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# A QUASISTATIC CONTACT PROBLEM WITH SLIP DEPENDENT COEFFICIENT OF FRICTION FOR ELASTIC MATERIALS

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**Abstract.** We consider a mathematical model which describes the frictional contact between a deformable body and an obstacle, say a foundation. The body is assumed to be linear elastic and the contact is modeled with a version of Coulomb's law of dry friction in which the normal stress is prescribed on the contact surface. The novelty consists here in the fact that we consider a slip dependent coefficient of friction and a quasistatic process. We present two alternative yet equivalent formulations of the problem and establish existence and uniqueness results. The proofs are based on a new result obtained in [10] in the study of evolutionary variational inequalities.

## 1. Introduction

Contact phenomena among deformable bodies abound in industry and everyday life and play an important role in structural and mechanical systems. The complicated surface structure, physics and chemistry involved in

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contact processes make it necessary to model them with highly nonlinear initial-boundary value problems.

An early attempt to study frictional contact problems within the theory of variational inequalities was made in [4]. An excellent reference on analysis and numerical approximations of contact problems involving elastic materials with or without friction is [8]. The mathematical, mechanical and numerical state of the art can be found in the proceedings [15] and in the special issue [17].

Quasistatic contact process arise when the forces applied to a system vary slowly in time so that acceleration is negligible. The mathematical treatment of quasistatic contact process is recent. The reason lies in the considerable difficulties that the nonlinear evolutionary inequalities modeling the quasistatic contact problems present in the variational analysis. Existence and uniqueness results in the study of quasistatic contact problems can be found for instance in [1, 2, 3, 9] within linearized elasticity. There, the friction has been modeled with versions of Coulomb's law in which the coefficient of friction was assumed to be constant. A dynamic frictional elastic problem in which the coeficient of friction is assumed to depend on the slip rate has been studied in [5]. Recent existence and uniqueness results for a class of evolutionary variational inequalities arising in quasistatic frictional contact problems for linear elastic materials were obtained in [10].

The aim of this paper is to study a problem of frictional contact between an elastic body and a foundation. We model the contact with a version of Coulomb's law of dry friction in which the normal stress is prescribed on the contact surface and the coefficient of friction depends on the slip. The static version of the model was already considered in [6]. There, the existence of the weak solution of the problem has been proved using a Weierstrass type theorem, based on lower semicontinuity arguments. The uniqueness was derived under an inequality assumption involving two scalar parameters: the first one depends on the geometry and on the elastic coefficients of the material, the second one measures the slip weakening and the normal stress.

The novelty in the present paper consists in the fact that here we consider a quasistatic process, which leads to a new and nonstandard mathematical model. We derive two variational formulations of the problem, denoted $\mathcal{P}_1$ and $\mathcal{P}_2$. Problem $\mathcal{P}_1$ is formulated in terms of displacements while problem $\mathcal{P}_2$ is formulated in terms of stress. For the variational problem $\mathcal{P}_1$ we prove the existence of the solution using the abstract result obtained recently in [10]. Then, we present an equivalence result which allows us to deduce the existence of the solution for the variational problem $\mathcal{P}_2$. We also investigate the uniqueness of the solutions for problems $\mathcal{P}_1$ and $\mathcal{P}_2$ as well as their dependence with respect to the data.

The paper is organized as follows. In Section 2 we introduce the notation and some preliminary material. In the Section 3 we present the mechanical problem and discuss the contact boundary conditions. In Section 4 we list the assumptions on the given data, derive variational formulations to the model and state our main results, Theorems 4.2–4.4. The proofs of the theorems are established in Section 6 and are based on the abstract result on evolutionary variational inequalities that we recall in Section 5. Finally, in Section 7 we present some concluding remarks and we compare our results with the results obtained in [6] in the case of the corresponding static process.

## 2. Notation and preliminaries

In this section we present the notation we shall and some preliminary material. For futher details, we refer the reader to [4, 7, 13].

We denote by $\mathbb{S}_d$ the space of second order symmetric tensors on $\mathbb{R}^d$ ($d = 2, 3$), while "$\cdot$" and $|\cdot|$ will represent the inner product and the Euclidean norm on $\mathbb{S}_d$ and $\mathbb{R}^d$, respectively, i.e.

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i, \quad |\boldsymbol{v}| = (\boldsymbol{v}.\boldsymbol{v})^{1/2} \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij} \quad |\boldsymbol{\tau}| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2} \qquad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}_d.$$

Here and below the indices $i$ and $j$ run between 1 and $d$ and the summation convention over repeated indices is adopted. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary $\Gamma$ and let $\boldsymbol{\nu}$ denote the unit outer normal on $\Gamma$.

We shall use the notation

$$H = L^2(\Omega)^d = \{\, \boldsymbol{u} = (u_i) \mid u_i \in L^2(\Omega)\,\},$$
$$\mathcal{H} = \{\, \boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\,\},$$
$$H_1 = \{\, \boldsymbol{u} = (u_i) \mid \boldsymbol{\varepsilon}(\boldsymbol{u}) \in \mathcal{H}\,\},$$
$$\mathcal{H}_1 = \{\, \boldsymbol{\sigma} \in \mathcal{H} \mid \mathrm{Div}\,\boldsymbol{\sigma} \in H\,\}.$$

Here $\boldsymbol{\varepsilon} : H_1 \longrightarrow \mathcal{H}$ and Div $: \mathcal{H}_1 \longrightarrow H$ are the *deformation* and the *divergence* operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\boldsymbol{\varepsilon}_{ij}(\boldsymbol{u})), \quad \boldsymbol{\varepsilon}_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \mathrm{Div}\ \boldsymbol{\sigma} = (\sigma_{ij,j}),$$

where the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. The spaces $H$, $\mathcal{H}$, $H_1$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the cannonical inner products given by

$$(\boldsymbol{u},\boldsymbol{v})_H = \int_\Omega u_i v_i\,dx, \quad (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}\,dx,$$
$$(\boldsymbol{u},\boldsymbol{v})_{H_1} = (\boldsymbol{u},\boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}},$$
$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} + (\mathrm{Div}\,\boldsymbol{\sigma},\mathrm{Div}\,\boldsymbol{\tau})_H.$$

The associated norms on this spaces are denoted by $|\cdot|_H$, $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H_1}$ and $|\cdot|_{\mathcal{H}_1}$, respectively.

Let $H_\Gamma = H^{1/2}(\Gamma)^d$ and let $\gamma : H_1 \longrightarrow H_\Gamma$ be the trace map. For every element $\boldsymbol{v} \in H_1$ we also write $\boldsymbol{v}$ for the trace $\gamma\boldsymbol{v}$ of $\boldsymbol{v}$ on $\Gamma$ and we denote by $v_\nu$ and $\boldsymbol{v}_\tau$ the *normal* and the *tangential* components of $\boldsymbol{v}$ on the boundary $\Gamma$ given by

$$v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}, \quad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}. \tag{2.1}$$

Let $H'_\Gamma$ be the dual of $H_\Gamma$ and let $(\cdot,\cdot)$ denote the duality pairing between $H'_\Gamma$ and $H_\Gamma$. For every $\boldsymbol{\sigma} \in \mathcal{H}_1$ there exists an element $\boldsymbol{\sigma\nu} \in H'_\Gamma$ such that

$$(\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\mathrm{Div}\,\boldsymbol{\sigma},\,\boldsymbol{v})_H = (\boldsymbol{\sigma\nu},\gamma\boldsymbol{v}) \quad \forall \boldsymbol{v} \in H_1. \tag{2.2}$$

