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AN EXPLICIT FORMULA FOR SOLUTIONS OF SOME SYSTEM OF LINEAR DELAY DIFFERENCE EQUATIONS

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Abstract. The aim of this paper is to provide an explicit formula for solutions of the following system of delay difference equations

$$F_{n+1} = A_n F_n + B_n F_{\gamma_n} + f_n, \quad n \in \mathbb{N},$$

where $\gamma_n = k\alpha_n$, $\alpha_n = [n/k]$ (the symbol $[x]$ stands for entire part of the real number x and k is a fixed positive integer). (A_n), (B_n), $n \in \mathbb{N}$, are sequences of square matrices of order m, (f_n) is a sequence of vectors from $\mathbb{R}^m$. From this formula conditions for the stability and asymptotic stability of solutions are derived.

1. Introduction

Difference equations and systems of difference equations have a wide use in many branches of science such as mathematics, physics, chemistry, engineering, computer science, biology, economy, social science and others.

Linear difference equations and systems of such equations are the simplest ones which are used to describe dynamical processes being an object of interest.

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The standard linear system of difference equations has the form

$$F_{n+1} = A_n F_n + f_n, \quad n \in \mathbb{N} = \{0, 1, \dots\}, \tag{1}$$

where (A_n) is a given sequence of square matrices of order m, (f_n) is a sequence of given vectors from $\mathbb{R}^m$, (F_n) is unknown sequence of vectors from $\mathbb{R}^m$.

The theory of the systems of the form (1) is very well developed and described in a number of textbooks (see for more recent: [1], [2], [5], [6]).

However, sometimes when modeling some dynamical processes one has to take into account delay effects. This take place even in such simple case as that when we are modeling the problem of capital deposits (see [3], [7]). In this case we arrive to the scalar difference delay equation of the form

$$F_{n+1} = a_n F_n + b_n F_{\gamma_n} + g_n, \quad n \in \mathbb{N}, \tag{2}$$

where $\gamma_n = k\alpha_n$, $\alpha_n = [n/k]$, k is a given positive integer and the symbol $[x]$ stands for the entire part of the real number x. An explicit formula for solutions of equation (2) is given in [8].

Observe that explicit formula is very useful when one wants to investigate the qualitative behavior of solutions of the equation under consideration.

When modeling more complicated economical problems which are described by systems of difference equations (see for instance [4], [9], [10] and others) we may have to take some delay effects which will lead us to linear systems of delay difference equations of the form

$$F_{n+1} = A_n F_n + B_n F_{\gamma_n} + f_n, \quad n \in \mathbb{N}, \tag{3}$$

where (A_n), (B_n) is a sequence of square matrices of order m, (f_n) is a sequence of vectors from $\mathbb{R}^m$ and γ_n has meaning as above.

The aim of the present paper is to give an explicit formula for solutions of equation (3) and to show how it can be used for the investigation of the behavior of the solutions of this equation when $n \to \infty$.

2. Main theorem

In the paper [8] it is shown that any solution of equation (2) is expressed by the formula

$$F_n = \left[F_0 \prod_{l=0}^{\alpha_n - 1} p_l + \sum_{s=0}^{\alpha_n - 1} q_s \prod_{j=s+1}^{\alpha_n - 1} p_j \right] \times \left\{ \prod_{s=k\alpha_n}^{n-1} a_s + \sum_{s=k\alpha_n}^{n-1} b_s \prod_{j=s+1}^{n-1} a_j \right\} + \sum_{s=k\alpha_n}^{n-1} g_s \prod_{j=s+1}^{n-1} a_j, \quad n \in \mathbb{N}, \tag{4}$$

where

$$p_l = \prod_{s=0}^{k-1} a_{kl+s} + \sum_{s=0}^{k-1} b_{kl+s} \prod_{j=s+1}^{k-1} a_{kl+j}, \quad q_l = \sum_{s=0}^{k-1} f_{kl+s} \prod_{j=s+1}^{k-1} a_{kl+j}, \quad l \in \mathbb{N}.$$

Here as usually it is assumed that

$$\sum_{s=p}^{p-1} d_s = 0 \qquad \text{and} \qquad \prod_{s=p}^{p-1} d_s = 1$$

for any d_s and $p \in \mathbb{N}$.

