$\varphi\big((-x,y)\big) = \varphi\big((x,y)\big) \quad \forall (x,y) \in \mathbb{R}^2. \tag{3.3}$$

Let $j \in \mathbb{N}^*$, $h = 1/2^j$ and $n = 2^j$, we consider the following subdivision of $(0,1)$

$$a_i = ih; \quad i = 0, 1, \dots, n. \tag{3.4}$$

Then we consider the grid on Ω with vertices (see Figure 1)

$$a_{ik} = (a_i, a_k); \quad i,k = 0, 1, \dots, n. \tag{3.5}$$

We denote by $\mathcal{T}_j$ any triangulation of Ω such that its nodes belong to the segments

$$[a_{0i}, a_{ni}]; \quad i = 0, 1, \dots, n. \tag{3.6}$$

Moreover, we assume that this triangulation is uniformly symmetric with respect to the segments

$$[a_{0\{1/2^k\}}, a_{n\{1/2^k\}}]; \quad k = 1, \dots, j.$$

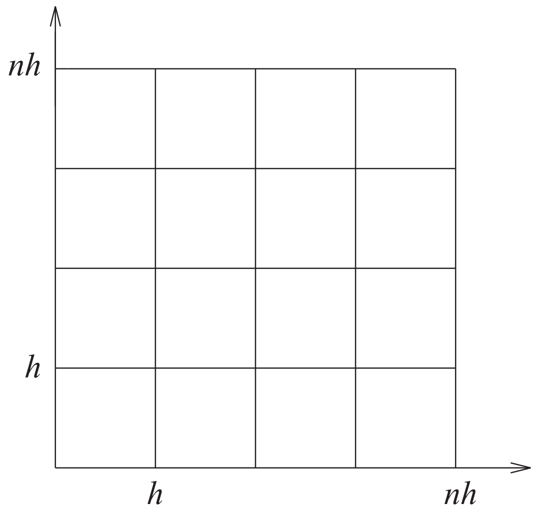

FIGURE 1.

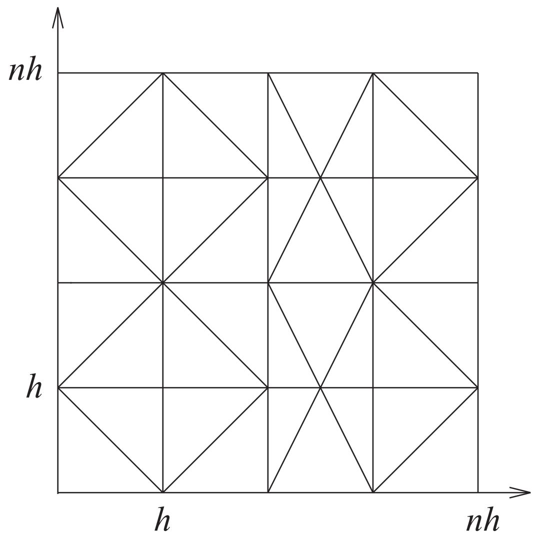

FIGURE 2.

We suppose that the elements of $\mathcal{T}_j$ are triangles (see Figure 2) but one can consider other types of finite elements. One can take for example 2-parallelotops of type (1) (see [14]) and substitute P_1 by Q_1 in V_0^j defined as follows

$$V_0^j = V_0^j(\Omega) = \{v\colon \Omega \to \mathbb{R},\ \text{continuous},\ v = 0 \ \text{ on } \Gamma,\ v/_T \in P_1 \quad \forall T \in \mathcal{T}_j\}$$

where P_1 denotes the space of polynomials of degree 1, $v/_T$ the restriction of v to the triangle T. We approach the continuous problem (1.2) by the following discrete one

$$I^j = I^j(\Omega) = \inf_{v \in V_0^j} \int_\Omega \varphi(\nabla v(x,y))\, dxdy. \tag{3.7}$$

