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ON TRIGONOMETRIC-LIKE DECOMPOSITIONS OF FUNCTIONS WITH RESPECT TO THE CYCLIC GROUP OF ORDER n

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Abstract. The cyclic group labelled family of α-projection operators implicitly present in [28] is used as in [5]–[9], [23] for investigation of decomposition of functions with respect to the cyclic group of order n. Series of new identities thus arising are demonstrated and new perspectives for further investigation are indicated as for example in the case of q-extended special polynomials. The paper constitutes an example of the application of the method of projections introduced in [26]; see also references [5]–[9].

1. Introduction

In the past century Ungar had introduced in his Indian J. Pure Appl. Math. paper [28] higher order α-hyperbolic functions which are denoted here as $\mathbf{Z}_n$ cyclic group labelled family $\{h_s^{\alpha}(z)\}_{s\in\mathbf{Z}_n}$. These functions are specific examples of eigenfunctions of the scaling Ω operator. Ω operator is used in this note to define a family of mutually orthogonal α-projection

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operators $\{\Pi_i^\alpha\}_{l\in\mathbf{Z}_n}$ and then these eigenfunctions themselves — including Ungar's α-hyperbolic functions (see: further examples below). For more information on the history of rediscovering standard $\alpha = 1$-hyperbolic $\{h_s(z)\}_{s\in\mathbf{Z}_n}$ and $\alpha = -1$-hyperbolic i.e. circular $\{f_s(z)\}_{s\in\mathbf{Z}_n}$ functions see first of all *The Mathematics Magazine* article [26] and for further references see also [19]–[20] and [4], [27]. If one takes $n = 2$, and $\alpha = \pm 1$ then one obtains cosh, sinh or cos and sin functions. Here one proposes an expedition a little bit more farther then *"Beyond Sin and Cos?"* [26]. In our story $\mathbf{Z}_n = \{0, 1, \dots, n-1\}$ denotes cyclic group under the addition i.e. for $k, l \in \mathbf{Z}_n$: $k \dot{+} l$ denotes addition mod n and $k \dot{-} l$ denotes subtraction mod n; $\omega = \exp\big(i(2\pi/n)\big)$; $n > 1$. While extending the range of *"Beyond Sin and Cos"* [26] we use group $\mathbf{Z}_n$ labelled family of α-projection $\{\Pi_l^\alpha\}_{l\in\mathbf{Z}_n}$ operators. Projection operators $\{\Pi_l\}_{l\in\mathbf{Z}_n}$ i.e. $\{\Pi_l^\alpha\}_{l\in\mathbf{Z}_n}$ with $\alpha = 1$, were used in [5]–[9], [23] for investigation of decomposition of functions with respect to the cyclic group of order n. We arrive at the *"Beyond Sin and Cos"* while α-decomposing exp function. Does then decomposing from [9], [23] of a function L given by Laurent series L leads too far *"Far Beyond Sin and Cos?"*. Perhaps this would be the better title of this article.

2. Expected elementary background

Apart from *The Mathematics Magazine* article [26] also monograph [10] on circulant matrices is recommended. As for Laurent series L considered here these may be treated also as formal Laurent series. This includes algebras of formal series (formal power series, exponential formal power series, Dirichlet series etc.) — as used in combinatorics [31]. This aspect is not pursued here — let us however remark that projection operators $\{\Pi_l\}_{l\in\mathbf{Z}_n}$ are ready to be applied for a might be desirable study of $\mathbf{Z}_n$ labelled subsequences of counting sequences in combinatorics. The use of circulant matrices enables one to introduce $\mathbf{Z}_n - L$-correspondents of trigonometric formulas for hyperbolic $\{h_s(z)\}_{s\in\mathbf{Z}_2}$ functions of second order in a manner this was done for hyperbolic $\{h_s(z)\}_{s\in\mathbf{Z}_n}$ functions of n-th order in [4], [27], [19]–[20].

It is easy to see that for $\{h_s(z)\}_{s\in\mathbf{Z}_2} \equiv \{\cosh z, \sinh z\}$, $z \in \mathbb{C}$ from the group property

$$\begin{pmatrix} \cosh z & \sinh z \\ \sinh z & \cosh z \end{pmatrix} \begin{pmatrix} \cosh w & \sinh w \\ \sinh w & \cosh w \end{pmatrix} = \begin{pmatrix} \cosh(z+w) & \sinh(z+w) \\ \sinh(z+w) & \cosh(z+w) \end{pmatrix} \quad \forall w, z \in \mathbb{C}$$

de Moivre formulas in their matrix form follow:

$$\begin{pmatrix} \cosh z & \sinh z \\ \sinh z & \cosh z \end{pmatrix}^n = \begin{pmatrix} \cosh nz & \sinh nz \\ \sinh nz & \cosh nz \end{pmatrix} \quad \forall w, z \in \mathbb{C}.$$

$\mathbf{Z}_n$ – exp-counterparts of the hyperbolic-trigonometric identity: $\forall w, z \in \mathbb{C}$;

$$(\cosh z)^2 - (\sinh z)^2 = 1, \text{ i.e. } \det \begin{pmatrix} \cosh z & \sinh z \\ \sinh z & \cosh z \end{pmatrix} = 1,$$

is introduced with help of circulants as in [19]–[20] (see also [2]). In the sequel we shall try to answer the question: are there $\mathbf{Z}_n - L$-counterparts also available?

The set of matrices

$$\begin{pmatrix} \cosh z & \sinh z \\ \sinh z & \cosh z \end{pmatrix}; \quad z \in \mathbb{R}$$

under matrix multiplication constitutes $SO(1,1)$ group. This is the group of two dimensional special relativity transformations.

The set of matrices

$$\begin{pmatrix} \cos z & -\sin z \\ \sin z & \cos z \end{pmatrix}; \quad z \in \mathbb{R}$$

under matrix multiplication constitutes $SO(2)$ group; this is of course the group of two dimensional rotations.