The result of the paper is a generalization of formula (4) for the systems of the form (3).

We define:

$$\prod_{s=0}^{n-1} A_s = \begin{cases} A_{n-1} \cdot A_{n-2} \cdots A_1 \cdot A_0, & n \in \mathbb{N} \setminus \{0\}, \\ I, & n = 0, \end{cases} \tag{5}$$

where I denotes identity matrix.

Assume as in the scalar case

$$\sum_{s=p}^{p-1} D_s = 0 \qquad \text{and} \qquad \prod_{s=p}^{p-1} D_s = I \tag{6}$$

for any D_s and $p \in \mathbb{N}$.

In order to find an explicit formula for solutions of equation (3) first we consider the case when $B_n \equiv 0$, i.e. B_n is zero matrix for all $n \in \mathbb{N}$. In this case equation (3) can be written as equation (1).

For this case we have the following result (see for instance [2]).

Theorem 1. *Any solution of equation* (1) *is expressed by the formula*

$$F_n = \left(\prod_{s=0}^{n-1} A_s \right) F_0 + \sum_{s=0}^{n-1} \left(\prod_{j=s+1}^{n-1} A_j \right) f_s, \quad n \in \mathbb{N}. \tag{7}$$

In order to formulate a theorem giving the formula for all solutions of system (3) we write $n = k\alpha_n + \beta_n$ where β_n is the remainder in division of n by k. For simplicity we will write $n = kl + r$ with $l = \alpha_n$ and $r = \beta_n$.

Put

$$P_l = \prod_{s=0}^{k-1} A_{kl+s} + \sum_{s=0}^{k-1} \left(\prod_{j=s+1}^{k-1} A_{kl+j} \right) B_{kl+s}, \quad l \in \mathbb{N}, \tag{8}$$

$$Q_l = \sum_{s=0}^{k-1} \left(\prod_{j=s+1}^{k-1} A_{kl+j} \right) f_{kl+s}, \quad l \in \mathbb{N}. \tag{9}$$

Now we can formulate the following theorem

Theorem 2. *Any solution of equation* (3) *is given by the formula*

$$F_n = \left[\prod_{s=k\alpha_n}^{n-1} A_s + \sum_{s=k\alpha_n}^{n-1} \left(\prod_{j=s+1}^{n-1} A_j \right) B_s \right] \times \left\{ \left(\prod_{s=0}^{\alpha_n - 1} P_s \right) F_0 + \sum_{s=0}^{\alpha_n - 1} \left(\prod_{j=s+1}^{\alpha_n - 1} P_j \right) Q_s \right\} + \sum_{s=k\alpha_n}^{n-1} \left(\prod_{j=s+1}^{n-1} A_j \right) f_s, \quad n \in \mathbb{N}. \tag{10}$$

Proof. The proof is quite similar to that for the scalar case. For a fixed nonnegative integer $n = kl + r$ put

$$\Phi_r = F_{kl+r}, \quad r = \{0, 1, \dots, k\},$$

where (F_n), $n \in \mathbb{N}$, is a solution of equation (3). Now for $r = \{0, 1, \dots, k-1\}$, from equation (3) we get

$$\Phi_{r+1} = A_{kl+r} \Phi_r + B_{kl+r} \Phi_0 + f_{kl+r}.$$

Using formula (7) with corresponding changes we find

$$\Phi_r = \left(\prod_{s=0}^{r-1} A_{kl+s} \right) \Phi_0 + \sum_{s=0}^{r-1} \left(\prod_{j=s+1}^{r-1} A_{kl+j} \right) B_{kl+s} \Phi_0 + \sum_{s=0}^{r-1} \left(\prod_{j=s+1}^{r-1} A_{kl+j} \right) f_{kl+s}$$

for $r = \{0, 1, \dots, k\}$. This means that

$$F_{kl+r} = \left[\prod_{s=0}^{r-1} A_{kl+s} + \sum_{s=0}^{r-1} \left(\prod_{j=s+1}^{r-1} A_{kl+j} \right) B_{kl+s} \right] F_{kl} + \sum_{s=0}^{r-1} \left(\prod_{j=s+1}^{r-1} A_{kl+j} \right) f_{kl+s} \tag{11}$$

for $r = \{0, 1, \dots, k\}$.

