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ON THE STATIONARY FLOW OF THE POWER LAW FLUID IN 2D

W. SADOWSKI

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Abstract. We consider the stationary flow of a heat conducting Power Law shear thinning fluid in a bounded domain in $\mathbb{R}^2$. We present an elementary proof of existence of at least one weak solution.

1. Introduction

Various mathematical aspects of the flow of a heat conducting incompressible nonNewtonian fluid was studied recently by many author ([2], [3], [4], [5]). A considerable attention was paid to the flow of a very important class of nonNewtonian fluids called Power Law fluids.

The general governing system of equations for the flow of heat conducting incompressible fluids consists of the following equations:

$$\operatorname{div} u = 0, \tag{1.1}$$

$$\rho\frac{\partial u_i}{\partial t} = -\rho u_j\frac{\partial u_i}{\partial x_j} + f_i\rho + \operatorname{div} T_i \qquad i = 1, 2, \dots, d, \tag{1.2}$$

$$\rho\frac{\partial \theta}{\partial t} = -\rho u_j\frac{\partial \theta}{\partial x_j} + \frac{\partial u_i}{\partial x_j}T_{ij} - \operatorname{div}\,(-K\nabla\theta). \tag{1.3}$$

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Here $u = (u_1, u_2, \dots, u_d)$ is the velocity of the fluid, θ is the temperature, T_i is the i-th column of the stress tensor T, $f = (f_1, f_2, \dots, f_d)$ represents external forces, K is the thermal conductivity and d is the space dimension. In special case of the Power Law fluids the stress tensor is of the form

$$T_{ij} = k(\theta)|e(u)|^{r-2} e_{ij}(u) - \delta_{ij} p, \tag{1.4}$$

where $k(\theta)$ is a function of temperature, δ is the Kronecker's symbol and $e(u)$ is a matrix defined by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

The parameter r is a real number bigger than 1. If $1 < r < 2$ then the fluid is shear thinning. If $r = 2$ the fluid is Newtonian and if $r > 2$ we have the case of the shear thickening fluid.

In this paper we consider the stationary flow of incompressible Power Law shear thinning fluid in a 2-dimensional bounded domain Ω. For simplicity we set $\rho = K = 1$. Then the system (1.1)–(1.3) takes the form:

$$\operatorname{div} u = 0 \qquad \text{in } \Omega, \tag{1.5}$$

$$-\frac{\partial}{\partial x_j}\left(k(\theta)|e(u)|^{r-2} e_{ij}(u)\right) - \frac{\partial p}{\partial x_i} + u_j \frac{\partial u_i}{\partial x_j} = f_i \qquad (i = 1, 2) \quad \text{in } \Omega, \tag{1.6}$$

$$-\triangle\theta + u_j \frac{\partial \theta}{\partial x_j} = \frac{\partial u_i}{\partial x_j}(k(\theta)|e(u)|^{r-2} e_{ij}(u) - \delta_{ij} p) \qquad \text{in } \Omega, \tag{1.7}$$

$$u|_{\partial\Omega} = u_0 \qquad \theta|_{\partial\Omega} = \theta_0. \tag{1.8}$$

The existence of solutions for this system with nonhomogeneous Dirichlet boundary conditions was proved in [4]. In a special case of this system for $1 < r < 2$ and with homogeneous boundary conditions we prove the existence of solutions in a more elementary way. We also get slightly better regularity of temperature since our solution belongs not only to each of the Sobolev spaces $W^{1,s}(\Omega)$, $1 \le s < 2$, but also to Sobolev space $H^1(\Omega)$.

The uniqueness of solutions for the system (1.5)–(1.8) remains an open problem. Only for the system without a convection term in dynamical equation the uniqueness of solutions was proved in [2].