Let us first adopt the following notation:

$$\Omega_k = (0,1) \times (0, \frac{1}{2^k}); \quad k = 0, 1, \dots, j.$$

$$V_0^j(\Omega_k) = \big\{ v \colon \Omega_k \to \mathbb{R},\ \text{continuous},\ v = 0 \ \text{ on } \partial\Omega_k,$$
$$v/_T \in P_1 \quad \forall T \in \mathcal{T}_j \cap \Omega_k \big\}$$

$$I^j(\Omega_k) = \inf_{v \in V_0^j(\Omega_k)} \int_{\Omega_k} \varphi(\nabla v(x,y))\, dxdy. \tag{3.8}$$

where $\partial\Omega_k$ denotes the boundary of Ω_k. Then we have the following theorem

Theorem 3.1. *We assume that φ is bounded on bounded subsets of $\mathbb{R}^2$. If φ verifies* (1.14), (3.3) *then for j large enough the discrete problem I^j cannot admit a unique minimizer.*

Before giving the proof of Theorem 3.1, we will need some preparatory lemmas. First we have:

Lemma 3.1. *If φ satisfies* (3.3) *and u is the unique minimizer of $I^j(\Omega_k)$ then we have*

$$u(x,y) = -u(x, \frac{1}{2^k} - y) \quad \forall (x,y) \in \Omega_k \tag{3.9}$$

Proof. We consider the following function:

$$v(x,y) = -u(x, \frac{1}{2^k} - y) \quad \forall (x,y) \in \Omega_k. \tag{3.10}$$

Using the symmetry of $\mathcal{T}_j \cap \Omega_k$ with respect to the segment $[a_{0\{1/2^{k+1}\}}, a_{n\{1/2^{k+1}\}}]$, it is easy to see that $v \in V_0^j(\Omega_k)$. To prove (3.9), it suffices to verify that v realizes the infimum in (3.8). Indeed one has:

$$\int_{\Omega_k} \varphi(\nabla v(x,y))\, dxdy = \int_{\Omega_k} \varphi\big(-\frac{\partial u}{\partial x}(x, \frac{1}{2^k} - y), \frac{\partial u}{\partial y}(x, \frac{1}{2^k} - y)\big)\, dxdy$$
$$= \int_{\Omega_k} \varphi\big(\frac{\partial u}{\partial x}(x, \frac{1}{2^k} - y), \frac{\partial u}{\partial y}(x, \frac{1}{2^k} - y)\big)\, dxdy, \tag{3.11}$$

since φ is assumed to verify (3.3). By a change of variables we obtain

$$\int_{\Omega_k} \varphi(\nabla v(x,y))\, dxdy = \int_{\Omega_k} \varphi(\nabla u(x,y))\, dxdy. \tag{3.12}$$

Since u is a unique minimizer we deduce that $v = u$ which is (3.9). □

Remark 3.1. Let $u \in V_0^j(\Omega_k)$ verifying (3.9) then the restriction of u to Ω_{k+1} that we will also denote by u belongs to $V_0^j(\Omega_{k+1})$. Indeed one only has to show that $u(x, 1/2^{k+1}) = 0$, but by (3.9) we have

$$\begin{aligned} u(x, \frac{1}{2^{k+1}}) &= -u(x, \frac{1}{2^k} - \frac{1}{2^{k+1}}) \\ &= -u(x, \frac{1}{2^{k+1}}). \end{aligned} \tag{3.13}$$

Then $u(x, 1/2^{k+1}) = 0$. We also have

$$\int_{\Omega_k} \varphi(\nabla u(x,y))\ dxdy = 2 \int_{\Omega_{k+1}} \varphi(\nabla u(x,y))\ dxdy. \tag{3.14}$$

Then we have the following lemma:

Lemma 3.2. *If u is a minimizer of $I^j(\Omega_k)$ verifying* (3.9) *then u is also a minimizer of $I^j(\Omega_{k+1})$. Moreover if u is the unique minimizer for $I^j(\Omega_k)$ then u is the unique minimizer of $I^j(\Omega_{k+1})$.*

Proof. Let $\tilde{v} \in V_0^j(\Omega_{k+1})$, we consider the extension v of $\tilde{v}$ to Ω_k defined by

$$v(x,y) = \begin{cases} \tilde{v}(x,y) & \text{if } (x,y) \in \Omega_{k+1}, \\ -\tilde{v}(x, \frac{1}{2^k} - y) & \text{if } (x,y) \in \Omega_k \backslash \Omega_{k+1}. \end{cases} \tag{3.15}$$