Moreover, we denote by $\sigma_\nu$ and $\boldsymbol{\sigma}_\tau$ the *normal* and *tangential* traces of $\boldsymbol{\sigma}$ and we recall that, when $\boldsymbol{\sigma}$ is a regular (say $C^1$) function, then

$$(\boldsymbol{\sigma\nu},\gamma\boldsymbol{v}) = \int_\Gamma \boldsymbol{\sigma\nu}\cdot\boldsymbol{v}\,da \quad \forall \boldsymbol{v} \in H_1, \tag{2.3}$$

$$\sigma_\nu = (\boldsymbol{\sigma\nu})\cdot\boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu\boldsymbol{\nu}. \tag{2.4}$$

Finally, for every real Hilbert space $X$ and $T > 0$, we use the classical notation for $L^p(0,T;X)$ spaces $(1 \le p \le +\infty)$. We also use the Sobolev space $W^{1,\infty}(0,T;X)$, with the norm

$$|\boldsymbol{u}|_{W^{1,\infty}(0,T;V)} = |\boldsymbol{u}|_{L^\infty(0,T;V)} + |\dot{\boldsymbol{u}}|_{L^\infty(0,T;V)}.$$

Here and everywhere in this paper a dot above represents the weak derivative with respect to the time variable.

## 3. Problem statement

We consider a deformable body, which occupies a domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ with outer Lipschitz surface $\Gamma$ that is divided into three disjoint measurable parts $\Gamma_i$, $i = 1,2,3$, such that $meas\,\Gamma_1 > 0$. Let $[0,T]$ be the time interval of interest, where $T > 0$. The body is clamped on $\Gamma_1 \times (0,T)$ and therefore the displacement field vanishes there. A volume force of density $\boldsymbol{f}_0$ acts in $\Omega \times (0,T)$ and surface tractions of density $\boldsymbol{f}_2$ act on $\Gamma_2 \times (0,T)$. We assume that the body forces and tractions vary slowly in time, so the inertial terms may be neglected in the equation of motion, leading to a quasistatic problem. The body is in contact on $\Gamma_3 \times (0,T)$ with an obstacle, the so called foundation. The contact is frictional and it is modeled with a version of Coulomb's law in which the normal stress is prescribed on the contact surface and the coefficient of friction depends on the slip.

With these assumptions, the classical formulation of the frictional contact problem of the elastic body is the following.

**Problem $\mathcal{P}$.** Find a displacement field $\boldsymbol{u} : \Omega \times [0,T] \longrightarrow \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times [0,T] \longrightarrow \mathbb{S}_d$ such that

$$\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) \qquad \text{in } \Omega \times (0,T), \tag{3.1}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \qquad \text{in } \Omega \times (0,T), \tag{3.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma_1 \times (0,T), \tag{3.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \qquad \text{on } \Gamma_2 \times (0,T), \tag{3.4}$$

$$\sigma_\nu = S \qquad \text{on } \Gamma_3 \times (0,T), \tag{3.5}$$

$$\begin{cases} |\boldsymbol{\sigma}_\tau| \le \mu(|\boldsymbol{u}_\tau|)|S|, \\ |\boldsymbol{\sigma}_\tau| < \mu(|\boldsymbol{u}_\tau|)|S| \Rightarrow \dot{\boldsymbol{u}}_\tau = \boldsymbol{0} \\ |\boldsymbol{\sigma}_\tau| = \mu(|\boldsymbol{u}_\tau|)|S| \Rightarrow \exists\, \lambda \ge 0,\ \boldsymbol{\sigma}_\tau = -\lambda\, \dot{\boldsymbol{u}}_\tau, \end{cases} \qquad \text{on } \Gamma_3 \times (0,T), \tag{3.6}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \text{in } \Omega. \tag{3.7}$$

In (3.1)–(3.7) and below, in order to simplify the notation, we do not indicate explicitely the dependence of various functions on the variables $\boldsymbol{x} \in \Omega \cup \Gamma$ and $t \in [0,T]$. The equation (3.1) represents the linear elastic constituve law in which $\mathcal{E}$ is the fourth order tensor of elastic coefficients and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the small strain tensor. Equation (3.2) is the equilibrium equation, while conditions (3.3), (3.4) are the displacement and traction boundary conditions, respectively. Finally, (3.7) represents the initial condition in which $\boldsymbol{u}_0$ is the initial displacement.

We make some comments on the contact conditions (3.5) and (3.6) in which our interest lies. Condition (3.5) states that the normal stress $\sigma_\nu$ is prescribed on the contact surface, since $S$ is a given datum. Such kind conditions arise in the study of some mechanisms and were already used in [4, 13]. Conditions (3.6) represent the Coulomb's law of dry friction. Here $\boldsymbol{\sigma}_\tau$ is the tangential stress, $\boldsymbol{u}_\tau$ and $\dot{\boldsymbol{u}}_\tau$ are the tangential displacement and velocity, respectively. The function $\mu$ represents the coefficient of friction. When the strong inequality holds the surface of the body adheres to the foundation and is in the so-called *stick* state and when equality holds, there is relative sliding, the so-called *slip* state. Therefore, the contact surface $\Gamma_3$ is divided at each time moment into two zones : the stick zone and the slip zone. The boundaries of these zones are unknown *a priori* and therefore their determination represents a part of the problem. We note that in (3.6) the coefficient of friction is assumed to depend on the slip $|\boldsymbol{u}_\tau|$. Such kind of dependence was pointed out in [14] in order to take into account the changes in the contact surface structure that result from sliding. It was used afterwards in a number of papers (see, e.g. [6, 16] and the references therein) in the geophysical context of earthquakes modeling. In this context it is usual to suppose that the slip rate has a single direction and a single sense during the slip and the slip dependent friction models the stick-slip-stick motions on the geological scales. Generally speaking, the dependence of the friction forces upon the surface displacements is usually accepted when the slip is very small on laboratory scales (see for instance [12] and [16]).

The variational analysis of the frictional contact problem (3.1)–(3.7) will be presented in the next sections, where the existence of the weak solutions to the problem will be proved.

## 4. Variational formulations and main results

To obtain variational formulations of the problem $\mathcal{P}$, we need additional notations. To this end, we introduce the closed subspace of $H_1$ defined by

$$V = \{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \Gamma_1\}.$$

Since $meas\,\Gamma_1 > 0$, Korn's inequality holds, thus there exists $C_K > 0$ which depends only on $\Omega$ and $\Gamma_1$ such that

$$|\boldsymbol{\varepsilon}(\boldsymbol{v})|_{\mathcal{H}} \geq C_K |\boldsymbol{v}|_{H_1} \qquad \forall \boldsymbol{v} \in V. \tag{4.1}$$

A proof of Korn's inequality (4.1) may be found in [11, p. 79]. On $V$ we consider the inner product given by

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \qquad \forall \boldsymbol{u},\, \boldsymbol{v} \in V, \tag{4.2}$$

and let $|\cdot|_V$ be the associated norm, i.e.

$$|\boldsymbol{v}|_V = |\boldsymbol{\varepsilon}(\boldsymbol{v})|_{\mathcal{H}} \qquad \forall \boldsymbol{v} \in V. \tag{4.3}$$

It follows from Korn's inequality and (4.3) that $|\cdot|_{H_1}$ and $|\cdot|_V$ are equivalent norms on $V$ and therefore $(V, |\cdot|_V)$ is a real Hilbert space. Moreover, by the Sobolev's trace theorem and Korn's inequality, there exists $C_0 > 0$ depending only on the domain $\Omega$, $\Gamma_1$ and $\Gamma_3$ such that

$$|\boldsymbol{v}|_{L^2(\Gamma_3)^d} \leq C_0 |\boldsymbol{v}|_V \qquad \forall \boldsymbol{v} \in V. \tag{4.4}$$