2. Notation

In this paper we use the following notation:

Ω – an open bounded set of a class C^2, $\Omega \subset \mathbb{R}^2$

L^q – the usual Lebesgue's space $L^q(\Omega)$, with the standard norm denoted by $|\cdot|_q$

$W_0^{1,r}$ – closure of the set of smooth functions with compact support in Ω (denoted by $C_0^\infty(\Omega)$) in the norm

$$||u||_r = \left(\int_\Omega |u|^r + |\nabla u|^r dx\right)^{1/r}$$

$$\tilde{V} = \{u \in C_0^\infty(\Omega)^2 \colon u = (u_1, u_2),\ \mathrm{div}\ u = 0 \text{ in } \Omega\}$$

$$V^{1,r} = \text{ closure of } \tilde{V} \text{ in } W_0^{1,r}.$$

3. Setting of the problem

In this section we define a weak form of the system (1.5)–(1.7) with the homogeneous boundary conditions. Multiplying the i-th equation of (1.6) by smooth function $\phi_i \in C_0^\infty$, $i = 1, 2$, div $(\phi_1, \phi_2) = 0$ in Ω we obtain after adding equations and integration by parts:

$$\int_\Omega k(\theta)|e(u)|^{r-2} e_{ij}(u) e_{ij}(\phi) - \int_\Omega \frac{\partial p}{\partial x_i}\phi_i + \int_\Omega u_j \frac{\partial u_i}{\partial x_j}\phi_i = \int_\Omega f_i \cdot \phi_i \qquad i = 1, 2. \tag{3.1}$$

Similarly, for (1.7) and $\xi \in C_0^\infty(\Omega)$ we obtain:

$$\int_\Omega \nabla\theta \cdot \nabla\xi + \int_\Omega u_j \frac{\partial \theta}{\partial x_j}\xi = \int_\Omega \frac{\partial u_i}{\partial x_j} k(\theta)|e(u)|^{r-2} e_{ij}(u)\xi - \int_\Omega \frac{\partial u_i}{\partial x_j}\delta_{ij} p\xi. \tag{3.2}$$

Since the first term on the right is equal to $\int_\Omega k(\theta)|e(u)|^r \xi$ and the second term vanishes we have:

$$\int_\Omega \nabla\theta \cdot \nabla\xi + \int_\Omega u_j \frac{\partial \theta}{\partial x_j}\xi = \int_\Omega k(\theta)|e(u)|^r \xi. \tag{3.3}$$

The term $k(\theta)|e(u)|^r$ naturally belongs to L^1, when k is a bounded function and u belongs to Sobolev space $W^{1,r}$. However, using again the equation of dynamics, we get more convenient expression for this term. Multiplying the i-th equation of (1.6) by $u_i \cdot \xi$, adding equations and integrating over Ω we get:

$$\int_\Omega k(\theta)|e(u)|^r \xi = \int_\Omega (f \cdot u)\xi - \int_\Omega u_j \frac{\partial u_i}{\partial x_j} u_i \xi - \int_\Omega p u \cdot \nabla\xi - \int_\Omega k(\theta)|e(u)|^{r-2} e_{ij}(u) \cdot \frac{\partial \xi}{\partial x_j} u_i. \tag{3.4}$$

Finally, the weak form of (1.7), which we shall use is (cf. [5]):

$$\int_\Omega \nabla\theta\cdot\nabla\xi + \int_\Omega u_j\frac{\partial\theta}{\partial x_j}\xi = \int_\Omega (f\cdot u)\xi - \int_\Omega u_j\frac{\partial u_i}{\partial x_j}u_i\xi - \int_\Omega pu\cdot\nabla\xi - \int_\Omega k(\theta)|e(u)|^{r-2}e_{ij}(u)\cdot\frac{\partial\xi}{\partial x_j}u_i. \tag{3.5}$$

Definition 3.1. We call a pair of functions $(u,\theta)\in V^{1,r}\times H_0^1$ a weak solution of (1.5)–(1.8) with $u_0=\theta_0=0$ if (3.1) holds for all $\phi\in V^{1,r}$ and (3.5) holds for all $\xi\in H_0^1$.

The aim of this paper is to prove

Theorem 3.1. *Let $\Omega\subset\mathbb{R}^2$ be a bounded set of class C^2. If $r\in(3/2,2)$, the function $k(\theta)$ is positive, bounded and separated from 0:*

$$k(x)\geq k_1>0\quad\forall x\in\mathbb{R} \tag{3.6}$$

and

$$f\in L^{2r/(3r-2)+\varepsilon}$$

for some $\varepsilon>0$, then there exists a weak solution of the system (1.5)–(1.7) with homogenous boundary condition in the sense of Definition 3.1.