The function v verifies

$$v(x,y) = -v(x, \frac{1}{2^k} - y) \quad \forall (x,y) \in \Omega_k \tag{3.16}$$

so that we have

$$\int_{\Omega_k} \varphi(\nabla v(x,y))\ dxdy = 2 \int_{\Omega_{k+1}} \varphi(\nabla \tilde{v}(x,y))\ dxdy. \tag{3.17}$$

The triangulation $\mathcal{T}_j \cap \Omega_k$ is symmetric with respect to the axis $y = 1/2^{k+1}$ so that $v \in V_0^j(\Omega_k)$. Since u is a minimizer we have

$$\begin{aligned} \int_{\Omega_k} \varphi(\nabla v(x,y))\ dxdy &\geq \int_{\Omega_k} \varphi(\nabla u(x,y))\ dxdy \\ &= 2 \int_{\Omega_{k+1}} \varphi(\nabla u(x,y))\ dxdy. \end{aligned} \tag{3.18}$$

The last equality is due to Lemma 3.1 and Remark 3.1. Combining (3.17) and (3.18) we conclude that u is a minimizer of $I^j(\Omega_{k+1})$. Now, let $\tilde{w}$ be another minimizer of $I^j(\Omega_{k+1})$. We denote by w its extension defined by (3.15). Then we have

$$\int_{\Omega_{k+1}} \varphi(\nabla \tilde{w}(x,y))\, dxdy = \int_{\Omega_{k+1}} \varphi(\nabla \tilde{u}(x,y))\, dxdy. \tag{3.19}$$

Since u and w verify (3.9) we have

$$\int_{\Omega_k} \varphi(\nabla w(x,y))\, dxdy = \int_{\Omega_k} \varphi(\nabla u(x,y))\, dxdy \tag{3.20}$$

therefore w is another minimizer of $I^j(\Omega_k)$. This completes the proof of the second part of the Lemma 3.2. □

Now we are able to prove:

Theorem 3.2. *Assume that φ verifies* (1.14) *and* (3.3). *If $I^j(\Omega) := I^j(\Omega_0)$ admits a unique minimizer, then it is identically equal to zero.*

Proof. Let u be the minimizer of $I^j(\Omega_0)$. By induction and Lemma 3.2 we deduce that u is also the unique minimizer of $I^j(\Omega_k)$ for $k = 0, 1, \dots, j$. Lemma 3.1 implies that

$$u(x,y) = -u(x, \frac{1}{2^{j-1}} - y) \quad \forall (x,y) \in \Omega_{j-1}. \tag{3.21}$$

Using repeatedly (3.14) we have

$$\int_{\Omega} \varphi(\nabla u(x,y))\, dxdy = 2^j \int_{\Omega_j} \varphi(\nabla u(x,y))\, dxdy. \tag{3.22}$$

On the other hand (3.21) implies that

$$u(x, \frac{1}{2^j}) = 0 \quad \forall x \in (0,1). \tag{3.23}$$

Therefore u is equal to the zero function in Ω_j and (3.22) becomes

$$\int_{\Omega} \varphi(\nabla u(x,y))\, dxdy = 2^j |\Omega_j| \varphi(0) = \varphi(0). \tag{3.24}$$

Thus the zero function also realizes the infimum. Since u is assumed to be the unique minimizer we conclude that u is equal to 0. □

Let us now return to the proof of Theorem 3.1.

Proof of Theorem 3.1. We assume that I^j admits a unique minimizer. This minimizer is equal to zero by Theorem 3.2, but for j large enough this is not possible. Thus I^j cannot have a unique minimizer when j is choosen so large. □

Remark 3.2. Unlike the one dimensional case, we considered very special energies φ in higher dimensions. Nevertheless, they cover a class of problems encountered in practice. Namely the energy density φ can be an even function in the sense of (3.3) vanishing at some wells. We can take for example

$$\varphi(x, y) = (|x| + |y - 1|)(|x| + |y - 2|).$$

We know that $\varphi^{**}(0) = 0$ so that (1.14) is verified. The infimum of the continuous problem (1.2) is not attained but the infimum of the discrete problem (3.8) is attained. When j is large enough there exist at least two discrete minimizers.