3. $\mathbf{Z}_n$ cyclic group labelled α-projection operators and $\mathbf{Z}_n$ decomposition of functions

In this section $\mathbf{Z}_n$ labelled α-projection operators are used for decomposition of functions with respect to the cyclic group of order n [28], [26], [5]–[9], [23]. Let us then define this family of α-projection operators.

Definition 3.1. $\{\Pi_l^\alpha\}_{l\in\mathbf{Z}_n}$ acting on the linear space of functions of complex variable are defined according to

$$\Pi_k^\alpha := \frac{1}{n}\alpha^{-k/n} \sum_{s\in\mathbf{Z}_n} \omega^{-ks}\Omega^s S(\sqrt[n]{\alpha}) \tag{1}$$

where $\sqrt[n]{\alpha}$ is an arbitrarily specified n-th root of α and Ω, $S(\lambda)$ are scaling operators:

$$(\Omega f)(z) := f(\omega z), \quad (S(\lambda)f)(z) := f(\lambda z), \quad \text{i.e. } \Omega = S(\omega). \tag{2}$$

The family of α-projection operators $\{\Pi_l^{(\alpha)}\}_{l\in\mathbf{Z}_n}$ extends the set of families of projection operators $\{V_k\}_k \in \mathbf{Z}_n$; $V_k \cdot V_l = V_l\delta_{kl}$ introduced in [18]. $\{\Pi_l^{(\alpha)}\}_{l\in\mathbf{Z}_n}$ is an easy generalization of the family of projection operators used under notation $\{\Pi_{[n,k]}\}_{k\in\mathbf{Z}_n}$ in [5]–[9] for a decomposition of various special functions with respect to the cyclic group of order n in analogy to the decomposition of exp function standard hyperbolic functions of n-th as was done and used under the notation $\{\Delta_k\}_{k\in\mathbf{Z}_n}$ in [23] in order to investigate higher order recurrences for analytical functions of Tchebysheff type [23], [1]. As $\{\Delta_k\}_{k\in\mathbf{Z}_n} \equiv \{\Pi_{[n,k]}\}_{k\in\mathbf{Z}_n}$ we shall use notation $\{\Pi_k\}_{k\in\mathbf{Z}_n} \equiv \{\Pi_{[n,k]}\}_{k\in\mathbf{Z}_n} \equiv \{\Pi_k^{\alpha=1}\}_{k\in\mathbf{Z}_n} \equiv \{\Delta_k\}_{k\in\mathbf{Z}_n}$ in conformity with all the papers mentioned and also this note. Of course (one arguments like in [18]) $\Pi_l\Pi_m = \delta_{lm}\Pi_l$ and from (1) and (2) one sees that $\Pi_k^{(\alpha)} = \alpha^{-k/n}S(\sqrt[n]{\alpha})\Pi_k$. Hence we infer what follows.

Observation 3.1.

$$\Pi_l^{(\alpha)}\Pi_m^{(\alpha)} = \delta_{lm}\Pi_l^{(\alpha)}\alpha^{-m/n}S(\sqrt[n]{\alpha}) \tag{3}$$

$$\sum_{k\in\mathbf{Z}_n}\alpha^{\frac{k}{n}}\Pi_k^{(\alpha)} = S(\sqrt[n]{\alpha}) \quad \textit{and} \quad \sum_{k\in\mathbf{Z}_n}\Pi_k = \mathrm{id}. \tag{4}$$

Although — as seen from formulas (1), (2), (3) and (4) — the α-projection operators $\Pi_k^{(\alpha)}$ differ from projection operators Π_k only by rescaling we keep introducing them because of reasons α-hyperbolic functions were introduced in [26], [28]. Namely $\alpha = -1, 0, +1$ cases may be treated with the same method and then formulas specified. This will therefore include $\alpha = -1$-hyperbolic i.e. circular $\{f_s(z)\}_{s\in\mathbf{Z}_n}$ functions, $\alpha = 0$-hyperbolic i.e. “binomial” [26] $\{h_s^0(z) = z^s/s!\}_{s\in\mathbf{Z}_n}$ functions and $\alpha = 1$-hyperbolic, i.e. hyperbolic $\{h_s(z)\}_{s\in\mathbf{Z}_n}$ functions. (In the $\alpha = 0$ case one uses after [28], [26] the convention $0^0 = 1$ — see Example 3.1.). Moreover, with help of these α-projection operators $\{\Pi_l^{(\alpha)}\}_{l\in\mathbf{Z}_n}$ one may define new families of eigenfunctions of the Ω operator. Here there are some introductory examples based on [23].

Example 3.1. Let $\{h_s^{(\alpha)}(z)\}_{s\in\mathbf{Z}_n}$ where $h_s^\alpha := \Pi_s^{(\alpha)}\exp$ then

$$h_s^\alpha = \sum_{k\geq 0}\frac{\alpha^k z^{nk+s}}{(nk+s)!} = \alpha^{-s/n}\sum_{k\geq 0}\frac{(\sqrt[n]{\alpha}z)^{nk+s}}{(nk+s)!}$$

and $\Omega h_s^\alpha = \omega^s h_s^\alpha$; $s \in \mathbf{Z}_n$. We shall call: h_l^α the $l-\alpha$-hyperbolic series (compare with [28], [26]). Of course

$$h_s^\alpha = \frac{1}{n}\alpha^{-s/n}\sum_{k\in\mathbf{Z}_n}\omega^{-ks}\exp(\omega^k\sqrt[n]{\alpha}z). \tag{5}$$

Note also [19]–[20] for future use that for $h_l \equiv h_l^{\alpha=1}$

$$\exp(\omega_l z) = \sum_{k\in\mathbf{Z}_n}\omega^{kl}h_k(z). \tag{6}$$

Let $\{g_l^\alpha(z)\}_{l\in\mathbf{Z}_n}$ where $g_l^\alpha := \Pi_l^{(\alpha)}1/(1-\mathrm{id})$ with $1/(1-\mathrm{id})(z) \equiv 1/(1-z)$ and $l \in \mathbf{Z}_n$ then $g_l^\alpha(z) = \sum_{k\geq 0}\alpha^k z^{nk+1}$ and $\Omega g_l^\alpha = \omega^l g_l^\alpha$. We shall call: g_l

the $l-\alpha$-geometric series; (compare with [23]). Of course

$$g_s^{\alpha}(z) = \frac{1}{n}\alpha^{-s/n}\sum_{k\in\mathbf{Z}_n}\omega^{-ks}g(\omega^k\sqrt[n]{\alpha}z). \tag{7}$$