4. Auxiliary results

Below we state some lemmas which we will need in the proof of the main theorem.

Lemma 4.1 ([8])**.** *For $u,w,s\in V$ and*

$$b(u,w,s)=\int_\Omega u_j\frac{\partial w_i}{\partial x_j}s_i$$

we have: $b(u,w,s)=-b(u,s,w)$.

Lemma 4.2 ([7])**.** *For $x,y\in\mathbb{R}^n$ the following inequality holds:*

$$(|x|^{r-2}x-|y|^{r-2}y)\cdot(x-y)\geq\frac{|x-y|^2}{(|x|+|y|)^{2-r}}\quad \textit{for}\quad 1<r<2.$$

Lemma 4.3. *The expression*

$$||u||_* = (\int\limits_\Omega |e(u)|^r)^{1/r}$$

is a norm in $V^{1,r}$ equivalent to the standard norm in $V^{1,r}$, introduced above.

Lemma 4.4. *For $r > 3/2$ we have compact imbedding $W^{1,r}$ in $L^{2r/(r-1)+\varepsilon}$ for some $\varepsilon > 0$.*

5. The proof of existence

We start with the definition of an operator $K\colon V^{1,r} \times H_0^1 \to V^{1,r} \times H_0^1$.

Definition 5.1. For the operator $K\colon V^{1,r} \times H_0^1 \to V^{1,r} \times H_0^1$ we have

$$K(u,\theta) = (u^*, \theta^*)$$

if and only if the following equalities hold:

$$\int\limits_\Omega k(\theta)|e(u^*)|^{r-2} e_{ij}(u^*) e_{ij}(\phi_i) + \int\limits_\Omega u_j \frac{\partial u_i^*}{\partial x_j}\phi_i = \int\limits_\Omega f_i \cdot \phi_i \qquad i = 1,2 \tag{5.1}$$

for all ϕ in $V^{1,r}$ and

$$\int\limits_\Omega \nabla\theta^* \cdot \nabla\xi + \int\limits_\Omega u_j^* \frac{\partial\theta^*}{\partial x_j}\xi = \int\limits_\Omega f \cdot u^*\xi - \int\limits_\Omega u_j^* \frac{\partial u_i^*}{\partial x_j} u_i^* \xi - \int\limits_\Omega p u^* \cdot \nabla\xi - \int\limits_\Omega k(\theta)|e(u^*)|^{r-2} e_{ij}(u^*) \cdot \frac{\partial \xi}{\partial x_j} u_i^* \tag{5.2}$$

for all $\xi \in H_0^1$, where p is the pressure associated with equation (5.1).

Remark 5.1. In the equation (5.2) we need to know the corresponding pressure function p. This function is a unique (up to a constant) solution of (1.6) on Ω. Moreover, $p \in L^{r/(r-1)}$ and the norm $|p|_{r/(r-1)}$ is bounded by the norm of

$$-\frac{\partial}{\partial x_j}\left(k(\theta)|e(u)|^{r-2} e_{ij}(u)\right) + u_j \frac{\partial u_i}{\partial x_j} - f_i$$

in the dual space $(V^{1,r})^*$. (For more details see [8] and [4].)

Lemma 5.1. *The operator K defined above is well defined.*

Proof. The existence of a unique solution u^* to the equation (5.1) follows from Browder-Minty theorem. Indeed, the operator A defined by

$$(A(u^*),\phi)=\int_\Omega k(\theta)|e(u^*)|^{r-2}e_{ij}(u^*)e_{ij}(\phi)+\int_\Omega u_j\frac{\partial u_i^*}{\partial x_j}\phi,$$

where u is given, is bounded, strictly monotonous, coercive and hemicontinuous. (For more details see [6].)