In the study of the mechanical problem (3.1)–(3.7), we assume that the elasticity tensor $\mathcal{E} = (\mathcal{E}_{ijkl})$ satisfies

$$(4.5)\qquad \begin{cases} \text{(a)} & \mathcal{E} : \Omega \times \mathbb{S} \to \mathbb{S}. \\ \text{(b)} & \mathcal{E}_{ijkl} \in L^\infty(\Omega),\ 1 \le i, j, k, l \le d. \\ \text{(c)} & \mathcal{E}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathcal{E}\boldsymbol{\tau} \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S},\ \text{a.e. in } \Omega. \\ \text{(d)} & \text{There exists } m > 0 \text{ such that } \ \mathcal{E}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \ge m|\boldsymbol{\tau}|^2 \quad \forall \boldsymbol{\tau} \in \mathbb{S}, \\ & \ \text{a.e. in } \Omega. \end{cases}$$

The coefficient of friction satisfies

$$(4.6)\qquad \begin{cases} \text{(a)} & \mu\ :\ \Gamma_3 \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+. \\ \text{(b)} & \text{There exists } L_\mu > 0 \text{ such that} \\ & |\mu(\boldsymbol{x}, r_1) - \mu(\boldsymbol{x}, r_2)| \le L_\mu |r_1 - r_2| \quad \forall\, r_1,\, r_2 \in \mathbb{R}_+, \\ & \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c)} & \text{The mapping } \boldsymbol{x} \longmapsto \mu(\boldsymbol{x}, r) \text{ is Lebesgue measurable on } \Gamma_3, \\ & \forall\, r \in \mathbb{R}_+. \\ \text{(d)} & \text{The mapping } \boldsymbol{x} \longmapsto \mu(\boldsymbol{x}, 0) \in L^2(\Gamma_3). \end{cases}$$

We note that the assumptions (4.6) on the coefficient of friction are fairly general and are weaker than those considered in [6], where $\mu$ was assumed to be bounded and continuous differentiable with respect to the second argument. However, to provide existence results, we need to impose an additional smallness assumption involving the coefficient of friction $\mu$, which is not needed in the static case treated in [6]. Also, to provide uniqueness results, we need to replace assumption (4.6) by a stronger condition. Thus, we alternatively consider that $\mu$ does not depend on the slip $|\boldsymbol{u}_\tau|$, i.e.

$$(4.7)\qquad \begin{cases} \mu \text{ is a given function which satisfies} \\ \mu \in L^2(\Gamma_3) \quad \text{and} \quad \mu(x) \ge 0 \quad \text{a.e. on } \Gamma_3. \end{cases}$$

The forces and tractions are asumed to satisfy

$$(4.8)\qquad \boldsymbol{f}_0 \in W^{1,\infty}(0, T; H), \qquad \boldsymbol{f}_2 \in W^{1,\infty}(0, T; L^2(\Gamma_2)^d),$$

and the given normal stress is such that

$$(4.9)\qquad S \in L^\infty(\Gamma_3).$$

Next we define the bilinear form $a : V \times V \to \mathbb{R}$ by

$$(4.10)\qquad a(\boldsymbol{u}, \boldsymbol{v}) = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}},$$

and the functional $j : V \times V \to \mathbb{R}$ by

$$(4.11)\qquad j(\boldsymbol{\eta}, \boldsymbol{v}) = \int_{\Gamma_3} \mu(|\boldsymbol{\eta}_\tau|)\, |S|\, |\boldsymbol{v}_\tau|\, da.$$

Using conditions (4.6) and (4.9), it follows that the integral in (4.11) is well defined.

Let $\boldsymbol{f} : [0,T] \to V$ be given by

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\, da + \int_{\Gamma_3} S\, v_\nu\, da. \tag{4.12}$$

The definition of (4.12) is based on Riesz's representation theorem, and we note that conditions (4.8) and (4.9) imply

$$\boldsymbol{f} \in W^{1,\infty}(0,T;V). \tag{4.13}$$

We assume that the initial data satisfies

$$\boldsymbol{u}_0 \in V, \tag{4.14}$$

$$a(\boldsymbol{u}_0, \boldsymbol{v}) + j(\boldsymbol{u}_0, \boldsymbol{v}) \geq (\boldsymbol{f}(0), \boldsymbol{v})_V \qquad \forall \boldsymbol{v} \in V. \tag{4.15}$$

To derive a variational formulation of problem $\mathcal{P}$, in terms of stress, we denote the family of sets $\Sigma(t, \boldsymbol{\eta})$, defined for all $\boldsymbol{\eta} \in V$ and $t \in [0,T]$ by

$$\Sigma(t,\boldsymbol{\eta}) = \{\boldsymbol{\tau} \in \mathcal{H} \mid (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + j(\boldsymbol{\eta}, \boldsymbol{v}) \geq (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \forall\ \boldsymbol{v} \in V\}. \tag{4.16}$$

Taking $\boldsymbol{v} = \pm\boldsymbol{\varphi}$ with $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)^d$ in (4.16) and using (4.11) and (4.12), it follows that

$$\boldsymbol{\tau} \in \Sigma(t,\boldsymbol{\eta}) \Rightarrow \mathrm{Div}\, \boldsymbol{\tau} + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{ in } \Omega. \tag{4.17}$$

We also denote by $\mathcal{D}(A)$ the subspace of $\mathcal{H}$ given by

$$\mathcal{D}(A) = \{\boldsymbol{z} \in \mathcal{H} \mid \exists\, \boldsymbol{v} \in V \text{ such that } \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{v})) = \boldsymbol{z}\} \tag{4.18}$$

and we note that from (4.5) and Korn's inequality (4.1), it follows that the operator $\mathcal{E} \circ \boldsymbol{\varepsilon} : V \to \mathcal{D}(A)$ is invertible. Therefore, we denote by $A : \mathcal{D}(A) \to V$ its inverse and we find

$$\boldsymbol{v} = A(\boldsymbol{\tau}) \iff \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{v})) = \boldsymbol{\tau}. \tag{4.19}$$

Finally, keeping in mind (3.1) and (3.7), we introduce the initial stress $\boldsymbol{\sigma}_0$ by

$$\boldsymbol{\sigma}_0 = \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_0)) \tag{4.20}$$

and we assume that it satisfies

$$\boldsymbol{\sigma}_0 \in \mathcal{D}(A) \cap \Sigma(0, A\boldsymbol{\sigma}_0). \tag{4.21}$$

We have the following result.