Now we are going to show that the number of minimizers may tend to infinity when the size of the triangulation goes to zero. Let $\varphi \colon \mathbb{R}^2 \to \mathbb{R}$ be a nonnegative function such that

$$\varphi(0, 1) = \varphi(0, -1) = 0, \tag{3.25}$$

$$\varphi(x, 0) \geq \varphi(1, 0) = \varphi(-1, 0) \quad \forall x \in \mathbb{R}; \tag{3.26}$$

one can take for example $\varphi(x, y) = x^2 + (y^2 - 1)^2(x^2 - 3/2)^2$. We denote by $\mathcal{T}$ the following triangulation of $\Omega = (0, 1) \times (0, 1)$ (see Figure 3)

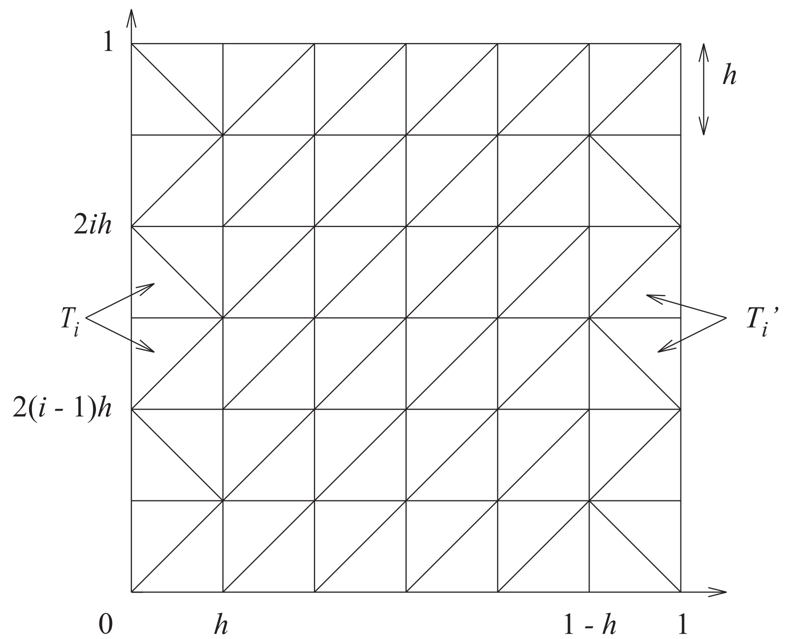

FIGURE 3.

where $h = 1/N$, and N is an even positive integer. We consider the following problem:

$$I_h = \inf_{v \in V_0^h} \int_\Omega \varphi(\nabla v(x,y))\, dxdy \tag{3.27}$$

where $V_0^h = \{v\colon \Omega \to \mathbb{R}$ continuous : $v/_T \in P_1 \quad \forall T \in \mathcal{T},\ v = 0$ on $\partial\Omega\}$ ($v/_T$ denotes the restriction of v to the triangle T). We denote by T_i and T_i' ($i = 1, \dots, N/2$) the following triangles (see Figure 3)

$$T_i = \{(x,y) \in \Omega : x \in (0,h),\ x \leq y - 2(i-1)h \leq 2h - x\} \tag{3.28}$$

$$T_i' = \{(x,y) \in \Omega : x \in (1-h,1),\ 1-x \leq y-2(i-1)h \leq x-1+2h\}. \tag{3.29}$$

Then we consider the function $u_h \in V_0^h$ defined by

$$u_h(x,y) = \begin{cases} x & \text{if } (x,y) \in T_i, \\ 1-x & \text{if } (x,y) \in T_i', \\ y - 2(i-1)h & \text{if } (x,y) \in \\ & (0,1) \times (2(i-1)h, (2i-1)h) \backslash \{T_i, T_i'\}, \\ 2ih - y & \text{if } (x,y) \in \\ & (0,1) \times ((2i-1)h, 2ih) \backslash \{T_i, T_i'\}. \end{cases} \tag{3.30}$$