Now, we need to prove existence of a unique solution to the equation (5.2). We will use Lax-Milgram lemma. It is easy to see that for given u^* the left hand side of (5.2) defines the form

$$C(\theta,\xi)=\int_\Omega \nabla\theta\cdot\nabla\xi+\int_\Omega u^*\cdot\nabla\theta\xi,$$

which is coercive, bilinear and continuous on $H_0^1\times H_0^1$. All we need to prove is that the linear operator

$$F(\xi)=\int_\Omega(f\cdot u^*)\xi-\int_\Omega u_j^*\frac{\partial u_i^*}{\partial x_j}u_i^*\xi-\int_\Omega pu^*\cdot\nabla\xi-\int_\Omega k(\theta)|e(u^*)|^{r-2}e_{ij}(u^*)\cdot\frac{\partial\xi}{\partial x_j}u_i^*$$

is continuous on H_0^1. Using Lemmas 4.3, 4.4, Hölder's inequality and the Sobolev imbedding theorem (cf. [1]) we obtain the following estimates (here we use the assumption on Ω being two dimensional):

$$\int_\Omega(f\cdot u^*)\xi\le|f|_\alpha\cdot|u^*|_{2r/(2-r)}\cdot|\xi|_q\le C||u^*||_r\cdot|f|_\alpha\cdot||\xi||_2,$$

where $\alpha=2r/(3r-2)+\delta$, $\delta>0$ and q satisfies $1/q+1/\alpha+(2-r)/2r=1$,

$$\int_\Omega u_j^*\frac{\partial u_i^*}{\partial x_j}u_i^*\xi\le|u^*|_\beta^2\cdot|\nabla u^*|_r|\xi|_\gamma\le C||u^*||_r^3\cdot||\xi||_2,$$

where $\beta=2r/(r-1)+\varepsilon$, $\varepsilon>0$ and γ satisfies $1/\beta+1/\gamma+1/r=1$,

$$\int_\Omega p\cdot u^*\cdot\nabla\xi\le|p|_{r/(r-1)}\cdot|u^*|_{2r/(r-1)}\cdot|\nabla\xi|_2\le C|p|_{r/(r-1)}\cdot||u^*||_r\cdot||\xi||_2,$$

$$\begin{aligned}\int_\Omega k(\theta)|e(u^*)|^{r-2}e_{ij}(u^*)\cdot\frac{\partial\xi}{\partial x_j}u_i^*&\le C_0|e(u^*)|_r^{r-1}\cdot|u^*|_{2r/(r-1)}\cdot|\nabla\xi|_2\\&\le C||u^*||_r^r||\xi||_2.\end{aligned}$$

Then we have:

$$|F(\xi)|\le C(||u^*||_r,|p|_{r/(r-1)},\Omega,|f|_\alpha)\cdot||\xi||_2. \tag{5.3}$$

This finishes the proof. □

Lemma 5.2. *There exists a closed ball $B = B(0, R)$ centered at zero and of radius R in $V^{1,r} \times H_0^1$ such that operator K maps B into itself.*

Proof. Setting $\phi = u^*$ in (5.1) we obtain:

$$\int_\Omega k(\theta)|e(u^*)|^r + \int_\Omega u_j \frac{\partial u_i^*}{\partial x_j} u_i^* = \int_\Omega f_i \cdot u_i^*.$$

According to lemma 4.1 the second term in equation above vanishes. Moreover, using lemma 4.3 and assumption that $k(\cdot)$ is separated from zero we obtain:

$$k_1 C||u^*||_r^r \le |f|_* ||u^*||_r$$

where $|f|_*$ is the norm of function f in the dual space $(V^{1,r})^*$.

Then we have:

$$||u^*||_r \le C_1$$

where C_1 is a constant depending only on Ω, the norm $|f|_*$ and the constant k_1 which separates function k from zero. Now setting $\xi = \theta^*$ in (5.2) we easily obtain:

$$||\theta^*||_2^2 \le |F(\theta^*)|$$

and according to (5.3) we have:

$$||\theta^*||_2 \le C_2,$$

where C_2 depends on the norms of $||u^*||_r$, p and f and on Ω. From this, the bound for $||u^*||_r$ and the Remark 5.1 we obtain that the solutions θ^* of (5.2) are also bounded by some constant $C(\Omega, |f|_*, k_1)$. □