Let $\{L_l^{\alpha}(z)\}_{l\in\mathbf{Z}_n}$ where $L_l^{\alpha} := \Pi_l^{(\alpha)}L$ with $L(z) = \sum_{k\in\mathbb{Z}} a_k z^k$ and $l \in \mathbf{Z}_n$ then $L_l^{\alpha} = \sum_{k\in\mathbb{Z}} a_{nk+l}\alpha^k z^{nk+l}$ and $\Omega L_l^{\alpha} = \omega^l L_l^{\alpha}$. We shall call: L_l^{α} the $l-\alpha$-Laurent series; (compare with [23]). Of course

$$L_s^{\alpha}(z) = \frac{1}{n}\alpha^{-sn}\sum_{k\in\mathbf{Z}_n}\omega^{-ks}L(\omega^k\sqrt[n]{\alpha}z) \tag{8}$$

Indeed:

$$\Pi_l\sum_{k\in\mathbb{Z}}a_kz^k = \sum_{k\in\mathbb{Z}}a_k\frac{1}{n}\sum_{s\in\mathbf{Z}_n}\omega^{s(k\dot{-}l)}z^k = \sum_{k\in\mathbb{Z}}a_kz^k\delta(k\dot{-}l) = \sum_{m\in\mathbb{Z}}a_{nm+l}z^{mn+l}$$

and now act with $\alpha(-1/n)S(\sqrt[n]{\alpha})$ on both sides in order to see that $L_l^{\alpha}(z) = \sum_{k\in\mathbb{Z}} a_{nk+l}\alpha^k z^{nk+l}$. This simple method of decomposition ([28], [26] and [5]–[9], [23]) of functions with respect to $\mathbf{Z}_n$ just by acting on them as in this note by α projection operators $\{\Pi_l^{(\alpha)}\}_{l\in\mathbf{Z}_n}$ may be then extended to explore special properties of new special functions $L_l^{\alpha}(z)$; $L \in \mathbf{Z}_n$ where L is any function expandable around complex $0 \in \mathbb{C}$ into Laurent series. In view of (4) functions $L_l(z) \equiv L_l^{\alpha=1}(z)$; $l \in \mathbf{Z}_n$ "*preserve the flavour of striking results like Euler's formula*" [26]. Indeed — in our $\alpha = 1$ case the generalised Euler formula is just this:

$$\sum_{l\in\mathbf{Z}_n} L_l(z) = L(z). \tag{9}$$

Also analogue of de Miovre formulas presented here in their matrix form [19]–[20] holds as well as correspondents of $(\cosh z)^2 - (\sinh z)^2 = 1$.

Example 3.2. $n = 3$; $\alpha = 1$ case.
Let us consider generalizations of cosh and sinh hyperbolic functions of the second order known since a long time (see [26], [4], [27], [19]–[20]). They are defined according to ($h_l \equiv h_l^{\alpha=1}$)

$$h_i(x) = \frac{1}{3}\sum_{k\in\mathbf{Z}_3}\omega^{-ki}\exp\{\omega^k x\};\ i\in\mathbf{Z}_3;\ \omega = \exp\left\{i\frac{2\pi}{3}\right\}. \tag{10}$$

One may call also (10) — Euler's formulas for hyperbolic functions of n-th order with $n = 3$. We put $n = 3$ only for convenience of easy presentation. In [19]–[20] identities for $\{h_i\}_{i\in\mathbf{Z}_n}$ hyperbolic functions are derived from properties of "de Moivre" groups which for $z \in \mathbb{R}$ and $m = 2$ coincide with $SO(1,1)$ (hyperbolic case). Let us then introduce at first $\gamma = (\delta_{i,k\dot{-}1})$; $k,i \in \mathbf{Z}_3$ i.e.

$$\gamma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

— the matrix generator of this "de Moivre" one parameter group. Now the following is obvious (check it): $\gamma^n = (\delta_{i,k\dot{-}i})^n = I$ and $Tr\gamma = Tr(\delta_{i,k\dot{-}i}) = 0$. Hence $\det\exp\{\gamma z\} = \exp\{Tr\gamma z\} = 1$ and $\{H(z) = \exp\{\gamma z\}\}_{z\in\mathbb{C}}$ forms what we call de Moivre group because $H(z)H(w) = H(z+w)$ where $H(z) = \exp\{\gamma z\}$; $\gamma = (\delta_{i,k\dot{-}i})$; $k,i \in \mathbf{Z}_n$ and $\det H(z) = 1$. Thus we arrive at the following observation.

Observation 3.2. *de Moivre formulas for $n = 3$ in their matrix may by written as follows: $\forall\phi \in \mathbb{C}$ and $\forall n \in \mathbb{Z}$*

$$H(n\phi) = \begin{pmatrix} h_0(n\phi) & h_1(n\phi) & h_2(n\phi) \\ h_2(n\phi) & h_0(n\phi) & h_1(n\phi) \\ h_1(n\phi) & h_2(n\phi) & h_0(n\phi) \end{pmatrix}. \tag{11}$$

Due to the group property of $\{H(z) = \exp\{\gamma z\}\}_{z\in\mathbb{C}}$ one easily gets series of identities [19]–[20], [23].