Note that u_h is a periodic function in the y-direction and $\nabla u_h = (0,1)$ or $(0,-1)$ except in the triangles T_i and T_i' where the gradient of u_h takes respectively the values (1,0) and $(-1,0)$ so that one has

$$\begin{aligned} \int_\Omega \varphi(\nabla u_h(x,y))\, dxdy &= \frac{N}{2} h^2 (\varphi(1,0) + \varphi(-1,0)) \\ &= h\varphi(1,0). \end{aligned} \tag{3.31}$$

Then we have

Theorem 3.3. *Let φ be a nonnegative function such that* (3.25) *and* (3.26) *hold, then u_h is a minimizer of I_h.*

Proof. Let $v \in V_0^h$ and denote by α_i and α_i' its values at the vertices of T_i and T_i' which do not belong to the boundary of Ω. Since we have

$$\sum_{i=1}^{N/2} \int_{T_i \cup T_i'} \varphi(\nabla v(x,y))\, dxdy = \sum_{i=1}^{N/2} h^2 \big(\varphi(\frac{\alpha_i}{h}, 0) + \varphi(\frac{\alpha_i'}{h}, 0)\big) \tag{3.32}$$

using (3.26) one gets

$$\int_\Omega \varphi(\nabla v(x,y))\, dxdy \geq h\varphi(1,0). \tag{3.33}$$

Combining (3.31) and (3.33) we deduce that u_h is a minimizer. $\square$

Remark 3.3. If one changes u_h into $-u_h$ in the strip $(0,1)\times(2(i-1)h, 2ih)$, for $i = 1, \ldots, N/2$, one obtains another minimizer. Therefore we have

Theorem 3.4. *I_h admits at least $2^{N/2}$ minimizers.*

Proof. Indeed we have $2^{N/2}$ possibilities to change u_h by $-u_h$ in the strips $(0,1) \times (2(i-1)h, 2ih)$ $i = 1, \ldots, N/2$. □

Let us assume now that

$$\varphi(x,y) = 0 \Longleftrightarrow (x,y) = (0,1) \text{ or } (0,-1), \tag{3.34}$$

$$\varphi(x,0) > \varphi(1,0) \quad \forall x \neq 1, -1. \tag{3.35}$$

Then we have

Theorem 3.5. *Assume that φ verifies* (3.25)*,* (3.26)*,* (3.34) *and* (3.35) *then the problem I_h admits exactly $2^{N/2}$ minimizers.*

Proof. Let us denote by Ω_b the set

$$\Omega_b = \bigcup_{i=1}^{N/2} (T_i \cup T_i')$$

and let v_h be a minimizer of I_h such that α_i and α_i' are its values at the interior vertices of T_i and T_i'. It is clear that

$$\int_{\Omega_b} \varphi(\nabla v_h(x,y))\, dxdy = \sum_{i=1}^{N/2} h^2 \left(\varphi(\frac{\alpha_i}{h}, 0) + \varphi(\frac{\alpha_i'}{h}, 0) \right). \tag{3.36}$$

By (3.26) we deduce that

$$\int_{\Omega_b} \varphi(\nabla v_h(x,y))\, dxdy \geq h\varphi(1,0) = I_h \tag{3.37}$$

so that

$$\int_{\Omega\backslash\Omega_b} \varphi(\nabla v_h(x,y))\, dxdy = 0, \quad \int_{\Omega_b} \varphi(\nabla v_h(x,y))\, dxdy = I_h. \tag{3.38}$$

By (3.34) we deduce that

$$\nabla v_h = (0,1) \text{ or } (0,-1) \text{ a.e. in } \Omega\backslash\Omega_b. \tag{3.39}$$

Using (3.26), (3.35), (3.36) and the second term of (3.38) we have

$$\alpha_i = \pm 1 \text{ and } \alpha_i' = \pm 1. \tag{3.40}$$

Combining (3.39) and (3.40) one gets

$$\alpha_i = \alpha_i' = \pm 1 \tag{3.41}$$

and v_h belongs to the class of $2^{N/2}$ minimizers of Theorem 3.4. This completes the proof of Theorem 3.5. $\square$

Remark 3.4. The above results are still valid if one changes in the parallelogram $(h, 1-h) \times (0,1)$ the triangulation $\mathcal{T}$ by another one provided that its nodes stand in the lines $\{(x, kh) : x \in (h, 1-h)\}$; $k = 1, \dots, N$.

