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GRADIENT-FINITE ELEMENT METHOD FOR NONLINEAR NEUMANN PROBLEMS

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Abstract. We consider the numerical solution of quasilinear elliptic Neumann problems. The basic difficulty is the non-injectivity of the operator, which can be overcome by suitable factorization. We extend the gradient-finite element method (GFEM), introduced earlier by the authors for Dirichlet problems, to the Neumann problem. The algorithm is constructed and its convergence is proved.

1. Introduction

This paper is devoted to the introduction and theoretical investigation of a numerical method for quasilinear elliptic boundary value problems of the form

$$\begin{cases} T(u) & \equiv -\mathrm{div}\,(f(x,\nabla u)\nabla u) \;=\; g(x) \\ \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} & = 0\,. \end{cases} \tag{1}$$

(The exact conditions on the domain Ω and the functions f, g are given in Section 2.)

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The most frequently used numerical methods for elliptic problems are the finite difference and finite element methods (e.g. [8, 21]). The basic difficulty in the discretized Neumann problem is the non-injectivity of the operator (inherited from the original problem). This difficulty can be overcome by suitable factorization (see [9, 10] for linear equations). The solution of the obtained system of algebraic equations is generally some iterative method, improved by suitable preconditioning.

Our aim is to extend the gradient-finite element method (GFEM), introduced by the authors for Dirichlet problems in [5], to the Neumann problem (1).

The GFEM for Dirichlet problems involves the infinite-dimensional generalization of the gradient method (GM). The latter was first developed by Kantorovich [11] for linear problems; further results are found among others in [6, 12, 20] for nonlinear equations. We underline that the Sobolev space background helps constructing effective natural preconditioners [1, 17]. The Sobolev space GM reduces the solution of the nonlinear equation to the sequence of auxiliary linear Poisson problems. The numerical solution of these auxiliary linear problems by a suitable finite element method yields the GFEM [5].

The extension of the GFEM to Neumann problems requires suitable Hilbert space background. The authors have investigated the GM for non-injective operators in Hilbert space [13]. Based on this, in the present paper the GFEM is constructed and its convergence is proved for the Neumann problem. The estimates are carried out in a factor space where the operator is injective.

The main advantages of the obtained GFEM are easy algorithmization and preserving the ellipticity bounds of the factorized differential operator in the ratio of linear convergence.

The paper is organized as follows. In Chapter 2 theoretical background is given. The next chapter describes the Sobolev space GM. Finally, Chapter 4 is devoted to the construction and error estimate of our method.

2. Formulation of the problem

In this section theoretical background is given for the BVP. The conditions and some Sobolev space properties of the problem are formulated.

In the sequel $\Omega \subset \mathbb{R}^N$ is a given bounded domain, and we use the notations

$$\begin{aligned} D &\equiv \left\{ u \in H^2(\Omega): \frac{\partial u}{\partial \nu}\big|_{\partial\Omega} = 0 \text{ in trace sense} \right\}, \\ \tilde{L}^2(\Omega) &\equiv \left\{ u \in L^2(\Omega): \int_\Omega u\,dx = 0 \right\}. \end{aligned} \tag{2}$$

Definition 1. The bounded domain $\Omega \subset \mathbb{R}^N$ is called *regular* if the range of the operator $-\Delta$ on D equals $\tilde{L}^2(\Omega)$.

For example, the following classes of domains are regular:

(a) Ω has C^2 boundary [3];
(b) Ω is a cube [4].

Remark 1. If $\Omega \subset \mathbb{R}^2$ is a polygon, then a related result for $p > 2$ is that the range of the operator $-\Delta$ on D equals $\tilde{L}^p(\Omega)$ if in (2) we replace $H^2(\Omega)$ by $W^{2,p}$ (see [7]).

We consider the Neumann problem (1)

$$\begin{cases} T(u) & \equiv -\mathrm{div}\,(f(x,\nabla u)\nabla u) \;=\; g(x) \\ \dfrac{\partial u}{\partial \nu}\big|_{\partial\Omega} & = 0 \end{cases}$$

with the following conditions:

(C1) Ω is a regular domain.