**Lemma 4.1.** *If* $\{\boldsymbol{u},\boldsymbol{\sigma}\}$ *are sufficiently smooth functions satifying* (3.1)–(3.7) *then, for all* $t \in [0,T]$,

$$\boldsymbol{u}(t) \in V,$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))_{\mathcal{H}} + j(\boldsymbol{u}(t),\boldsymbol{v}) - j(\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \qquad \forall \boldsymbol{v} \in V, \tag{4.22}$$

$$\boldsymbol{\sigma}(t) \in \mathcal{D}(A) \cap \Sigma(t, A\boldsymbol{\sigma}(t)),$$
$$(\boldsymbol{\tau} - \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))_{\mathcal{H}} \geq 0 \qquad \forall \boldsymbol{\tau} \in \Sigma(t, A\boldsymbol{\sigma}(t)). \tag{4.23}$$

**Proof.** Let $t \in [0,T]$ and $\boldsymbol{v} \in V$. Using (3.2)–(3.5) and (2.1)–(2.4) we have

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} = \int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{v}\,dx + \int_{\Gamma_2} \boldsymbol{f}_2(t)\cdot\boldsymbol{v}\,da + \int_{\Gamma_3} S\,v_\nu\,da + \int_{\Gamma_3} \boldsymbol{\sigma}_\tau(t)\cdot\boldsymbol{v}_\tau\,da, \tag{4.24}$$

and, using (3.6), we deduce that

$$\int_{\Gamma_3} \boldsymbol{\sigma}_\tau(t)\cdot\boldsymbol{v}_\tau\,da \geq -\int_{\Gamma_3} \mu(|\boldsymbol{u}_\tau(t)|)\,|S|\,|\boldsymbol{v}_\tau|\,da. \tag{4.25}$$

Therefore, from (4.11), (4.12), (4.24) and (4.25), we find

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + j(\boldsymbol{u}(t),\boldsymbol{v}) \geq (\boldsymbol{f}(t),\boldsymbol{v})_V. \tag{4.26}$$

The regularity $\boldsymbol{\sigma}(t) \in \mathcal{D}(A) \cap \Sigma(t, A\boldsymbol{\sigma}(t))$ follows from (3.1), (4.16), (4.18) and (4.26). Moreover, we note that (3.6) and (4.11) imply

$$\int_{\Gamma_3} \boldsymbol{\sigma}_\tau(t)\cdot\dot{\boldsymbol{u}}_\tau(t)\,da = -j(\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)). \tag{4.27}$$

Thus, taking $\boldsymbol{v} = \dot{\boldsymbol{u}}(t)$ in (4.24) and using again (4.12) and (4.27), we deduce that

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_{\mathcal{H}} + j(\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) = (\boldsymbol{f}(t),\dot{\boldsymbol{u}}(t))_V. \tag{4.28}$$

The inequatilies in (4.22) and (4.23) are now a consequence of (4.16), (4.26) and (4.28). □

Lemma 4.1, (3.1), (3.7), (4.10), (4.19) and (4.20) leads us to consider the following two variational problems.

**Problem $\mathcal{P}_1$.** Find a displacement field $\boldsymbol{u} : [0,T] \to V$ such that

$$a(\boldsymbol{u}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)) + j(\boldsymbol{u}(t),\boldsymbol{v}) - j(\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \qquad \forall\,\boldsymbol{v} \in V, \quad a.e. \quad t \in (0,T), \tag{4.29}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0. \tag{4.30}$$

**Problem $\mathcal{P}_2$.** Find a stress field $\boldsymbol{\sigma} : [0,T] \to \mathcal{H}$ such that

$$\boldsymbol{\sigma}(t) \in \mathcal{D}(A) \cap \Sigma(t, A\boldsymbol{\sigma}(t)) \quad \forall t \in [0,T],$$

$$(\mathcal{E}^{-1}\dot{\boldsymbol{\sigma}}(t), \boldsymbol{\tau} - \boldsymbol{\sigma}(t))_{\mathcal{H}} \geq 0 \quad \forall \boldsymbol{\tau} \in \Sigma(t, A\boldsymbol{\sigma}(t)), \tag{4.31}$$

a.e. on $(0,T)$,

$$\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0. \tag{4.32}$$

Our main results, which we establish in Section 6, are the followings.

**Theorem 4.2.** *Assume that conditions* (4.5)*,* (4.8)*,* (4.9)*,* (4.14) *and* (4.15) *hold. Then:*

(i) *Under the assumption* (4.6)*, there exists $L_0 > 0$ depending only on $\Omega$, $\Gamma_1$, $\Gamma_3$ and $\mathcal{E}$ such that if $L_\mu |S|_{L^\infty(\Gamma_3)} < L_0$ then problem $\mathcal{P}_1$ has at least a solution $\boldsymbol{u}$ which satisfies $\boldsymbol{u} \in W^{1,\infty}(0,T;V)$.*

(ii) *Under the assumption* (4.7)*, there exists a unique solution $\boldsymbol{u} \in W^{1,\infty}(0,T;V)$ for problem $\mathcal{P}_1$. Moreover the mapping $(\boldsymbol{f}, \boldsymbol{u}_0) \longmapsto \boldsymbol{u}$ is Lipschitz continuous from $W^{1,\infty}(0,T;V) \times V$ to $L^\infty(0,T;V)$.*

**Theorem 4.3.** *Assume that conditions* (4.5)*,* (4.6)*,* (4.8) *and* (4.9) *hold. Then:*

(i) *If $\boldsymbol{u} \in W^{1,\infty}(0,T;V)$ is a solution of problem $\mathcal{P}_1$ then $\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u})$ represents a solution of problem $\mathcal{P}_2$ which satisfies $\boldsymbol{\sigma} \in W^{1,\infty}(0,T;\mathcal{H})$.*

(ii) *Conversely, if $\boldsymbol{\sigma} \in W^{1,\infty}(0,T;\mathcal{H})$ is a solution of problem $\mathcal{P}_2$ then $\boldsymbol{u} = A\boldsymbol{\sigma}$ represents a solution of problem $\mathcal{P}_1$ which satisfies $\boldsymbol{u} \in W^{1,\infty}(0,T;V)$.*

**Theorem 4.4.** *Assume that conditions* (4.5)*,* (4.8)*,* (4.9) *and* (4.21) *hold.*

(i) *Let* (4.6) *hold and let $L_0$ be defined as in Theorem* 4.2*. Then problem $\mathcal{P}_2$ has at least a solution if $L_\mu |S|_{L^\infty(\Gamma_3)} < L_0$. Moreover the solution satisfies $\boldsymbol{\sigma} \in W^{1,\infty}(0,T;\mathcal{H})$.*

(ii) *Under the assumption* (4.7)*, there exists a unique solution $\boldsymbol{\sigma} \in W^{1,\infty}(0,T;\mathcal{H})$ for problem $\mathcal{P}_2$. Moreover the mapping $(\boldsymbol{f}, \boldsymbol{\sigma}_0) \longmapsto \boldsymbol{\sigma}$ is Lipschitz continuous from $W^{1,\infty}(0,T;\mathcal{H}) \times V$ to $L^\infty(0,T;\mathcal{H})$.*

The proofs of Theorems 4.2–4.4 are based on an abstract result in the study of a class of evolutionary variational inequalities, recently obtained in [10], that we recall in Section 5. Here, to end this section, we present the mechanical interpretation of our results.