Observation 3.3. *For $n = 3$: $\forall k,m \in \mathbb{Z}$ and $\forall l \in \mathbf{Z}_3$ the following three identities hold:*

$$h_i((n+k)z) = 3h_0(nz)h_l(kz) - h_l((n+k\omega)z) - h_l((n+k\omega^2)z) \tag{12}$$

as well as $\forall x \in \mathbb{C}$

$$h_0(3x) = h_0^3(x) + h_1^3(x) + h_2^3(x) + 3!h_0(x)h_1(x)h_2(x) \tag{13}$$

and (see [21])

$$h_0(x)h_1(x)h_1(x) = \frac{1}{9}(h_0(3x) - 1). \tag{14}$$

Observation 3.4. *The identity corresponding to $(\cosh\alpha)^2 - (\sinh\alpha)^2 = 1$ identity for $n = 2$ is the following:*

$$h_0^3(\phi) + h_1^3(\phi) + h_2^3(\phi) - 3h_0(\phi)h_1(\phi)h_2(\phi) = 1 \tag{15}$$

which is equivalent to $\det H(\phi) = 1$; $\forall\phi \in \mathbb{C}$ i.e.

$$\begin{vmatrix} h_0(\phi) & h_1(\phi) & h_2(\phi) \\ h_2(\phi) & h_0(\phi) & h_1(\phi) \\ h_1(\phi) & h_2(\phi) & h_0(\phi) \end{vmatrix} = 1. \tag{16}$$

Naturally some of these identities are easy to be written for arbitrary n; for example (12) is a specification of (17) (see: (2.1) in [23]).

Observation 3.5. *$\forall \alpha, \beta \in \mathbb{C}$ and $\forall l \in \mathbf{Z}_n$:*

$$h_0(\alpha) h_l(\beta) = \frac{1}{n} \sum_{k \in \mathbf{Z}_n} h_l(\alpha + \omega^k \beta). \tag{17}$$

(Compare the formula (17) with (4.2) formulas in [4] and [27].)

From $H(z)H(w) = H(z+w)$ and cyclicity of $H(z)$ matrix we derive: $\forall k \in \mathbf{Z}_n$

$$h_k(x+y) = \sum_{i \in \mathbf{Z}_n} h_i(x) h_{k \dot{-} i}(y). \tag{18}$$

For real parameter $\phi \in \mathbb{R}$ the elements of the de Moivre one parameter group might be represented by points $(h_0(\phi), h_1(\phi), h_2(\phi))$ of the curve defined by (16). This curve runs on the surface defined by the equation $x^3 + y^3 + z^3 - 3xyz = 1$; see [3]. For $m = 4$ case due to $\det H(\phi) = 1$ — the corresponding hyper-surface is defined by

$$-x^4 + y^4 - z^4 + t^4 + 4x^2yt - 4xy^2z + 4z^2yt - 4t^2xz + 2x^2z^2 - 2y^2t^2 = 1.$$

Example $\boldsymbol{\alpha}$-3.2 . $n = 3$ — Ungar's α-hyperbolic case.
Let us consider Ungar's α-hyperbolic functions of the $n = 3$ order. They are defined — by (5)

$$h_i^\alpha(z) = \frac{1}{3} \alpha^{-i/n} \sum_{k \in \mathbf{Z}_3} \omega^{-ki} \exp\{\omega^k \sqrt[n]{\alpha} z\};\ i \in \mathbf{Z}_3;\ \omega = \exp\{i \frac{2\pi}{3}\}. \tag{α-10}$$

One may call (α-10) — Euler's formulas for Ungar's α-hyperbolic functions of n-th order with $n = 3$. (We put $n = 3$ only for convenience of easy presentation.) As in [23] identities for $\{h_s^\alpha(z)\}_{s \in \mathbf{Z}_n}$ Ungar's α-hyperbolic functions might be derived from properties of α-de Moivre groups which for $z \in \mathbb{R}$ and $m = 2$ coincide with $SO(2)$ ($\alpha = -1$; elliptic case) or $SO(1,1)$ ($\alpha = +1$; hyperbolic case). The matrix generator of this α-de Moivre one parameter group is the matrix

$$\gamma(\alpha) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha & 0 & 0 \end{pmatrix}.$$

The following is obvious: $\gamma(\alpha)^n = \alpha I$, $\mathrm{Tr}\gamma = 0$ and as $\det\{\exp A\} = \exp\{\mathrm{Tr} A\}$ then $\det \exp\{\gamma(\alpha) z\} = 1$ and $\{H^\alpha(z) = \exp\{\gamma(\alpha) z\}\}_{z \in \mathbb{C}}$ forms what we call an α-de Moivre group. Sure; $H^\alpha(z) H^\alpha(w) = H^\alpha(z+w)$ takes place for arbitrary n where $H^\alpha(\phi) = \exp\{\gamma(\alpha)\phi\}$, $\gamma(\alpha) = (\delta_{i, k \dot{-} i} + (\alpha - 1)\delta_{n-1,0})$; $k, i \in \mathbf{Z}_n$ and $\det H^\alpha(z) = 1$. Therefore we observe what follows.

Observation 3.6. *α-de Moivre formulas in the matrix form are given by:*

$$\forall\alpha,\phi\in\mathbb{C},\forall n\in\mathbb{Z}, H^\alpha(n\phi)=\begin{pmatrix} h_0^\alpha(n\phi) & h_1^\alpha(n\phi) & h_2^\alpha(n\phi)\\ \alpha h_2^\alpha(n\phi) & h_0^\alpha(n\phi) & h_1^\alpha(n\phi)\\ \alpha h_1^\alpha(n\phi) & \alpha h_2^\alpha(n\phi) & h_0^\alpha(n\phi)\end{pmatrix} \equiv (H^\alpha(\phi))^n. \tag{α-11}$$

For real group parameter $\phi\in\mathbb{R}$ and $\alpha\in\mathbb{R}$ the elements of the de Moivre one parameter group might be represented by points $\big(h_0^\alpha(\phi),h_1^\alpha(\phi),h_2^\alpha(\phi)\big)$ of the curve defined by $\det H^\alpha(\phi)=1$. This curve runs on the surface defined by the equation

$$x^3+\alpha y^3+\alpha^2z^3-\alpha 3xyz=1.$$

Due to the group property of $\big\{H^\alpha(\phi)=\exp\{\gamma(\alpha)\phi\}\big\}_{z\in\mathbb{C}}$ one may obtain series of identities as in [19]–[20], [23].