Lemma 5.3. *The operator K is weakly continuous in $V^{1,r} \times H_0^1$.*

Proof. Let $u_m \rightharpoonup u$ weakly in $V^{1,r}$ and $\theta_m \rightharpoonup \theta$ weakly in H_0^1. We need to show that $u_m^* \rightharpoonup u^*$ weakly in $V^{1,r}$ and $\theta_m^* \rightharpoonup \theta^*$ weakly in H_0^1, where $K(u_m, \theta_m) = (u_m^*, \theta_m^*)$ and $K(u, \theta) = (u^*, \theta^*)$. Observe that since u_m^* is bounded in $V^{1,r}$ it contains a subsequence weakly convergent to some $u_+^* \in V^{1,r}$. We will show that $u_+^* = u^*$ following the idea presented in [5]. First, we set $\phi = u_m^* - u_+^*$ in (5.1) with u^* replaced by u_m^* and then we subtract $\int_\Omega k(\theta_m)|e(u*_+)|^{r-2} e_{ij}(u_+^*) e_{ij}(u_m^* - u_+^*)$ from both sides to obtain

$$\int_\Omega k(\theta_m)(|e(u_m^*)|^{r-2} e_{ij}(u_m^*) - |e(u_+^*)|^{r-2} e_{ij}(u_+^*)) e_{ij}(u_m^* - u_+^*)$$

$$= -\int_\Omega (u_m)_j \frac{\partial (u_m^*)_i}{\partial x_j}((u_m^*)_i - (u_+^*)_i) + \int_\Omega f \cdot (u_m^* - u_+^*)$$

$$- \int_\Omega k(\theta_m)|e(u_+^*)|^{r-2} e_{ij}(u_+^*) e_{ij}(u_m^* - u_+^*). \tag{5.4}$$

Due to Lemma 4.2 and (3.6) the left hand side of (5.4) is not less than $k_1\ C_1\ ||u_m^* - u_+^*||_r$ (for more details see [3]). We will show that the right hand side of (5.4) tends to zero as $m \to \infty$. We have, eventually for a subsequence,

$$\int_\Omega (u_m)_j \frac{\partial (u_m^*)_i}{\partial x_j}((u_m^*)_i - (u_+^*)_i)$$

$$\leq |u_m|_{2r/(r-1)} \cdot ||u_m^*||_r \cdot |u_m^* - u_+^*|_{2r/(r-1)} \to 0 \tag{5.5}$$

since $||u_m^*||_r$ is bounded (Lemma 5.2) and $||u_m||_r$ is bounded too (because of weak convergence of u_m) and, taking into account that Ω is a set on the plane and $3/2 < r < 2$, it follows from the Rellich-Kondrachov theorem that we can choose a subsequence strongly convergent in $L^{2r/(r-1)}$. Moreover, from the weak convergence of $u_m^* \rightharpoonup u_+^*$ we get

$$\int_\Omega f \cdot (u_m^* - u_+^*) \to 0. \tag{5.6}$$

To show that also the third term on the right hand side of (5.4) tends to zero, we split it into three terms below and we show that:

$$\int_\Omega [k(\theta_m) - k(\theta)]|e(u_+^*)|^{r-2} e_{ij}(u_+^*) e_{ij}(u_m^*) \to 0, \tag{5.7}$$

$$\int_\Omega k(\theta)|e(u_+^*)|^{r-2} e_{ij}(u_+^*) e_{ij}(u_m^* - u_+^*) \to 0, \tag{5.8}$$

$$\int_\Omega [k(\theta) - k(\theta_m)]|e(u_+^*)|^r \to 0. \tag{5.9}$$

To obtain the convergence (5.7) we observe that $\theta_m \to \theta$ almost everywhere for a subsequence. Indeed, the functions θ_m are bounded in H_0^1 and it follows from the Rellich-Kondrachov theorem that there exist a subsequence strongly convergent in L^2 and, eventually for another subsequence, we have convergence a.e. Since k is continuous we have $k(\theta_m) \to k(\theta)$ a.e. Now, using Lebesgue's dominated convergence theorem, we get that $k(\theta_m)|e(u^*)|^{r-1}$ is, eventually for a subsequence, strongly convergent in $L^{r/(r-1)}$ and (5.7) follows from Hölder's inequality. The convergence in (5.8) is due to the weak convergence of u_m^*. Finally, (5.9) follows from a.e. convergence of $k(\theta_m)$ and