We note that problems $\mathcal{P}_1$ and $\mathcal{P}_2$ represent *variational formulations* of the frictional contact problem $\mathcal{P}$ in terms of displacements and stress, respectively. Theorems 4.2 and 4.4 show the solvability of the mechanical problem $\mathcal{P}$ in terms of displacements and stress, respectively, under the smallness assumption $L_\mu |S|_{L^\infty(\Gamma_3)} < L_0$. Notice that here $L_0$ represents a scalar parameter which depends only on the elasticity operator and on the geometry of the problem but does not depend on the external forces, nor on the initial displacement. From mathematical point of view the previous inequality represents a sufficient condition for the solvability of the variational problems $\mathcal{P}_1$ and $\mathcal{P}_2$, and it is requested by the abstract theorem we use in solving these problems. From mechanical point of view this inequality shows that problem $\mathcal{P}$ can be solved if either the slip weakening or the given normal stress on the contact surface are small enough. Notice that a similar condition was used in [6] in order to derive the uniqueness of the solution in the static model. Theorem 4.3 shows that the operator $A : \mathcal{D}(A) \longrightarrow V$ represents a bijective correspondence between the solutions of the variational problems $\mathcal{P}_2$ and $\mathcal{P}_1$. We also note that if the coefficient of friction does not depend on the slip (i.e. assumption (4.6) is replaced by the stronger condition (4.7)), then Theorems 4.2 and 4.4 show the unique solvability of problems $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. Moreover, in this case the solutions are connected by the elastic constitutive law $\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u})$. Also, keeping in mind (4.17) and (4.8), we remark that if $\boldsymbol{\sigma} \in W^{1,\infty}(0,T;\mathcal{H})$ solves problem $\mathcal{P}_2$ then $\boldsymbol{\sigma}$ has stronger regularity, i.e. $\boldsymbol{\sigma} \in W^{1,\infty}(0,T;\mathcal{H}_1)$.

Finally, we note that the choice of the homogeneous boundary condition (3.3) in the mechanical problem $\mathcal{P}$ was made for simplicity. This choice is not restrictive. Indeed, in the case when non homogeneous Dirichlet boundary conditions in displacements are assumed on $\Gamma_1 \times (0,T)$ then, using a change of variable in the unknown functions $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ similar to that used in [6], we obtain a homogenized problem with friction in the study of which our arguments hold and lead to similar results as those presented in Theorems 4.2–4.4.

## 5. An abstract existence and uniqueness result

Let $V$ be a real Hilbert space endowed with the inner product $(\cdot,\cdot)_V$ and the associated norm $|\cdot|_V$. We denote by "$\rightharpoonup$" and "$\rightarrow$" the weak convergence and the strong convergence on $V$, respectively. In the sequel $0_V$ will represent the zero element of $V$. Let $a : V \longrightarrow V$ be a bilinear form on $V$, $j : V \times V \longrightarrow \mathbb{R}$, $f : [0,T] \longrightarrow V$ and $u_0 \in V$. With these data, we consider the following quasivariational problem: find $u : [0,T] \longrightarrow V$ such that

$$
\begin{aligned}
&a(u(t), v-\dot u(t)) + j(u(t), v) - j(u(t), \dot u(t)) \\
&\geq (f(t), v-\dot u(t))_V \quad \forall v\in V, \text{ a.e. } t\in(0,T),
\end{aligned} \tag{5.1}
$$

$$
u(0) = u_0. \tag{5.2}
$$

In order to solve (5.1)–(5.2), we consider the following assumptions.

$$
\begin{cases}
a: V\times V\to\mathbb{R} \text{ is a bilinear symmetric form and}\\
\text{(a)} \quad \text{there exists } M>0 \text{ such that}\\
\qquad |a(u,v)|_V \leq M|u|_V|v|_V \quad \forall\, u,v\in V;\\
\text{(b)} \quad \text{there exists } m>0 \text{ such that } a(v,v)\geq m|v|_V^2 \quad \forall v\in V.
\end{cases} \tag{5.3}
$$

$$
\begin{cases}
j: V\times V\to\mathbb{R} \text{ and for every } \eta\in V,\ j(\eta,\cdot): V\to\mathbb{R}\\
\text{is a positively homogenuous subadditive functional, i.e.}\\
\text{(a)} \quad j(\eta,\lambda u) = \lambda j(\eta,u) \quad \forall u\in V,\ \lambda\in\mathbb{R}_+;\\
\text{(b)} \quad j(\eta,u+v)\leq j(\eta,u)+j(\eta,v) \quad \forall u,v\in V.
\end{cases} \tag{5.4}
$$

$$
f\in W^{1,\infty}(0,T;V). \tag{5.5}
$$

$$
u_0\in V. \tag{5.6}
$$

$$
a(u_0,v)+j(u_0,v)\geq (f(0),v)_V \qquad \forall v\in V. \tag{5.7}
$$

Keeping in mind (5.4), it results that for all $\eta\in V$, $j(\eta,\cdot): V\to V$ is a convex functional. Therefore, there exists the directional derivative $j'_2$ given by

$$
j'_2(\eta,u;v) \ = \ \lim_{\lambda\to 0^+}\frac{1}{\lambda}\Big[j(\eta,u\ +\ \lambda v)\ -\ j(\eta,u)\Big] \quad \forall \eta,\ u,\ v\ \in\ V. \tag{5.8}
$$

We consider now the following additional assumptions on the functional $j$.

$$
\begin{cases}
\text{For every sequence } \{u_n\}\subset V \text{ with } |u_n|_V\to\infty,\\
\text{every sequence } \{t_n\}\subset[0,1] \text{ and each } \overline{u}\in V \text{ one has}\\
\liminf_{n\to\infty}\ \Big[\dfrac{1}{|u_n|_V^2}j'_2(t_nu_n, u_n-\overline{u}; -u_n)\Big] < m.
\end{cases} \tag{5.9}
$$

$$
\begin{cases}
\text{For every sequence } \{u_n\}\subset V \text{ with } |u_n|_V\to\infty,\\
\text{every bounded sequence } \{\eta_n\}\subset V \text{ and each } \overline{u}\in V \text{ one has}\\
\liminf_{n\to\infty}\ \Big[\dfrac{1}{|u_n|_V^2}j'_2(\eta_n, u_n-\overline{u}; -u_n)\Big] < m.
\end{cases} \tag{5.10}
$$

$$
(5.11)\quad \begin{cases} \text{For all sequences } \{u_n\} \subset V \text{ and } \{\eta_n\} \subset V \text{ such that} \\ u_n \rightharpoonup u \in V,\ \eta_n \rightharpoonup \eta \in V \text{ and for every } v \in V, \text{ the inequality} \\ \text{below holds} \quad \limsup_{n\to\infty} \,[j(\eta_n, v) - j(\eta_n, u_n)] \le j(\eta, v) - j(\eta, u). \end{cases}
$$

$$
(5.12)\quad \begin{cases} \text{There exists } c_0 \in (0, m) \text{ such that} \\ j(u, v-u) - j(v, v-u) \le c_0 |u - v|_V^2 \quad \forall u, v \in V. \end{cases}
$$

$$
(5.13)\quad \begin{cases} \text{There exist two functions } a_1 : V \to \mathbb{R} \text{ and } a_2 : V \to \mathbb{R}, \\ \text{which map bounded sets in } V \text{ into bounded sets in } \mathbb{R} \\ \text{such that} \quad |j(\eta, u)| \le a_1(\eta)|u|_V^2 + a_2(\eta) \quad \forall \eta, u \in V, \\ \text{and } a_1(0_V) < m - c_0. \end{cases}
$$

$$
(5.14)\quad \begin{cases} \text{For every sequence } \{\eta_n\} \subset V \text{ with } \eta_n \rightharpoonup \eta \in V, \\ \text{and every bounded sequence } \{u_n\} \subset V \text{ one has} \\ \lim_{n\to\infty} \,[j(\eta_n, u_n) - j(\eta, u_n)] = 0. \end{cases}
$$

$$
(5.15)\quad \begin{cases} \text{For every } s \in (0, T] \text{ and every functions } u, v \in W^{1,\infty}(0, T; V) \\ \text{with } u(0) = v(0),\ u(s) \ne v(s), \text{ the inequality below holds} \\ \displaystyle\int_0^s [j(u(t), \dot v(t)) - j(u(t), \dot u(t)) + j(v(t), \dot u(t)) - j(v(t), \dot v(t))]\, dt \\ < \dfrac{m}{2} |u(s) - v(s)|_V^2. \end{cases}
$$

$$
(5.16)\quad \begin{cases} \text{There exists } \alpha \in (0, m/2) \text{ such that for every } s \in (0, T] \text{ and} \\ \text{every functions } u, v \in W^{1,\infty}(0, T; V) \text{ with } u(s) \ne v(s), \\ \text{the inequality below holds} \\ \displaystyle\int_0^s [j(u(t), \dot v(t)) - j(u(t), \dot u(t)) + j(v(t), \dot u(t)) - j(v(t), \dot v(t))]\, dt \\ < \alpha |u(s) - v(s)|_V^2. \end{cases}
$$

In the study of the evolutionary problem (5.1)–(5.2), we recall the following result.