Take $R_l = F_{kl}$. Then from the last formula for $r = k$ we get

$$R_{l+1} = P_l R_l + Q_l, \quad l \in \mathbb{N}.$$

This is an equation of form (1) so according to Theorem 1 we have

$$R_l = \left(\prod_{s=0}^{l-1} P_s\right) R_0 + \sum_{s=0}^{l-1} \left(\prod_{j=s+1}^{l-1} P_j\right) Q_s, \quad l \in \mathbb{N}\,.$$

Because $R_l = F_{kl}$ then from formula (11) we get

$$\begin{aligned} F_{kl+r} = &\left[\prod_{s=0}^{r-1} A_{kl+s} + \sum_{s=0}^{r-1} \left(\prod_{j=s+1}^{r-1} A_{kl+j}\right) B_{kl+s}\right] \times \left\{ \left(\prod_{s=0}^{l-1} P_s\right) F_0 \right. \\ &\left. + \sum_{s=0}^{l-1} \left(\prod_{j=s+1}^{l-1} P_j\right) Q_s \right\} + \sum_{s=0}^{r-1} \left(\prod_{j=s+1}^{r-1} A_{kl+j}\right) f_{kl+s} \end{aligned}$$

for $l \in \mathbb{N}$ and $r = \{0, 1, \dots, k\}$.

If we use this formula for $l = 0, 1, \dots,$ $r = \{0, 1, \dots, k-1\}$ and $n = kl+r$ then we have $l = \alpha_n$, $r = \beta_n$ and we can write

$$\begin{aligned} F_n = &\left[\prod_{s=0}^{\beta_n-1} A_{ka_n+s} + \sum_{s=0}^{\beta_n-1} \left(\prod_{j=s+1}^{\beta_n-1} A_{ka_n+j}\right) B_{ka_n+s}\right] \times \left\{ \left(\prod_{s=0}^{a_n-1} P_s\right) F_0 \right. \\ &\left. + \sum_{s=0}^{a_n-1} \left(\prod_{j=s+1}^{a_n-1} P_j\right) Q_s \right\} + \sum_{s=0}^{\beta_n-1} \left(\prod_{j=s+1}^{\beta_n-1} A_{ka_n+j}\right) f_{ka_n+s}, \quad n \in \mathbb{N}. \end{aligned}$$

Finally after corresponding shifts of indices in the product and summation symbols we arrive at the formula of the Theorem. The proof is complete. □

3. Asymptotic behavior of solutions

Observe that the solution (10) can be written in the form

$$F_n = \Phi_n F_0 + \Psi_n, \quad n \in \mathbb{N}, \tag{12}$$

where

$$\Phi_n = \left[\prod_{s=k\alpha_n}^{n-1} A_s + \sum_{s=k\alpha_n}^{n-1} \left(\prod_{j=s+1}^{n-1} A_j\right) B_s\right] \cdot \left(\prod_{s=0}^{\alpha_n-1} P_s\right), \quad n \in \mathbb{N}, \tag{13}$$

and

$$\Psi_n = \left[\prod_{s=k\alpha_n}^{n-1} A_s + \sum_{s=k\alpha_n}^{n-1} \left(\prod_{j=s+1}^{n-1} A_j \right) B_s \right] \cdot \left(\sum_{s=0}^{\alpha_n - 1} \left(\prod_{j=s+1}^{\alpha_n - 1} P_j \right) Q_s \right) + \sum_{s=k\alpha_n}^{n-1} \left(\prod_{j=s+1}^{n-1} A_j \right) f_s, \quad n \in \mathbb{N}. \tag{14}$$

The matrix Φ_n we will call the fundamental matrix of homogeneous system

$$F_{n+1} = A_n F_n + B_n F_{\gamma_n}, \quad n \in \mathbb{N}. \tag{15}$$

It is easy to see that Ψ_n is a particular solution of nonhomogeneous system (3). Having formula (12) we can answer easily questions about the behavior of solutions of system (3). We see that all solutions of system (3) are bounded if there are constants $M_0 > 0$ and $M_1 > 0$ such that

$$\|\Phi_n\| \le M_0, \quad n \in \mathbb{N}, \tag{16}$$

and

$$\|\Psi_n\| \le M_1, \quad n \in \mathbb{N}. \tag{17}$$

Also one can see easily that the trivial solution of system (15) is stable if condition (16) holds.