(C2) $f \in C^1(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^+)$ and $g \in L^2(\Omega)$ are real scalar-valued functions.

(C3) $|\dfrac{\partial f}{\partial x_i}(x,p)| \le c|p| \quad ((x,p) \in \overline{\Omega} \times \mathbb{R}^N)$ with some constant $c > 0$.

(C4) Let $\Phi : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ be defined by $\Phi(x,p) = f(x,p)p$. Then there exist constants $0 < m \le M$ such that for any $(x,p) \in \overline{\Omega} \times \mathbb{R}^N$ the matrix $\dfrac{\partial \Phi}{\partial p}(x,p)$ is symmetric and has eigenvalues between m and M.

Remark 2. An important special case of T satisfying conditions (C2)–(C4) is of the form

$$T(u) \equiv -\mathrm{div}\,(a(|\nabla u|)\nabla u)\,, \tag{3}$$

where $a \in PC^1(\mathbb{R}^+)$, $0 < m \le a(r) \le (ra(r))' \le M \quad (r > 0)$. Here $\Phi(x,p) = a(|p|)p$ satisfies

$$\frac{\partial}{\partial p}\,(a(|p|)p)\;\xi \cdot \xi = a(|p|)|\xi|^2 + \frac{a'(|p|)}{|p|}(p \cdot \xi)^2,$$

whence

$$\begin{aligned} m|\xi|^2 \le a(|p|)|\xi|^2 &\le \frac{\partial \Phi(x,p)}{\partial p}\,\xi \cdot \xi \\ &\le \big(a(|p|) + a'(|p|)|p|\big)\,|\xi|^2 \le M|\xi|^2. \end{aligned} \tag{4}$$

For example, the function a, being the stress-strain connection in plasticity theory obtained using the measurements in [14]:

$$a(r) = \begin{cases} \dfrac{1.02}{1+\sqrt{1-r^2/3}} & \text{if } 0 \le r \le r_0 = 1.66; \\ a(r_0) = 0.7951 & \text{if } r \ge r_0 = 1.66 \end{cases} \tag{5}$$

satisfies the above conditions.

Besides plasticity theory, the kind of operator (3) also arises in connection with magnetic potential, see e.g. [6, 15]. We note that in these physical examples the unknown function u is a potential type quantity (magnetic or stress). The boundary condition $\partial u/\partial \nu = 0$ on $\partial\Omega$ expresses that the corresponding field ∇u is parallel to the boundary.

The domain of T is the set D defined in formula (2).

We introduce the real Hilbert space $H = L^2(\Omega)$ with scalar product $\langle u, v\rangle \equiv \int_\Omega uv\,dx$, further, the real Sobolev space $H^1(\Omega)$ with scalar product

$$\langle u, v\rangle_m \equiv \int_\Omega \nabla u \cdot \nabla v\,dx + \frac{1}{|\Omega|^2}\Big(\int_\Omega u\,dx\Big)\Big(\int_\Omega v\,dx\Big) \tag{6}$$

(where $|\Omega|$ stands for the Lebesgue measure of Ω), i.e. the mean values of u and v are used instead of the usual scalar product

$$\langle u, v\rangle_1 \equiv \int_\Omega \nabla u \cdot \nabla v\,dx + \int_\Omega uv\,dx\,.$$

The following properties are found in [13].

Remark 3. The norms $\|\ \|_1$ and $\|\ \|_m$ (corresponding to the above scalar products) are equivalent due to the Poincaré inequality.

We define the one-dimensional subspace

$$H_0 = \{u \in L^2(\Omega):\ u(x) \equiv \text{const on } \Omega\}. \tag{7}$$

Remark 4. (a) We have both in $L^2(\Omega)$ and in $H^1(\Omega)$

$$u \in H_0^\perp \Leftrightarrow \int_\Omega u\,dx = 0\,.$$

(b) If $u \in H^1(\Omega)$ and $u \in H_0^\perp$ then $\|u\|_m^2 = \int_\Omega |\nabla u|^2\,dx.$