Problem 3.1. *We ask: "May one obtain trigonometric-like identities* (10)–(18) *in the case when* exp *function is replaced by L function representing Laurent series?"*

For that to try to answer in $\alpha=1$ case let us at first recall again the identity (6) and let us note that it generalises to the case when exp function is replaced by L function representing Laurent series-just act with $\sum_{k\in\mathbf{Z}_n}\Pi_k=\mathrm{id}$ on $L(\omega^l z)$ — i.e.

$$L(\omega^l z)=\sum_{k\in\mathbf{Z}_n}L_k(\omega^l z)=\sum_{k\in\mathbf{Z}_n}\omega^{kl}L_k(x). \tag{19}$$

Now introduce circulant matrix — an analogue of $H(z)$ — according to

$$C(\vec{L})(z)=L\{\gamma z\} \tag{20}$$

where $\vec{L}\equiv\big(L_0(z),L_1(z),\dots,L_{n-1}(z)\big)$ i.e. consider the matrix of the form

$$C(\vec{L})(z)=\begin{pmatrix} L_0(z) & L_1(z) & \dots & L_{n-1}(z)\\ L_{n-1}(z) & L_0(z) & \dots & L_{n-2}(z)\\ \dots & \dots & \dots & \dots\\ L_1(z) & L_2(z) & \dots & L_0(z)\end{pmatrix}. \tag{21}$$

We know from textbooks [17] that $C(\vec{L})(z)=\sum_{k\in\mathbf{Z}_n}L_k(z)\gamma^k$ and the spectrum of γ matrix is just the multiplicative cyclic group $\hat{\mathbf{Z}}_n=\{\omega^k\}_{k\in\mathbf{Z}_n}$. Due to this elementary fact

$$\det C(\vec{L})(z)=\prod_{l\in\mathbf{Z}_n}\sum_{k\in\mathbf{Z}_n}L_k(z)\omega^{kl} \tag{22}$$

(see Remarks 3.1. bellow for links with discrete Fourier transform). Formulas (19) and (22) imply then another one and very important one (compare with (8.1) in [5]):

$$\det\begin{pmatrix} L_0(z) & L_1(z) & \dots & L_{n-1}(z) \\ L_{n-1}(z) & L_0(z) & \dots & L_{n-2}(z) \\ \dots & \dots & \dots & \dots \\ L_1(z) & L_2(z) & \dots & L_0(z) \end{pmatrix} = \prod_{l\in\mathbf{Z}_n} L(\omega^l z). \tag{23}$$

For $L = \exp$ we come back to $\{L_s(z)\}_{s\in\mathbf{Z}_n} = \{h_s(z)\}_{s\in\mathbf{Z}_n}$ i.e. hyperbolic functions of n-th order and the formula (23) coincides with $\det H(z) = 1$ because $\prod_{l\in\mathbf{Z}_n}\exp(\omega^l z) \equiv 1$.

Answer to Problem 3.1. Coming now back to our question above for $\alpha = 1$ case *"May one obtain trigonometric-like identities* (10) – (18) *in the case when* exp *function is replaced by* L *function representing Laurent series?"* we answer:

Observation 3.7. *The crucial trigonometric-like identities* (11), (16) – (18) *do not hold in the case when* exp *function is replaced by* $L \neq \exp$ *function representing Laurent series.*

We readily see that exp function is exceptional and irreplaceable because of the following.

Observation 3.8. *Only for* $L = \exp$ *(up to scaling of the argument) circulant matrices* $C(\hat{L})(z) = L\{\gamma z\}$ *form a group such that* $L(z)L(w) = L(z+w)$.

However not all is lost. One may find out many counterparts, analogue identities to those originating from exp decomposition with help of projection operators family $\{\Pi_l\}_{l\in\mathbf{Z}_n}$ [28], [26], [5]–[9], [23] even in arbitrary $\alpha\in\mathbb{C}$ case. For that to see let us recall again the formula (19):

$$L(\omega^l z) = \sum_{k\in\mathbf{Z}_n} L_k(\omega^l z) = \sum_{k\in\mathbf{Z}_n}\omega^{kl}L_k(x).$$

Now apply to both sides the operator $\sum_{k\in\mathbf{Z}_n}\alpha^{k/n}\Pi_k^{(\alpha)} = S(\sqrt[n]{\alpha})$ and recall that $L_l^\alpha := \Pi_l^{(\alpha)}L$. Then we get

$$L(\omega^l \sqrt[n]{\alpha} z) \;=\; \sum_{k\in\mathbf{Z}_n}\alpha^{k/n}L_k^\alpha(\omega^l z) \;=\; \sum_{k\in\mathbf{Z}_n}\alpha^{k/n}\omega^{kl}L_k^\alpha(x). \tag{α-19}$$

One may also introduce α-circulant matrix — an analogue of $H^\alpha(z)$ — according to

$$C^\alpha(\vec{L})(z) = L\{\gamma(\alpha)z\} \tag{α-20}$$

where $\vec{L} \equiv (L_0^\alpha(z), L_1^\alpha(z), \ldots, L_{n-1}^\alpha(z))$ so that we consider now the α-circulant matrix

$$C^\alpha(\vec{L})(z) = \begin{pmatrix} L_0^\alpha(z) & L_1^\alpha(z) & \ldots & L_{n-1}^\alpha(z) \\ \alpha L_{n-1}^\alpha(z) & L_0^\alpha(z) & \ldots & L_{n-2}^\alpha(z) \\ \ldots & \ldots & \ldots & \ldots \\ \alpha L_1^\alpha(z) & \alpha L_2^\alpha(z) & \ldots & L_0^\alpha(z) \end{pmatrix}. \tag{α-21}$$