4. A vectorial case

In this section we consider φ a function defined on $\mathbb{R}^{m\times n}$; $m, n \geq 2$. We denote also by $W_0^{1,\infty}(\Omega)$ the space

$$W_0^{1,\infty}(\Omega) = (W_0^{1,\infty}(\Omega))^m. \tag{4.1}$$

Then we consider the following problem

$$\begin{aligned} I &= \inf_{v \in W_0^{1,\infty}(\Omega)} \int_\Omega \varphi(\nabla v(x))\, dx \\ &= |\Omega| Q\varphi(0) \end{aligned} \tag{4.2}$$

where ∇v stands now for the Jacobian matrix $\left(\dfrac{\partial v^i}{\partial x_j}\right)$ and $Q\varphi$ for the quasi-convex envelope of φ (see [10]). Let $\mathcal{T}$ be a regular triangulation of Ω with simplices of diameters less than h. If K is a simplex of $\mathcal{T}$ we denote by $P_1(K)$ the space of polynomials of degree 1 on K and set

$$V_0^h = \{v \in W_0^{1,\infty}(\Omega) : v^i/_K \in P_1(K) \quad \forall i = 1, \dots, m, \quad \forall K \in \mathcal{T}\} \tag{4.3}$$

where $v^i/_K$ denotes the restriction of v^i (the ith component of v) to K. Then we approach the problem (4.2) by

$$I_h = \inf_{v \in V_0^h} \int_\Omega \varphi(\nabla v(x))\, dx. \tag{4.4}$$

Theorem 1.1 holds if φ is bounded on bounded subsets of $\mathbb{R}^{m\times n}$ (we refer to [1] for a proof). Thus if one assumes that

$$Q\varphi(0) < \varphi(0) \tag{4.5}$$

the discrete problem I_h cannot attain its infimum at 0 when h is small. We assume that there exists a matrix $A \in \mathbb{R}^{m\times m}$ not equal to the identity matrix I such that

$$\varphi(AW) = \varphi(W) \quad \forall W \in \mathbb{R}^{m\times n} \tag{4.6}$$

$$1 \text{ is not an eigenvalue of } A. \tag{4.7}$$

Then we have the following theorem:

Theorem 4.1. *Assume that φ is bounded on bounded subsets of $\mathbb{R}^{m\times n}$ then the discrete problem* (4.4) *cannot admit a unique minimizer when h is small. More precisely if u is a discrete minimizer then Au is another one.*

Proof. One proceeds by contradiction. We assume that I_h admits a unique minimizer $u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}$ then by (4.6) we have

$$\int_\Omega \varphi(\nabla u(x))\, dx = \int_\Omega \varphi(A\nabla u(x))\, dx. \tag{4.8}$$

Thus we get

$$\int_\Omega \varphi(\nabla u(x))\, dx = \int_\Omega \varphi(\nabla(Au)(x))\, dx \tag{4.9}$$

so that Au also realizes the infimum. Since u is assumed to be unique we have

$$Au(x) = u(x) \quad \text{for a.e. } x \in \Omega. \tag{4.10}$$

Hence

$$u(x) \in \operatorname{Ker}(I - A) \text{ for a.e. } x \in \Omega. \tag{4.11}$$

where Ker denotes the kernel of matrices. By (4.7) $\operatorname{Ker}(I - A)$ is reduced to the zero vector. Then u is equal to zero. This contradicts the fact that I_h cannot admit a zero minimizer when h is small. ☐

Example 4.1. (The Ericksen-James energy) Take $m = n = 2$ and consider the energy density

$$\varphi(F) = k_1(c_{11} + c_{22} - 2)^2 + k_2 c_{12}^2 + k_3\Big(\frac{(c_{11} - c_{22})^2}{2} - \varepsilon^2\Big)^2$$

where $C = F^T F = (c_{ij})$, F^T is the transpose of F. The problem I_h admits a minimizer. Remark that φ is an even function, we deduce that the problem I_h admits at least two opposite discrete minimizers. But φ is left invariant under the rotations which obviously verify (4.7). Then the discrete problem I_h admits an infinite number of minimizers. More precisely if u is a minimizer of I_h then $\{Ru : R \in \mathrm{SO}(2)\}$ is a family of minimizers of I_h.

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