The corresponding generalized differential operator $F : H^1(\Omega) \to H^1(\Omega)$ is defined by

$$\langle F(u), v\rangle_m = \int_\Omega f(x, \nabla u)\nabla u \cdot \nabla v \, dx \quad (v \in H^1(\Omega)). \tag{8}$$

Clearly, setting $v \equiv$ const implies that $F(u) \in H_0^\perp$. We note that, consequently, $z = F(u)$ is equivalent to

$$\begin{cases} \int_\Omega \nabla z \cdot \nabla v \, dx = \int_\Omega f(x, \nabla u)\nabla u \cdot \nabla v \, dx \qquad (v \in H^1(\Omega)), \\ \int_\Omega z \, dx = 0 \, . \end{cases}$$

Remark 5. Let $u \in D \cap H_0^\perp$, i.e. $u \in H^2(\Omega)$, $\frac{\partial u}{\partial \nu}\big|_{\partial\Omega} = 0$, $\int_\Omega u \, dx = 0$. Then $z = F(u)$ satisfies

$$\langle z, v\rangle_m = \int_\Omega \nabla z \cdot \nabla v \, dx = -\int_\Omega (\Delta z) v \, dx = \int_\Omega T(u) v \, dx \quad (v \in H^1(\Omega)),$$

hence z becomes the strong solution of the equation $-\Delta z = T(u)$. This implies the decomposition

$$F(u) = (-\Delta)^{-1} T(u) \, .$$

(When u is not as above, then we have only the nonconstructive definition (8).)

The conditions ensure that $F'_{|H_0^\perp}$ is uniformly elliptic, namely,

$$\begin{aligned} m\|h\|_m^2 &\le \langle F'(u)h, h\rangle_m \\ &\le M\|h\|_m^2 \qquad (u, h \in H^1(\Omega), \int_\Omega h dx = 0), \end{aligned} \tag{9}$$

where m and M are the bounds defined in condition (C4).

A *weak solution* of (1) is defined in the usual way as a function $u \in H^1(\Omega)$ satisfying

$$\langle F(u), v\rangle_m = \int_\Omega g v \, dx \quad (v \in H^1(\Omega)).$$

3. The Sobolev space gradient method

As mentioned in the introduction, the main difficulty to extend the gradient method to the Neumann problem is non-injectivity. This can be overcome by suitable factorization. We rely on our theoretical results formulated in [13].

Theorem 1. *Let the conditions* (C1)–(C4) *be fulfilled and assume that* $\int_\Omega g\,dx = 0$. *Then*

(1) *problem* (1) *has a unique weak solution* $u^* \in H^1(\Omega)$ *such that* $\int_\Omega u^*\,dx = 0$. *The set of solutions is* $\{u^* + c : c \in \mathbb{R}\}$. (*If assumption* $\int_\Omega g\,dx = 0$ *fails to hold then there exists no solution.*)

(2) *Let* $u_0 \in H^2(\Omega)$, $\frac{\partial u_0}{\partial \nu}\big|_{\partial\Omega} = 0$, $\int_\Omega u_0\,dx = 0$. *For any* $n \in \mathbb{N}$ *let*

$$\begin{cases} u_{n+1} = u_n - \dfrac{2}{M+m} z_n\,, \\ \\ \text{where } z_n \in H^2(\Omega) \text{ is the (unique) solution of equation} \\ -\Delta z_n = T(u_n) - g,\ \dfrac{\partial z_n}{\partial \nu}\Big|_{\partial\Omega} = 0,\ \displaystyle\int_\Omega z_n\,dx = 0. \end{cases}$$

Then (u_n) *converges to* u^* *according to the linear estimate*

$$\|\nabla u_n - \nabla u^*\|_{L^2(\Omega)} \le \frac{1}{mp^{1/2}} \|T(u_0) - g\|_{L^2(\Omega)} \left(\frac{M-m}{M+m}\right)^n \qquad (n \in \mathbb{N})\,, \tag{10}$$

where p *is the smallest positive eigenvalue of* $-\Delta$ *on the domain* D *defined in* (2).