We know from textbooks [17] that $C^\alpha(\vec{L})(z) = \sum_{k\in\mathbf{Z}_n} L_k^\alpha(z)\gamma(\alpha)^k$ and the spectrum of $\gamma(\alpha)$ matrix is just $\alpha^{1/n}\hat{\mathbf{Z}}_n \equiv \left\{\alpha^{1/n}\omega^k\right\}_{k\in\mathbf{Z}_n}$ because $\gamma(\alpha)^n = \alpha I$. Due to this simple fact and (α-19)

$$\det C^\alpha(\vec{L})(z) = \prod_{l\in\mathbf{Z}_n}\sum_{k\in\mathbf{Z}_n} L_k^\alpha(z)\alpha^{k/n}\omega^{kl}. \tag{α-22}$$

$$\det\begin{pmatrix} L_0^\alpha(z) & L_1^\alpha(z) & \ldots & L_{n-1}^\alpha(z) \\ \alpha L_{n-1}^\alpha(z) & L_0^\alpha(z) & \ldots & L_{n-2}^\alpha(z) \\ \ldots & \ldots & \ldots & \ldots \\ \alpha L_1^\alpha(z) & \alpha L_2^\alpha(z) & \ldots & L_0^\alpha(z) \end{pmatrix} = \prod_{l\in\mathbf{Z}_n} L(\omega^l \sqrt[n]{\alpha} z) \tag{α-23}$$

— so as we see — (α-23) for $\alpha \neq 1$ is also comfortable and handy as (23) (see (8.1) in [5]). For $L = \exp$ we come back to $\{L_s(z)\}_{s\in\mathbf{Z}_n} = \{h_s^\alpha(z)\}_{s\in\mathbf{Z}_n}$ Ungar's α-hyperbolic functions of n-th order and the formula (α-23) coincides with $\det H^\alpha = 1$ because $\prod_{l\in\mathbf{Z}_n}\exp(\omega^l|\alpha|z) \equiv 1$.

Miscellaneous Remarks 3.1.

1. γ matrix plays a crucial role in $\mathbf{Z}_n$-quantum mechanics (see: (2.3) in [22] and references therein and also see: (2.5) in [22]).
2. Columns of Sylvester matrix $S = (1/\sqrt{n})(\omega^{kl})_{kl\in\mathbf{Z}_n}$ are eigenvectors of γ matrix which makes a link to $\mathbf{Z}_n$-group discrete Fourier transform analysis and synthesis [21], where harmonic analysis is the passage from functional values to coefficients while harmonic synthesis is the passage from coefficients to functional values.
3. For a related simple generalisation of analytic function theory see [12].
4. It is obvious that $H^\alpha(z) = \exp\{\gamma(\alpha)z\}$, $\gamma(\alpha) = (\delta_{i,k\dot{-}1} + (\alpha-1)\delta_{n-1,0})$; $k, i \in \mathbf{Z}_n$ is the unique solution of the equation

$$\frac{d}{dz}H^\alpha(z) = \gamma(\alpha)H^\alpha(z)$$

with $H^\alpha(0) = I$. This is equivalent to say that γ is the generator of α-de Moivre group $H^\alpha(z) = \exp\{\gamma(\alpha)z\}$. Naturally from the above we conclude (compare with (5) in [26]) that

$$\frac{d^n}{dz^n}H^\alpha(z) = \alpha H^\alpha(z);\ \frac{d}{dz}h_s^\alpha(z) = (1 + (\alpha-1)\delta_{0,s})h_{s-1}^\alpha(z);\ s \in \mathbf{Z}_n).$$

5. Sylvester matrix of $\mathbf{Z}_n$-discrete Fourier transform analysis serves to diagonalize our hero-circulant matrix $C(\vec{L})(z)$ as well as α-hero: an α-circulant matrix (see definition below and [26], [28]).
6. Consider $\alpha \in \mathbb{C}$ case. It is like we went away too far from the source of trigonometric analogies i.e. from exp function when exp function is replaced by L function representing Laurent series. Therefore we shall consider now the so called $\psi -$ exp functions [24]–[25], [7].

4. On q-extension and ψ-extension of higher order α-hyperbolic functions

In this section we shall try to stay close to exp. At first we shall refer to what is known since a long time; see [13]–[15] from 1910 year and [11] for Heine and Gauss contribution and [16] for may be application to quantum processes description and overall theory of the so called non-commutative geometry. Therefore we shall consider here a specific example of such series L which are extensions of exp with some properties surviving or being mimicked. These are $\exp_q$ and $\exp_\psi$ functions. Before doing that we shall give some preliminaries.

We perform after Heine and Gauss [11] a replacement $x \rightarrowtail x_q$ thus arriving at the standard by now deformation of the variable $x \in \mathbb{R}$ [11], [16] according to the prescription:

$$x \rightarrowtail x_q \equiv \frac{1-q^x}{1-q} \xrightarrow[q \to 1]{} x.$$

Then consequently we have for n_q, q-factorial and q-binomial coefficients:

$$\binom{n}{k}_q \equiv \frac{n_q^{\underline{k}}}{k_q!} \quad \text{where} \quad n_\psi^{\underline{k}} = n_\psi (n-1)_\psi (n-2)_\psi \dots (n-k+1)_\psi .$$

Also integration and derivation [13]–[15] might be q-extended. Here we introduce only — what is called — Jackson's derivative ∂_q — a kind of difference operator.

Definition 4.1. Jackson's derivative ∂_q is defined as follows. Let ϕ denote any Laurent series. Then

$$(\partial_q \phi)(x) = \frac{\phi(x) - \phi(qx)}{(1-q)x}.$$

Naturally

$$\partial_q \xrightarrow[q \to 1]{} \frac{d}{dx}$$

and it is a mere of exercise to prove that Q-Leibnitz rule holds.

Observation 4.1. *Let f, g, ϕ denote Laurent series. Let $(Q\phi)(z) := \phi(qz)$. Then $\partial_q(f \cdot g)(\partial_q f) \cdot g + (Qf) \cdot (\partial_q g)$.*

It is a easy to see $\partial_q x^n = n_q x^{n-1}$ and $\partial_q \exp_q = \exp_q$; $\exp_q[z]_{z=0} = 1$ where $q-\exp$ function is defined by $\exp_q[z] := \sum_{k=0}^{\infty} z^n/n_q!$. Applying now projection operators $\{\Pi_l\}_{l\in\mathbf{Z}_n}$ to $\exp_q$ function we get the family $\{h_{q,s}(z)\}_{s\in\mathbf{Z}_n}$ of q-extended hyperbolic functions of order n.