**Theorem 5.1.** *Let* (5.3)–(5.7) *hold.*

(i) *If the assumptions* (5.9)–(5.14) *are satisfied then there exists at least a solution* $u \in W^{1,\infty}(0, T; V)$ *to the problem* (5.1)–(5.2).

(ii) *If the assumptions* (5.9)–(5.15) *are satisfied then there exists a unique solution* $u \in W^{1,\infty}(0, T; V)$ *to the problem* (5.1)–(5.2).

(iii) *If the assumptions* (5.9)–(5.14) *and* (5.16) *are satisfied then there exists a unique solution* $u = u(f, u_0) \in W^{1,\infty}(0,T;V)$ *to the problem* (5.1)–(5.2) *and the mapping* $(f, u_0) \longmapsto u$ *is Lipschitz continuous from* $W^{1,\infty}(0,T;V) \times V$ *to* $L^\infty(0,T;V)$.

The proof, which can be found in [10], is obtained in several steps and it is based on arguments of elliptic quasivaritional inequalities and a time discretization method.

## 6. Proofs

In this section we present the proofs of our main results, Theorems 4.2–4.4. We start with the proof of the Theorem 4.2, which will be carried out in several steps. We assume in the sequel that (4.5), (4.6) and (4.9) hold and, under these assumptions, we start by investigating the properties of the functional $j$ given by (4.11). We remark that $j$ satisfies condition (5.4). Moreover, we have the following results.

**Lemma 6.1.** *The functional $j$ satisfies the assumptions* (5.9) *and* (5.10).

**Proof.** Let $\boldsymbol{\eta}$, $\boldsymbol{u}$, $\overline{\boldsymbol{u}} \in V$ and let $\lambda \in ]0,1]$. Using (4.11), it results that

$$j(\boldsymbol{\eta}, \boldsymbol{u} - \overline{\boldsymbol{u}} - \lambda \boldsymbol{u}) - j(\boldsymbol{\eta}, \boldsymbol{u} - \overline{\boldsymbol{u}}) \leq \lambda \int_{\Gamma_3} \mu(|\boldsymbol{\eta}_\tau|)|S|\,|\overline{\boldsymbol{u}}_\tau|\; da.$$

Therefore, by (5.8) we obtain

$$j_2'(\boldsymbol{\eta}, \boldsymbol{u} - \overline{\boldsymbol{u}}; -\boldsymbol{u}) \leq \int_{\Gamma_3} \mu(|\boldsymbol{\eta}_\tau|)|S|\,|\overline{\boldsymbol{u}}_\tau|\; da \quad \forall \boldsymbol{\eta},\ \boldsymbol{u},\ \overline{\boldsymbol{u}} \in V. \tag{6.1}$$

Let now consider the sequences $\{\boldsymbol{u}_n\} \subset V$, $\{t_n\} \subset ]0,1]$ and $\overline{\boldsymbol{u}} \in V$. Using (4.4), (4.6), (4.9), and (6.1) we find

$$\begin{aligned} j_2'(t_n \boldsymbol{u}_n, \boldsymbol{u}_n - \overline{\boldsymbol{u}}; -\boldsymbol{u}_n) &\leq \int_{\Gamma_3} (L_\mu |\boldsymbol{u}_{n\tau}| + |\mu(0)|)|S|\,|\overline{\boldsymbol{u}}_\tau|\; da \\ &\leq C_0 |S|_{L^\infty(\Gamma_3)} (C_0 L_\mu |\boldsymbol{u}_n|_V + |\mu(0)|_{L^2(\Gamma_3)}) |\overline{\boldsymbol{u}}|_V. \end{aligned}$$

It follows from the previous inequality that if $|\boldsymbol{u}_n|_V \to \infty$ then

$$\liminf_{n\to\infty} \Big[ \frac{1}{|\boldsymbol{u}_n|_V^2} j_2'(t_n \boldsymbol{u}_n, \boldsymbol{u}_n - \overline{\boldsymbol{u}}; -\boldsymbol{u}_n) \Big] \leq 0,$$

and we conclude that $j$ satisfies assumption (5.9).

Let now consider the sequences $\{\boldsymbol{u}_n\} \subset V$, $\{\boldsymbol{\eta}_n\} \subset V$ such that

$$|\boldsymbol{u}_n|_V \to \infty, \tag{6.2}$$

$$|\boldsymbol{\eta}_n|_V \leq C \quad \forall n \in \mathbb{N}, \tag{6.3}$$

where $C > 0$. Let $\overline{\boldsymbol{u}} \in V$. Using (4.4), (4.6) and (6.1) we obtain

$$\begin{aligned} &j_2'(\boldsymbol{\eta}_n, \boldsymbol{u}_n - \overline{\boldsymbol{u}}; -\boldsymbol{u}_n) \\ &\leq C_0 |S|_{L^\infty(\Gamma_3)} \Big( C_0 L_\mu |\boldsymbol{\eta}_n|_V + |\mu(0)|_{L^2(\Gamma_3)} \Big) |\overline{\boldsymbol{u}}|_V \quad \forall n \in \mathbb{N}. \end{aligned} \tag{6.4}$$

Thus, from (6.2)–(6.4), we deduce that $j$ satisfies assumption (5.10). □

**Lemma 6.2.** *The functional $j$ satisfies the assumptions* (5.11) *and* (5.14).

**Proof.** Let $\{\boldsymbol{u}_n\} \subset V$, $\{\boldsymbol{\eta}_n\} \subset V$ be two sequences such that $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u} \in V$ and $\boldsymbol{\eta}_n \rightharpoonup \boldsymbol{\eta} \in V$. Using the compactness property of the trace map and the assumption (4.6), it follows that

$$\mu(|\boldsymbol{\eta}_{n\tau}|) \to \mu(|\boldsymbol{\eta}_\tau|) \quad \text{in } L^2(\Gamma_3), \tag{6.5}$$

$$\boldsymbol{u}_n \to \boldsymbol{u} \quad \text{in } L^2(\Gamma_3)^d. \tag{6.6}$$

Therefore, we deduce from (6.5) and (6.6) that

$$j(\boldsymbol{\eta}_n, \boldsymbol{v}) \to j(\boldsymbol{\eta}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in V, \qquad j(\boldsymbol{\eta}_n, \boldsymbol{u}_n) \to j(\boldsymbol{\eta}, \boldsymbol{u}),$$

which show that the functional $j$ satisfies condition (5.11).