Proof. It follows similarly as Theorem 3.1 in [13], now substituting the domain D as defined in (2) instead of

$$D = \left\{ u \in C^{2,\gamma}(\overline{\Omega}) : \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0 \right\}.$$

Since Ω is regular, the solution of the auxiliary equation satisfies $z_n \in D$ in each step. Using this, the proof goes on as in the cited theorem, as referred to also Remark 3.6 in [13]. □

Remark 6. We have linear convergence in the usual $H^1(\Omega)$ norm as well, due to Remarks 3 and 4, since by $u_n - u^* \in H_0^\perp$ the left side of (10) equals $\|u_n - u^*\|_m$.

Remark 7. For example, let us consider the problem quoted in Remark 2. The corresponding equation was examined in [5] with Dirichlet boundary conditions. Owing to (9) and Remark 4 (b), the ellipticity bounds obtained in that paper remain valid: $m = 0.51,\ M = 2.81$. Hence the corresponding convergence quotient for the GM is

$$\frac{M-m}{M+m} = 0.6929.$$

The algorithmic form of the GM in $H^1(\Omega)$ is the following:

$$\begin{cases} (1) \quad u_0 \in H^2(\Omega),\ \dfrac{\partial u_0}{\partial \nu}\big|_{\partial\Omega} = 0,\ \displaystyle\int_\Omega u_0 dx\ = 0; \\ \qquad \text{for any } \ n \in \mathbb{N}: \\ (2a) \quad r_n = T(u_n) - g, \\ (2b) \quad -\Delta z_n = r_n,\ \dfrac{\partial z_n}{\partial \nu}\big|_{\partial\Omega} = 0,\ \displaystyle\int_\Omega z_n\, dx\ = 0, \\ (2c) \quad u_{n+1} = u_n - \dfrac{2}{M+m} z_n\ . \end{cases} \tag{11}$$

The summary of this section is that the gradient method (11) gives a theoretically well-defined sequence which converges linearly. The solution of equation $T(u) = g$ is reduced to a sequence of linear Poisson equations:

$$-\Delta z_n = r_n,\ \frac{\partial z_n}{\partial \nu}\big|_{\partial\Omega} = 0,\ \int_\Omega z_n\, dx\ = 0, \tag{12}$$

where $z_n \in H^2(\Omega)$. To turn this theoretical algorithm into a numerical one, we must specify the way in which z_n is computed numerically.

Remark 8. We note that the algorithm (11) can also be defined for non-regular domains and initial guess $u \in H^1(\Omega)$ if in the auxiliary problem (2b) the weak form is considered. Moreover, the theorems to follow on the gradient-finite element method can also be generalized without modification of the reasoning to this setting. The strong form in the algorithm is proposed owing to the qualitative aspects to be described in Subsection 4.2.

4. The gradient-finite element method

From numerical aspect, the main point of realizing the GM algorithm (11) is the solution of the auxiliary Poisson equations (12). We will define a combined method in which these auxiliary equations are computed numerically by the finite element method. This method has been introduced in [5] for Dirichlet problems and called gradient-finite element method (GFEM). Now its analogue is developed for our Neumann problem (1).

In the following subsections first the construction of the GFEM algorithm is given, then the qualitative effect of strong solutions is briefly referred to, and finally the convergence of the GFEM is proved.

4.1. Construction.

The FEM is able to provide numerical solutions of the auxiliary equations in $H^2(\Omega)$ in which the exact solutions z_n are. Then in the next steps of the iteration we can apply the operator T directly to the numerically computed u_{n+1} to obtain r_{n+1}. In order to define completely the combined method, it remains to determine the accuracy of the numerical computation of the functions z_n.