Lebesgue's dominated convergence theorem. We showed that the right hand side of (5.4) tends to zero as m tends to infinity. Since the left hand side is not less than $k_1\, C_1\, ||u_m^* - u_+^*||_r$ we obtain that for a subsequence:

$$u_m^* \to u_+^* \quad \text{strongly in} \quad V^{1,r}. \tag{5.10}$$

The strong convergence (5.10) and the weak convergence of u_m allow us to pass to the limit in (5.1) with u^* replaced by u_m^* and conclude that u_+^* satisfies the equation

$$\int_\Omega k(\theta)|e(u_+^*)|^{r-2} e_{ij}(u_+^*) e_{ij}(\phi_i) + \int_\Omega u_j \frac{\partial (u_+^*)_i}{\partial x_j} \phi_i = \int_\Omega f_i \cdot \phi_i \qquad i = 1, 2. \tag{5.11}$$

Since this solution is unique, we get $u^* = u_+^*$.

Now we will show that $\theta_m^* \rightharpoonup \theta^*$ in H_0^1. Since θ_m^* is bounded, there exists a subsequence for which $\theta_m^* \rightharpoonup \theta_+^*$, weakly for some $\theta_+^* \in H_0^1$. We need to show that $\theta_+^* = \theta^*$, where θ^* is a solution to (5.2). For fixed $\xi \in H_0^1$ we have, due to the strong convergence of u_m^* in $V^{1,r}$

$$\int_\Omega (f \cdot u_m^*)\xi \to \int_\Omega (f \cdot u^*)\xi, \tag{5.12}$$

$$\int_\Omega (u_m^*)_j \frac{\partial (u_m)_i^*}{\partial x_j} (u_m^*)_i \xi \to \int_\Omega (u^*)_j \frac{\partial u_i^*}{\partial x_j} (u^*)_i \xi. \tag{5.13}$$

Moreover, from the strong convergence of u_m^* in $V^{1,r}$ and the weak convergence of p_m in $L^{r/(r-1)}(\Omega)$ (see Remark 5.1) it follows that

$$\int_\Omega p_m u_m^* \cdot \nabla \xi \to \int p u^* \nabla \xi. \tag{5.14}$$

Finally, due to Lebesgue's dominated convergence theorem and almost everywhere convergence for a subsequence of u_m^* and $k(\theta_m)$ we have

$$\int_\Omega k(\theta_m)|e(u_m^*)|^{r-2} e_{ij}(u_m^*) \cdot \frac{\partial \xi}{\partial x_j} (u_m^*)_i \to \int_\Omega k(\theta)|e(u^*)|^{r-2} e_{ij}(u^*) \cdot \frac{\partial \xi}{\partial x_j} u_i^* \tag{5.15}$$

From (5.12)–(5.15), the strong convergence of u_m^* in $V^{1,r}$ and the weak convergence of θ_m^* in H_0^1 follows that we can pass to the limit in the equation

$$\int_\Omega \nabla\theta_m^* \cdot \nabla\xi + \int_\Omega (u_m)_j^* \frac{\partial\theta_m^*}{\partial x_j}\xi = \int_\Omega f \cdot u_m^*\xi - \int_\Omega (u_m)_j^* \frac{\partial(u_m)_i^*}{\partial x_j}(u_m)_i^*\xi$$
$$- \int_\Omega p_m u_m^* \cdot \nabla\xi - \int_\Omega k(\theta_m)|e(u_m^*)|^{r-2} e_{ij}(u_m^*) \cdot \frac{\partial\xi}{\partial x_j}(u_m)_i^*$$

and conclude that θ_+^* is a solution of the equation (5.2). Since this solution is unique we get $\theta_+^* = \theta^*$. □

Summarizing the results of last three lemmas we obtain — due to Schauder-Tichonov theorem — the Theorem 3.1.

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Witold Sadowski
Institute of Applied Mathematics
Warsaw University
Banacha 22
02–097 Warsaw, Poland
witeks@mimuw.edu.pl