Now, let $\{\boldsymbol{u}_n\}$ be a bounded sequence of $V$, i.e.

$$|\boldsymbol{u}_n| \leq C \quad \forall n \in \mathbb{N}, \tag{6.7}$$

where $C > 0$. We have

$$|j(\boldsymbol{\eta}_n, \boldsymbol{u}_n) - j(\boldsymbol{\eta}, \boldsymbol{u}_n)| \leq \int_{\Gamma_3} |S| \Big( \mu(|\boldsymbol{\eta}_{n\tau}|) - \mu(|\boldsymbol{\eta}_\tau|) |\boldsymbol{u}_{n\tau}| \Big)\, da$$

and, using (4.4) and (4.9), we deduce that

$$\begin{aligned} &|j(\boldsymbol{\eta}_n, \boldsymbol{u}_n) - j(\boldsymbol{\eta}, \boldsymbol{u}_n)| \\ &\leq C_0 |S|_{L^\infty(\Gamma_3)} \big| \mu(|\boldsymbol{\eta}_{n\tau}|) - \mu(|\boldsymbol{\eta}_\tau|) \big|_{L^2(\Gamma_3)} |\boldsymbol{u}_n|_V. \end{aligned} \tag{6.8}$$

It follows now from (6.5), (6.7) and (6.8) that $j$ satisfies condition (5.14). □

**Lemma 6.3.** *The functional $j$ satisfies assumption* (5.13) *for all $c_0 \in (0, m)$. Moreover,*

$$j(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) - j(\boldsymbol{v}, \boldsymbol{v} - \boldsymbol{u}) \leq L_\mu C_0^2 |S|_{L^\infty(\Gamma_3)} |\boldsymbol{u} - \boldsymbol{v}|_V^2 \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V. \tag{6.9}$$

**Proof.** Let $\boldsymbol{\eta}, \boldsymbol{u} \in V$. Using (4.6), (4.9) and (4.11) it follows that

$$\begin{aligned} |j(\boldsymbol{\eta}, \boldsymbol{u})| &\leq \int_{\Gamma_3} |S| \mu(|\boldsymbol{\eta}_\tau|) |\boldsymbol{u}_\tau|\, da, \\ &\leq |S|_{L^\infty(\Gamma_3)} \Big( L_\mu |\boldsymbol{\eta}_\tau|_{L^2(\Gamma_3)^d} + |\mu(0)|_{L^2(\Gamma_3)} \Big) |\boldsymbol{u}_\tau|_{L^2(\Gamma_3)^d}, \end{aligned}$$

and, keeping in mind (4.4), we find

$$|j(\boldsymbol{\eta},\boldsymbol{u})| \le C_0|S|_{L^\infty(\Gamma_3)}\Big(L_\mu C_0|\boldsymbol{\eta}|_V + |\mu(0)|_{L^2(\Gamma_3)}\Big)|\boldsymbol{u}|_V,$$

which implies condition (5.13), for all $c_0 \in (0, m)$.

Let now $\boldsymbol{u}, \boldsymbol{v} \in V$. Using again (4.6), (4.9) and (4.11) it follows that

$$\begin{aligned} j(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) - j(\boldsymbol{v},\boldsymbol{v}-\boldsymbol{u}) &= \int_{\Gamma_3} |S|\Big(\mu(|\boldsymbol{u}_\tau|) - \mu(|\boldsymbol{v}_\tau|)\Big)|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau|\,da \\ &\le L_\mu|S|_{L^\infty(\Gamma_3)}\int_{\Gamma_3}|\boldsymbol{u}-\boldsymbol{v}|_V^2\,da. \end{aligned}$$

Using now (4.4) in the previous inequality, we deduce (6.9). □

We have now all the ingredients to prove the theorems.

**Proof of Theorem 4.2.**

(i) Using the conditions (4.2) and (4.5), we see that the bilinear form $a$ defined by (4.10) is symmetric and coercive, i.e.

$$a(\boldsymbol{v},\boldsymbol{v}) \ge m|\boldsymbol{v}|_V^2 \qquad \forall \boldsymbol{v} \in V. \tag{6.10}$$

Let $L_0 = m/C_0^2$. Clearly, $L_0$ depends only on $\Omega$, $\Gamma_1$, $\Gamma_3$ and $\mathcal{E}$. Let now assume that

$$L_\mu|S|_{L^\infty(\Gamma_3)} < L_0.$$

Then, there exits $c_0 \in \mathbb{R}$ such that

$$L_\mu C_0^2|S|_{L^\infty(\Gamma_3)} < c_0 < m.$$

Using (6.9), we obtain

$$j(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) - j(\boldsymbol{v},\boldsymbol{v}-\boldsymbol{u}) \le c_0|\boldsymbol{u}-\boldsymbol{v}|^2 \quad \forall \boldsymbol{u},\boldsymbol{v} \in V$$

and we conclude that the functional $j$ satisfies condition (5.12). Using now Lemmas 6.1–6.3, (4.13)–(4.15) and Theorem 5.1 (i), we deduce that problem $\mathcal{P}_1$ has at least a solution $\boldsymbol{u} \in W^{1,\infty}(0,T;V)$.

(ii) Let (4.7) holds. We note that in this case the functional $j$ is given by

$$j(\boldsymbol{\eta},\boldsymbol{v}) = \int_{\Gamma_3} \mu|S|\,|\boldsymbol{v}_\tau|\,da,$$

and therefore it does not depend on the first argument. It is obvious to see that in this case assumptions (5.15) and (5.16) hold. The conclusion follows now from Theorem 5.1 (ii) and (iii). □

**Proof of Theorem 4.3.**

(i) Let $\boldsymbol{u} \in W^{1,\infty}(0,T;V)$ be a solution of problem $\mathcal{P}_1$ and $\boldsymbol{\sigma} \in W^{1,\infty}(0,T;\mathcal{H})$ be the function defined by

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \quad \forall t \in [0,T]. \tag{6.11}$$

Using (4.18), (4.19), (6.11), we have

$$\boldsymbol{\sigma}(t) \in \mathcal{D}(A) \quad \text{and} \quad A\boldsymbol{\sigma}(t) = \boldsymbol{u}(t) \quad \forall t \in [0,T], \tag{6.12}$$

and, keeping in mind (4.20), (4.30), we find

$$\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0. \tag{6.13}$$

Choosing $\boldsymbol{v} = 2\dot{\boldsymbol{u}}$ and $\boldsymbol{v} = \mathbf{0}_V$ in (4.29), by (4.10) and (6.11) we obtain

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}))_{\mathcal{H}} + j(\boldsymbol{u}, \dot{\boldsymbol{u}}) = (\boldsymbol{f}, \boldsymbol{u})_V \quad \text{a.e. on } (0,T). \tag{6.14}$$

Using again (4.29), (6.14) and (4.16), we deduce that

$$\boldsymbol{\sigma}(t) \in \Sigma(t, \boldsymbol{u}(t)) \quad \text{a.e. on } (0,T),$$

and, keeping in mind (4.16), (6.12) and the time regularity of $\boldsymbol{\sigma}$ and $\boldsymbol{f}$, we obtain

$$\boldsymbol{\sigma}(t) \in \Sigma(t, A\boldsymbol{\sigma}(t)) \quad \forall t \in [0,T]. \tag{6.15}$$

It follows now from (4.16), (6.12) and (6.14) that

$$(\boldsymbol{\tau} - \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_{\mathcal{H}} \geq 0 \quad \forall \boldsymbol{\tau} \in \Sigma(t, A\boldsymbol{\sigma}(t)) \quad \text{a.e. on } (0,T). \tag{6.16}$$

We conclude by (6.11)–(6.13), (6.15) and (6.16) that $\boldsymbol{\sigma}$ is a solution of problem $\mathcal{P}_2$.