These considerations give the following algorithm: let $(\delta_n) \subset \mathbb{R}^+$ be a sequence such that $\delta_n \to 0$. Then

$$
\begin{cases}
(1) \quad \overline{u}_0 \in H^2(\Omega)\,, \dfrac{\partial \overline{u}_0}{\partial \nu}\Big|_{\partial\Omega} = 0, \displaystyle\int_\Omega \overline{u}_0 dx \,= 0; \\
\qquad \text{for any } \; n \in \mathbb{N}: \\
(2a) \quad \overline{r}_n = T(\overline{u}_n) - g, \\
(2b) \quad (z_n^* \in H^2(\Omega) \;\text{ denotes the exact solution of} \\
\qquad - \Delta z_n^* = \overline{r}_n,\; \dfrac{\partial z_n^*}{\partial \nu}\Big|_{\partial\Omega} = 0,\; \displaystyle\int_\Omega z_n^*\, dx \,= 0); \\
\qquad \overline{z}_n \approx z_n^* \text{ using FEM such that} \\
\qquad \overline{z}_n \in H^2(\Omega),\; \dfrac{\partial \overline{z}_n}{\partial \nu}\Big|_{\partial\Omega} = 0,\; \displaystyle\int_\Omega \overline{z}_n\, dx \,= 0 \text{ and } \; \|z_n^* - \overline{z}_n\|_m \le \delta_n\,, \\
(2c) \quad \overline{u}_{n+1} = \overline{u}_n - \dfrac{2}{M+m}\overline{z}_n\,.
\end{cases}
\tag{13}
$$

In other words, $\overline{z}_n$ is the numerically computed solution of the auxiliary equation

$$
-\Delta z \,=\, \overline{r}_n\,, \quad \frac{\partial z}{\partial \nu}\Big|_{\partial\Omega} = 0,\; \int_\Omega z\, dx \,= 0
$$

in $H^2(\Omega)$ with accuracy δ_n in $H^1(\Omega)$ norm.

4.2. Qualitative aspects.

Our problem (1) satisfies the following C^1-regularity result, following from [16], under slightly strengthened conditions on Ω.

Theorem 2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with piecewise C^2 boundary, locally convex at the corners. Then the weak solution of* (1) *satisfies* $u \in C^1(\overline{\Omega})$.

Consequently, from qualitative aspects, a reasonable numerical method is expected to produce a numerical solution which preserves the smoothness $C^1(\overline{\Omega})$ of the exact solution.

In course of the FEM solution, this requirement means that the derivatives of the corresponding polynomials must also agree on the boundary of the triangles (i.e. C^1-elements are used). The requirement $z_n \in C^1(\overline{\Omega})$ in the FEM subspaces (consisting of piecewise polynomials) is equivalent to $z_n \in H^2(\Omega)$. This suits the algorithm (13) which defines the numerical solution of the Poisson equations in $H^2(\Omega)$, and explains the choice of the strong formulation for the auxiliary problems.

As is clear, this higher order approximation leads to more arithmetic operations than for the more usual lower degree elements. However, the reason for its usage in general is presented in literature (see e.g. [2, 21]), it is also a basis for the *hp*-version [19]. In our case, the extra work with C^1-elements is above all justified from the above detailed qualitative aspects.

4.3. Convergence.

The main difficulty to overcome is non-injectivity, therefore we study the problem in the subspace $H_0^\perp$ of $H^1(\Omega)$ (defined in Section 2, see (7) and afterwards) in which the generalized differential operator F is one-to-one and the sequence $\overline{u}_n$ was defined.

The convergence estimates on $(\overline{u}_n)$ are obtained from the investigation of

$$E_n = \|\overline{u}_n - u_n\|_m$$

where (u_n) is defined by the theoretical GM (11) with $u_0 = \overline{u}_0$. Here $\|\,.\,\|_m$ denotes the norm on $H^1(\Omega)$ defined in (6), equivalent to the usual norm.

The proof of convergence is achieved by the suitable generalization of Lemmas 3.1–3.2 and Theorem 3.1 in [5].