Definition 4.2. $\{h_{q,s}(z)\}_{s\in\mathbf{Z}_n}$ are defined by

$$h_{q,s} = \Pi_s \exp_q; \quad s \in \mathbf{Z}_n; \quad h_{q,s}h_s; \quad s \in \mathbf{Z}_n. \tag{24}$$

Many formulas and identities q-extend almost automatically from the $q = 1$ case as for example those from [6] with q-extended Laguerre polynomials $L_{n,q}^{(\alpha=-1)}(x) \equiv L_{n,q}(x)$ replacing the standard ones which are the so called binomial type or convolution type depending on n-dependent factor. These q-identities yield automatically the corresponding ones for projected out functions $L_l^{\alpha} := \Pi_l^{(\alpha)} L$; for example for $L = L_{n,q}(x)$.

Example 4.1 (see [24]–[25])**.** As an example of q-extended polynomial sequences we present now the q-extended Laguerre polynomials $L_{n,q}^{(\alpha=-1)}(x) \equiv L_{n,q}(x)$.

$$L_{n,q}(x) = \frac{n_q}{n} \sum_{k=1}^{n} (-1)^k \frac{n_q!}{k_q!} \binom{n-1}{k-1}_q \frac{k}{k_q} x^k$$

form the so called *basic polynomial sequence* $\{L_{n,q}(X)\}_{n\geq 0}$ of the operator

$$Q(\partial_q) = -\sum_{k=0}^{\infty} \partial_q^{k+1} \equiv \frac{\partial_q}{\partial_q - 1} \equiv -[\partial_q + \partial_q^2 + \partial_q^3 + \dots]$$

which is equivalent to say that for polynomial sequence $p_n(x) = L_{n,q}(x)$; $\deg p_n(x) = n$ the following requirements are fulfilled: $p_0(x) = 1$; b) $p_n(0) = 0$; and c) $Q(\partial_q)p_n = n_p p_{n-1}$. One may show that the so called q-binomiality identity holds [24]–[25]:

$$p_n(x +_q y) = \sum_{k\geq 0} \binom{n}{k}_q p_k(x) p_{n-k}(x)$$

where

$$E^a(\partial_q) = \exp_q\{a\partial_q\} = \sum_{k=0}^{\infty} \frac{a^k}{k_q!} \partial_q^k \quad \text{and} \quad E^y(\partial_q)p_n(x) \equiv p_n(x +_q y).$$

The above q-binomial identity yield automatically the corresponding ones for projected out new special q-polynomials $L_l^{\alpha} := \Pi_l^{(\alpha)} L$; $L = L_{n,q}(x)$.

The same could be applied also to q-extensions of the well known hyperbolic functions of any order. Before considering this in the next example let us at first note that similarly to

$$\frac{dh_k(x)}{dx} = h_{k-1}(x);\ k \in \mathbf{Z}_n$$

from which it follows that

$$\frac{d^k h_l(x)}{dx^k} = h_{l \dot{-} k}(x);\ k, l \in \mathbf{Z}_n$$

— also the following holds.

Observation 4.2.

$$\partial_q^k h_{q,l}^\alpha = \prod_{s=0}^{k \dot{-} 1} \left(1 + (\alpha - 1)\delta_{0,l \dot{-} s}\right) h_{q,l \dot{-} k}^\alpha;\ k, l \in \mathbf{Z}_n. \tag{25}$$

where

$$h_{q,s}^\alpha \equiv \Pi_s^\alpha \exp_q;\ s \in \mathbf{Z}_n. \tag{26}$$

Example 4.2 (versus Example 3.1)**.**
Let $\{h_{q,s}^\alpha(z)\}_{s \in \mathbf{Z}_n}$ where $h_{q,s}^\alpha = \Pi_s^\alpha \exp_q$ then

$$h_{q,s}^\alpha(z) = \sum_{k \geq 0} \frac{\alpha^k z^{nk+s}}{(nk+s)_q!}$$

and $\Omega h_{q,s}^\alpha = \omega^s h_{q,s}^\alpha$; $s \in \mathbf{Z}_n$. We shall call: $h_{q,l}^\alpha$ the $l - \alpha - q$-hyperbolic series. Of course

$$h_{q,s}^\alpha(z) = \frac{1}{n}\alpha^{-s/n} \sum_{k \in \mathbf{Z}_n} \omega^{-ks} \exp_q(\omega^k \sqrt[n]{\alpha} z);\ s \in \mathbf{Z}_n. \tag{27}$$

Note also that $(h_{q,j} \equiv h_{q,l}^{\alpha=1})$; $\exp_q(\omega^l z) = (1/n)\sum_{k \in \mathbf{Z}_n} \omega^{kl} h_{q,k}(z)$; $l \in \mathbf{Z}_n$. The "ω-with" rescaling operator Ω has much more of eigenvectors apart from the family represented by

$$\Omega h_{q,s}^\alpha = \omega^s h_{q,s}^\alpha;\ s \in \mathbf{Z}_n \textit{ or by } \Omega L_l^\alpha = \omega^l L_l^\alpha \tag{28}$$

where L_l^α are the $l - \alpha$-Laurent series (see Example 3.1). Namely: consider the generalised factorial $n_\psi! \equiv n_\psi (n-1)_\psi (n-2)_\psi \cdots 2_\psi 1_\psi$; $0_\psi! = 1$ for an arbitrary sequence $\psi = \{\psi_n\}_{n \geq 1}$ with the condition, $\psi_n \neq 0$, $n \in \mathbb{N}$. Here n_ψ denotes the ψ-deformed number where in conformity with Viskov [29]–[30] notation $n_\psi \equiv \psi_{n-1}(q)\psi_n^{-1}(q)$ or equivalently $n_\psi! \equiv \psi_n^{-1}(q)$ [24]–[25]. One may now define linear operator ∂_ψ named ψ-derivative on — say — polynomials according to: $\partial_\psi x^n = n_\psi x^{n-1}$; $n > 0$, $\partial_\psi \text{const} = 0$. One defines then $\psi - \exp$ function $\exp_\psi[z] := \sum_{k=0}^\infty z^n / n_\psi!$ so that all other constructions and statements of this section "ψ-extend" automatically. An so

$$\Omega h^{\alpha}_{\psi,s} = \omega^s h^{\alpha}_{\psi,s};\ s \in \mathbf{Z}_n \tag{29}$$

with self-explanatory notation: $h^{\alpha}_{\psi,s} = \Pi^{\alpha}_s \exp_{\psi}$.