(ii) Conversely, let $\boldsymbol{\sigma} \in W^{1,\infty}(0,T;\mathcal{H})$ be a solution of problem $\mathcal{P}_2$ and consider $\boldsymbol{u} : [0,T] \longrightarrow V$ the function defined by

$$\boldsymbol{u}(t) = A\boldsymbol{\sigma}(t) \qquad \forall t \in [0,T]. \tag{6.17}$$

It follows from (4.19) that (6.11) hold and, moreover, $\boldsymbol{u} \in W^{1,\infty}(0,T;V)$. Next, from (4.32), (4.20), (4.5) and Korn's inequality (4.1), we obtain

$$\boldsymbol{u}(0) = \boldsymbol{u}_0. \tag{6.18}$$

Using (4.2) and the subdifferentiability of the seminorm $j(\boldsymbol{u}, \cdot) : V \longrightarrow \mathbb{R}_+$ in $\dot{\boldsymbol{u}}$, we deduce that there exists $\tilde{\boldsymbol{\tau}} : [0,T] \longrightarrow \mathcal{H}$ such that

$$\begin{aligned} &(\tilde{\boldsymbol{\tau}}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}))_{\mathcal{H}} + j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \dot{\boldsymbol{u}}) \geq (\boldsymbol{f}, \boldsymbol{v} - \dot{\boldsymbol{u}})_V \\ &\forall \boldsymbol{v} \in V, \ \text{a.e. on } (0,T). \end{aligned} \tag{6.19}$$

We replace $\boldsymbol{v} = 2\dot{\boldsymbol{u}}$ and $\boldsymbol{v} = \mathbf{0}_V$, both in $V$, in (6.19) to deduce

$$(\tilde{\boldsymbol{\tau}}, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}))_{\mathcal{H}} + j(\boldsymbol{u}, \dot{\boldsymbol{u}}) = (\boldsymbol{f}, \dot{\boldsymbol{u}})_V \quad \text{a.e. on } (0,T). \tag{6.20}$$

Using now (6.17), (6.19) and (6.20), it follows that

$$\tilde{\boldsymbol{\tau}}(t) \in \Sigma(t, A\boldsymbol{\sigma}(t)) \quad \text{a.e. on } (0,T).$$

Therefore, taking $\boldsymbol{\tau} = \tilde{\boldsymbol{\tau}}$ in (4.31) and using again (6.11) yields

$$(\tilde{\boldsymbol{\tau}}, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}))_{\mathcal{H}} \geq (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}))_{\mathcal{H}} \quad \text{a.e. on } (0,T),$$

and, keeping in mind (6.20), we obtain

$$(\boldsymbol{f}, \dot{\boldsymbol{u}})_V \geq (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}))_{\mathcal{H}} + j(\boldsymbol{u}, \dot{\boldsymbol{u}}) \quad \text{a.e. on } (0,T).$$

The converse inequality follows from (4.16) and (6.17), since $\boldsymbol{\sigma}(t) \in \Sigma(t, A\boldsymbol{\sigma}(t))$ for all $t \in [0,T]$. Thus, we conclude that

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}))_{\mathcal{H}} + j(\boldsymbol{u}, \dot{\boldsymbol{u}}) = (\boldsymbol{f}, \dot{\boldsymbol{u}})_V \quad \text{a.e. on } (0,T). \tag{6.21}$$

Using again (4.16) and (6.21) we deduce

$$\begin{aligned} &(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}))_{\mathcal{H}} + j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \dot{\boldsymbol{u}}) \geq (\boldsymbol{f}, \dot{\boldsymbol{u}})_V \\ &\forall \boldsymbol{v} \in V, \ \text{a.e. on } (0,T). \end{aligned} \tag{6.22}$$

It follows now from (4.10), (6.11), (6.18) and (6.22) that $\boldsymbol{u}$ is a solution to problem $\mathcal{P}_1$. □

**Proof of Theorem 4.4.**
(i) Let $\boldsymbol{u}_0 = A\boldsymbol{\sigma}_0$. Using (4.21) and (4.16) we obtain

$$(\boldsymbol{\sigma}_0, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + j(\boldsymbol{u}_0, \boldsymbol{v}) \geq (\boldsymbol{f}(0), \boldsymbol{v})_V \qquad \forall \boldsymbol{v} \in V,$$

and, by (4.10) and (4.19), it follows that $\boldsymbol{u}_0$ satisfies conditions (4.14) and (4.15). We now apply Theorem 4.2 (i) and find that there exists $L_0 > 0$ such that if $L_\mu |S|_{L^\infty(\Gamma_3)} < L_0$ then problem $\mathcal{P}_1$ has at least a solution $\boldsymbol{u} \in W^{1,\infty}(0,T;V)$. It follows now from Theorem 4.3 (i) that $\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u})$ is a solution of problem $\mathcal{P}_2$ which satisfies $\boldsymbol{\sigma} \in W^{1,\infty}(0,T;\mathcal{H})$.

(ii) Let (4.7) hold. The existence and uniqueness of the solution to problem $\mathcal{P}_2$ follows from arguments similar to those used in (i), using Theorem 4.2 (ii) and Theorem 4.3. Moreover, Theorem 4.2 (ii) shows that the mapping $(\boldsymbol{f}, \boldsymbol{u}_0) \longmapsto \boldsymbol{u}$ is Lipschitz continuous from $W^{1,\infty}(0,T;V) \times V$ to $L^\infty(0,T;V)$, and since $\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u})$, using (4.2), (4.5) and (4.20), we deduce that the mapping $(\boldsymbol{f}, \boldsymbol{\sigma}_0) \longmapsto \boldsymbol{\sigma}$ is Lipschitz continuous from $W^{1,\infty}(0,T;\mathcal{H}) \times \mathcal{H}$ to $L^\infty(0,T;\mathcal{H})$. □

## 7. Conclusion

Our main results in this paper concern the existence and the uniqueness of the weak solution in the study of a quasistatic frictional contact problem involving linear elastic materials. The contact is modeled with a version of Coulomb's law of dry friction in which the normal stress is prescribed on the contact surface and the coefficient of friction depends on the slip, that is $\mu = \mu(|\boldsymbol{u}_\tau|)$. The existence of the solution is obtained under the assumption $L_\mu |S|_{L^\infty(\Gamma_3)} < L_0$ and its uniqueness is proved in the case when the coefficient of friction $\mu$ does not depend on the slip $|\boldsymbol{u}_\tau|$. Here $L_0$ is a scalar parameter which depends on the elasticity operator and on the geometry of the problem, $L_\mu$ is the Lipschitz constant of the function $\mu = \mu(|\boldsymbol{u}_\tau|)$, and therefore it measures the slip weakening, and, finally, $S$ represents the given normal stress. From mathematical point of view, these restrictions are dicted from the structure of conditions (5.12) and (5.15) in Theorem 5.1 we use in order to prove the solvability and the unique solvability of the problem. The static version of the model was already studied in [6]. There, the existence of the solution was proved without imposing smallness assumptions on the problem data and the uniqueness was derived in the case when $\mu = \mu(|\boldsymbol{u}_\tau|)$, under a smallness assumption involving the coefficient of friction and the given normal stress on the contact surface. We conclude that our results in the study of the quasistatic process hold under more restrictive assumptions in comparison with those used in [6]. The reason is that, owing to inherent complicate nature, quasistatic frictional contact process are modeled by nonstandard variational inequalities which involve considerable difficulties in their mathematical analysis. An important extension of the results of this paper would remove the restrictions above and would allow for results on the qualitative behavior of the solution in the case when $\mu = \mu(|\boldsymbol{u}_\tau|)$, similar to those obtained in [6] in the study of the static process.

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