Lemma 1. *Let* $J : H_0^\perp \to H_0^\perp$, $J(u) = u - [2/(M+m)]F(u)$. *Then*

$$\|J(u) - J(v)\|_m \le \frac{M-m}{M+m}\|u-v\|_m \quad (u, v \in H_0^\perp).$$

Proof. The operator J satisfies

$$J'(u)h = h - \frac{2}{M+m}F'(u)h \quad (u, h \in H_0^\perp). \tag{14}$$

Let $u, v \in H_0^\perp$ be fixed. Then

$$\begin{aligned} J(u) - J(v) &= \int_0^1 J'(v + t(u-v))(u-v)dt \\ &= u - v - \frac{2}{M+m}\int_0^1 F'(v + t(u-v))(u-v)dt. \end{aligned} \tag{15}$$

Let $A : H_0^\perp \to H_0^\perp$,

$$Ar = r - \frac{2}{M+m} \int_0^1 F'(v + t(u-v))r dt.$$

Then A is a bounded self-adjoint linear operator since $F'(w)$ has these properties for all $w \in H_0^\perp$. Further, using (9),

$$\begin{aligned} -\frac{M-m}{M+m}\|r\|_m^2 &= \|r\|_m^2 - \frac{2}{M+m} M\|r\|_m^2 \\ &\le \langle Ar, r\rangle_m \le \|r\|_m^2 - \frac{2}{M+m} m\|r\|_m^2 = \frac{M-m}{M+m}\|r\|_m^2 \quad (r \in H_0^\perp). \end{aligned}$$

Hence $\|A\| \le (M-m)/(M+m)$. Since by (15) $J(u) - J(v) = A(u-v)$, the lemma is proved. □

Lemma 2. *For all $n \in \mathbb{N}$ there holds the estimate*

$$E_{n+1} \le \frac{M-m}{M+m} E_n + \frac{2}{M+m}\delta_n.$$

Proof. The sequences u_n and $\overline{u}_n$ satisfy

$$\begin{aligned} u_{n+1} - \overline{u}_{n+1} &= u_n - \frac{2}{M+m} z_n - (\overline{u}_n - \frac{2}{M+m}\overline{z}_n) \\ &= u_n - \overline{u}_n - \frac{2}{M+m}(z_n - z_n^*) - \frac{2}{M+m}(z_n^* - \overline{z}_n). \end{aligned}$$

Here $z_n - z_n^* = (-\Delta)^{-1}(T(u_n) - T(\overline{u}_n)) = F(u_n) - F(\overline{u}_n)$. Hence (using Lemma 1 and (13))

$$u_{n+1} - \overline{u}_{n+1} = J(u_n) - J(\overline{u}_n) - \frac{2}{M+m}(z_n^* - \overline{z}_n),$$

$$\begin{aligned} \|u_{n+1} - \overline{u}_{n+1}\|_m &\le \|J(u_n) - J(\overline{u}_n)\|_m + \frac{2}{M+m}\|z_n^* - \overline{z}_n\|_m \\ &\le \frac{M-m}{M+m}\|u_n - \overline{u}_n\|_m + \frac{2}{M+m}\delta_n. \end{aligned}$$

Now we are in the position to prove our main theorem. We will impose the requirement $\delta_n \le c_1 q^n$ for the errors in the auxiliary equations with some $0 < q < 1$. As can be expected, the estimate for the final error will satisfy

$$\|\overline{u}_n - u^*\|_m \le \text{const} \cdot \max\left\{q, \frac{M-m}{M+m}\right\}^n.$$

The proof of this will need different calculations depending on the relation of q and $(M-m)/(M+m)$. □

Theorem 3. *Let $0 < q < 1$ be fixed, $c_1 > 0$ and $\delta_n \le c_1 q^n \quad (n \in \mathbb{N})$. Then (with a suitable constant $c_2 > 0$) the following estimates hold for all n:*

(a) *If* $q > \dfrac{M-m}{M+m}$ *then* $\|\overline{u}_n - u^*\|_m \le c_2 q^n$.

(b) *If* $q < \dfrac{M-m}{M+m}$ *then* $\|\overline{u}_n - u^*\|_m \le c_2 \left(\dfrac{M-m}{M+m}\right)^n$.

Proof. Since Theorem 1 yields

$$\|\overline{u}_n - u^*\|_m \le \|\overline{u}_n - u_n\|_m + \|u_n - u^*\|_m \le \|\overline{u}_n - u_n\|_m + c_3 \left(\frac{M-m}{M+m}\right)^n ,$$

it suffices to verify our estimates (a)–(b) for $E_n = \|\overline{u}_n - u_n\|_m$ instead of $\|\overline{u}_n - u^*\|_m$.