From the linearity of system (3) it follows that the stability of any solution of this system is equivalent to the stability of the trivial solution of homogeneous system (15), so we can state that any solution of system (3) is stable if and only if the condition (16) holds.

To get the asymptotic stability of any solution of system (3) it is enough to assume that there exist two positive numbers M_2 and η, $\eta < 1$, such that

$$\|\Phi_n\| \le M_2 \eta^n, \quad n \in \mathbb{N}. \tag{18}$$

Let us now evaluate the norm of Φ_n. Assume that

$$\|A_n\| \le M \quad \text{and} \quad \|B_n\| \le b, \quad n \in \mathbb{N}, \tag{19}$$

than from the formulas (8) and (13) we have

$$\|P_l\| \le \prod_{s=0}^{k-1} \|A_{kl+s}\| + \sum_{s=0}^{k-1} \left(\prod_{j=s+1}^{k-1} \|A_{kl+j}\| \right) \|B_{kl+s}\|, \quad l \in \mathbb{N},$$

and

$$\|\Phi_n\| \le \left[\prod_{s=k\alpha_n}^{n-1} \|A_s\| + \sum_{s=k\alpha_n}^{n-1} \left(\prod_{j=s+1}^{n-1} \|A_j\| \right) \|B_s\| \right] \prod_{l=0}^{\alpha_n - 1} \|P_l\|, \quad n \in \mathbb{N}.$$

Hence

$$\|P_l\| \le M^k + b\frac{1-M^k}{1-M}, \quad l \in \mathbb{N},$$

and

$$\|\Phi_n\| \le \left(M^{\beta_n} + b\frac{1-M^{\beta_n}}{1-M}\right)\left(M^k + b\frac{1-M^k}{1-M}\right)^{\alpha_n}, \quad n \in \mathbb{N}.$$

Now assume that $M < 1$ then

$$M^{\beta_n} + b\frac{1-M^{\beta_n}}{1-M} \le 1 + b\frac{1-M^k}{1-M}, \quad n \in \mathbb{N},$$

and

$$\|\Phi_n\| \le \left(1 + b\frac{1-M^k}{1-M}\right)\left(M^k + b\frac{1-M^k}{1-M}\right)^{\alpha_n}, \quad n \in \mathbb{N}.$$

Because $\alpha_n = (n-\beta_n)/k$ then

$$\|\Phi_n\| \le \left(1 + b\frac{1-M^k}{1-M}\right)\left[\left(M^k + b\frac{1-M^k}{1-M}\right)^{1/k}\right]^{n-\beta_n}, \quad n \in \mathbb{N},$$

and

$$\|\Phi_n\| \le M_2\left[\left(M^k + b\frac{1-M^k}{1-M}\right)^{1/k}\right]^{n}, \quad n \in \mathbb{N}, \tag{20}$$

where

$$M_2 = \left(1 + b\frac{1-M^k}{1-M}\right)\max_{0\le i\le k-1}\left(M^k + b\frac{1-M^k}{1-M}\right)^{-i/k}. \tag{21}$$

Now we are in the position to state the following

Theorem 3. *If the conditions* (19) *are satisfied and* $M + b < 1$ *then all solutions of the system* (3) *are asymptotically stable.*

Proof. It is obvious that the inequality $M + b < 1$ is equivalent to the following one

$$M^k + b\frac{1-M^k}{1-M} < 1.$$

From relation (20) it follows that inequality (18) holds for

$$\eta = \left(M^k + b\frac{1-M^k}{1-M}\right)^{1/k}.$$

□

Remark 4. Notice that for $M \geq 0$, $b > 0$ satisfying condition $M + b < 1$ the inequality

$$\left(M^k + b\frac{1 - M^k}{1 - M} \right)^{1/k} > M + b \tag{22}$$

for $k > 1$ holds. This means that for such k the convergence of $\|\Phi_n\|$ to zero is in general slower than for the case $k = 1$ when there is no delay effect.