Remark 4.1 (see [29]–[30] and [24]–[25]). We may introduce now

$$\binom{n}{k}_{\psi} \equiv \frac{n^k_{\psi}}{k_{\psi}!}$$

where $n^k_{\psi} = n_{\psi}(n-1)_{\psi}\dots(n-k+1)_{\psi}$ and extend a very important notion of the polynomial sequence of binomial type. Here are examples: take for polynomial sequence $\{p_n\}_0^{\infty}$; $\deg p_n = n$; $p_n(x) = x^n$ or take $p_n(x) = x^{\underline{n}} = x(x-1)\dots(x-n+1)$. Then one easily checks that the following identity holds:

$$p_n(x+y) \equiv \sum_{k\geq 0} \binom{n}{k}_{\psi} p_k(x)p_{n-k}(y). \tag{30}$$

Polynomial sequences satisfying (30) are polynomial sequences of binomial type. Polynomial sequence $\{p_n\}_0^{\infty}$ is then of ψ-binomial type if it satisfies the recurrence

$$E^y(\partial_{\psi})p_n(x) \equiv \sum_{k\geq 0} \binom{n}{k}_{\psi} p_k(x)p_{n-k}(y)$$

where

$$E^y(\partial_{\psi}) \equiv \exp_{\psi}\{y\partial_{\psi}\} = \sum_{k=0}^{\infty} \frac{y^k \partial^k_{\psi}}{k_{\psi}!}$$

is a generalised translation operator [24]–[25]. Polynomials encompassing those of ψ-binomial type are the so called [24]–[25] Sheffer ψ-polynomials. In [29, Proposition 8] Viskov have proved that polynomial sequence $\{p_n\}_0^{\infty}$ is Sheffer ψ-polynomial if and only if its "ψ-generating function" is of the form:

$$\sum_{n\geq 0} \psi_n p_n(x) z^n = A(z)\psi(xg(z)); \tag{31}$$

$$\psi(z) = \sum_{n\geq 0} \psi_n z^n;\ \psi_n \neq 0;\ n = 0, 1, 2, \dots \tag{32}$$

where $A(z)$, $g(z)/z$ are formal series with constant terms different from zero. In the very important reference [7] Y. Ben Cheikh has given important examples of decomposition of the Boas-Buck polynomials with respect to the cyclic group $\mathbf{Z}_n$. In our notation [24]-[25] adapted to [29]–[30] these are Sheffer ψ-polynomials including polynomial sequences of ψ-binomial type.

Example 4.3. It is easy to check that for $\psi_n(q) = [R(q^n)!]^{-1}$ and $R(x) = (1-x)/(1-q)$ we get $\psi_n(q) = n_q$. In [29, Proposition 4] Viskov have proved also that polynomial sequence $\{p_n\}_0^\infty$ is of ψ-binomial type if and only if its "ψ-generating function" is of the form

$$\sum_{n\geq 0} \psi_n p_n(x) z^n = \exp_\psi(xg(z)) \tag{33}$$

for formal series g inverse to appropriate formal series (see [29]). Now for $\psi_n(q) = [n_q!]^{-1}$, $\psi(z) = \exp_q\{z\}$ and "$\exp_q$ generating function" (33) takes the form

$$\sum_{n\geq 0} \frac{z^n}{n_q!} p_n(x) = \exp_q(xg(z)). \tag{34}$$

If one denotes by $p_{n,s}^{\alpha,\psi}$ the following eigenpolynomials of Ω: $p_{n,s}^{\alpha,\psi} = \Pi_s^\alpha p_n$; $s \in \mathbf{Z}_n$ and if $A(z) = 1$ then for generating functions of these special polynomials we get from (33) the following expressions:

$$\sum_{n\geq 0} \psi_n p_{n,s}^{\alpha,\psi}(x) z^n = h_{\psi,s}^\alpha(xg(z)),\ s \in \mathbf{Z}_r. \tag{35}$$

If in addition $g = \mathrm{id}$ then

$$\sum_{n\geq 0} \psi_n p_{n,s}^{\alpha,\psi}(x) z^n = h_{\psi,s}^\alpha(xz),\ s \in \mathbf{Z}_r. \tag{36}$$

We call functions $h_{\psi,s}^\alpha$ the ψ-hyperbolic functions. Naturally Ω ω-rescales x argument of $p_{n,s}^{\alpha,\psi}$ and $h_{\psi,s}^\alpha(xg(z))$; $s \in \mathbf{Z}_r$ and both immense sets of these special functions are ω^s-homogeneous, $s \in \mathbf{Z}_r$, which is equivalent to say that these are eigenfunctions of scaling operator Ω corresponding to the eigenvalue ω^s; $s \in \mathbf{Z}_r$. For $\psi_n(q) = n_q$ one gets from (34) q-deformed ω^s-homogeneous special q-hyperbolic functions $h_{q,s}^\alpha$ and special ω^s-homogeneous q-deformed polynomials $p_{n,s}^{\alpha,q}$. In the limit case of $q = 1$ we end up with classical special polynomials — for example with Laguerre polynomials [6] — and other polynomial sequences — for example of binomial type.

Remark 4.2. Note that in the case of analytic functions instead of $f(x)$ one may consider also functions with matrix arguments $f(A)$; $A \in M_{kxk}(\mathbb{C})$ or arguments from associative algebras with unity over $\mathbb{C}$ equipped with norm in order to assure the possibility of convergence. Hyperbolic mappings of such type might be now equally well investigated.

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