We use notations $Q = (M-m)/(M+m)$ and $\alpha = 2/(M+m)$. Then Lemma 2 asserts

$$E_{n+1} \le QE_n + \alpha\delta_n .$$

(a) Let $c_2 = \alpha c_1/(q-Q)$. We prove by induction

$$E_n \le c_2 q^n \quad (n \in \mathbb{N}). \tag{16}$$

Since $E_0 = \|\overline{u}_0 - u_0\|_m = 0$, (16) is trivial for $n = 0$.
If (16) holds for fixed $n \in \mathbb{N}$, then

$$\begin{aligned} E_{n+1} &\le QE_n + \alpha\delta_n \le c_2 Q q^n + c_1 \alpha q^n \\ &= \left(\frac{\alpha c_1 Q}{q-Q} + c_1\alpha\right) q^n = \frac{\alpha c_1}{q-Q} q^{n+1} = c_2 q^{n+1}. \end{aligned}$$

(b) Let $r = q/Q$, $c_2 = (\alpha c_1 r)/[q(1-r)]$. We prove by induction

$$E_n \le c_2(1-r^n)Q^n \quad (n \in \mathbb{N}). \tag{17}$$

For $n = 0$ this is again trivial.
If (17) holds for fixed $n \in \mathbb{N}$, then

$$\begin{aligned} E_{n+1} &\le QE_n + \alpha\delta_n \le c_2(1-r^n)Q^{n+1} + c_1\alpha q^n \\ &= \frac{\alpha c_1 r(1-r^n)}{q(1-r)} Q^{n+1} + \frac{\alpha c_1}{q} r^{n+1} Q^{n+1} \\ &= \frac{\alpha c_1 r}{q}\left(\frac{1-r^n}{1-r} + r^n\right) Q^{n+1} \\ &= \frac{\alpha c_1 r}{q(1-r)}(1-r^{n+1})Q^{n+1} = c_2(1-r^{n+1})Q^{n+1}. \end{aligned}$$

The hereby verified inequality (17) yields the desired estimate

$$E_n \le c_2 Q^n \quad (n \in \mathbb{N}).$$

□

Remark 9. In the case $q = (M-m)/(M+m)$ the convergence estimate is faster than s^n for any $s > (M-m)/(M+m)$, but is slower than $[(M-m)/(M+m)]^n$.

Remark 10. Denoting by h_n the width of the mesh used in (2b) in (13), we have the estimate

$$\begin{aligned}\|\overline{z}_n - z_n^*\|_m &\le Ch_n\|z_n^*\|_{H^2(\Omega)} = Ch_n\|(-\Delta)^{-1}(T(\overline{u}_n) - f)\|_{H^2(\Omega)} \\ &\le C'h_n\left(\|T(\overline{u}_n) - f\|_{L^2(\Omega)} + \|z_n^*\|_{H^1(\Omega)}\right) \\ &\le C"h_n\left(\|T(\overline{u}_n) - f\|_{L^2(\Omega)} + \|F(\overline{u}_n) - b\|_m\right)\end{aligned}$$

with suitable constants $C, C', C" > 0$, where b denotes the weak form of the right side defined by $\langle b, v\rangle_m = \int_\Omega gv \quad (v \in H^1(\Omega))$. (See [18] for the corresponding FEM estimate and [7] for the Bernstein type inequality used.) The obtained expression on the right side plays the role of δ_n. If $(h_n) \to 0$ is chosen a geometric sequence, further, $\sup\{\|T(\overline{u}_n) - f\|_{L^2(\Omega)} : n \in \mathbb{N}\} < +\infty$ and $\sup\{\|F(\overline{u}_n) - b\|_m : n \in \mathbb{N}\} < +\infty$ (which can be assumed since $(\overline{u}_n)$ is constructed to converge to the solution of equation $T(u) = f$ or equivalently $F(u) = b$), then the condition of Theorem 3 on δ_n is fulfilled, i.e. (instead of estimating δ_n in the steps) the suitably prescribed refinement of the mesh yields the required order estimate of the convergence of δ_n.

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