Proof. Indeed, first observe that inequality (22) is equivalent to the following one

$$M^k + b\frac{1 - M^k}{1 - M} > (M + b)^k \, .$$

The last inequality can be proved by mathematical induction rule as follows. It is easy to see that the inequality holds for $k = 2$.
Now, if we assume that inequality (22) holds for a fixed k then one can check easily that

$$(M + b)^{k+1} < (M + b) \left(M^k + b\frac{1 - M^k}{1 - M} \right) < M^{k+1} + b\frac{1 - M^{k+1}}{1 - M}$$

what means that inequality holds for $k + 1$ and the conclusion is implied by the induction rule. □

4. Linear autonomous systems

Now let us consider the case of linear autonomous system

$$F_{n+1} = AF_n + BF_{\gamma_n} + f_n, \quad n \in \mathbb{N}. \tag{23}$$

In this case using the explicit formula for solutions of such systems we are able to establish necessary and sufficient conditions for asymptotic stability of solutions of autonomous systems.

It is obvious that we can consider only the homogeneous system corresponding to system (23)

$$F_{n+1} = AF_n + BF_{\gamma_n}, \quad n \in \mathbb{N}. \tag{24}$$

Let us list the special cases of equation (24):

a) $B = 0$, then

$$\Phi_n = A^n, \quad n \in \mathbb{N},$$

and the trivial solution of the corresponding system is asymptotically stable if and only if

$$\rho\,(A) < 1$$

(here $\rho\,(A)$ denotes the spectral radius of the matrix A).

b) $A = 0$, then

$$\Phi_n = \begin{cases} B^{\alpha_n}, & \beta_n = 0, \qquad n \in \mathbb{N}, \\ B^{\alpha_n+1}, & \beta_n \in \{1, 2, \dots, k-1\}, \quad n \in \mathbb{N}, \end{cases}$$

and the trivial solution of the corresponding system is asymptotically stable if and only if

$$\rho(B) < 1.$$

c) $A = I$, then

$$\Phi_n = (I + \beta_n B)(I + kB)^{\alpha_n}, \quad n \in \mathbb{N},$$

and the trivial solution of the corresponding system is asymptotically stable if and only if

$$\rho(I + kB) < 1.$$

d) $B = I$, then

$$\Phi_n = \left[A^{\beta_n} + (A-I)^{-1}\left(A^{\beta_n} - I\right)\right]\left[A^k + (A-I)^{-1}\left(A^k - I\right)\right]^{\alpha_n}$$

and the trivial solution of the corresponding system is asymptotically stable if and only if

$$\rho\left(A^k + (A-I)^{-1}\left(A^k - I\right)\right) < 1.$$

e) the general case, then

$$\Phi_n = \left[A^{\beta_n} + (A-I)^{-1}\left(A^{\beta_n} - I\right)B\right]\left[A^k + (A-I)^{-1}\left(A^k - I\right)B\right]^{\alpha_n}$$

and the trivial solution of the corresponding system is asymptotically stable if and only if

$$\rho\left(A^k + (A-I)^{-1}\left(A^k - I\right)B\right) < 1.$$

To prove this assertion it is enough to observe that the first term in the formula for Φ_n is bounded because β_n is bounded. On the other hand it is known that for any $\varepsilon > 0$ there is a norm $\|\cdot\|_\varepsilon$ in $\mathbb{R}^m$ such that

$$\left\|A^k + (A-I)^{-1}\left(A^k - I\right)\right\|_\varepsilon < \rho\left(A^k + (A-I)^{-1}\left(A^k - I\right)\right) + \varepsilon.$$

Taking ε small enough we have

$$\rho\left(A^k + (A-I)^{-1}\left(A^k - I\right)\right) + \varepsilon < 1.$$

Now we have

$$\|\Phi_n\|_\varepsilon \le Q\left(\rho\left(A^k + (A-I)^{-1}\left(A^k - I\right)\right) + \varepsilon\right)^{\alpha_n}$$

for some $Q > 0$. The rest of the proof is standard reasoning, the same concerns the proof of necessity of the condition mentioned above.

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