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# ON DENSITY TOPOLOGIES WITH RESPECT TO INVARIANT $\sigma$-IDEALS

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**Abstract.** The density topologies with respect to measure and category are motivation to consider the density topologies with respect to invariant $\sigma$-ideals on $\mathbb{R}$. The properties of such topologies, including the separation axioms, are studied.

## Notation

By $\mathbb{R}$ we shall denote the set of all reals numbers and by $\mathbb{N}$ the set of positive integers. Let $l$ stand for Lebesgue measure. The capitals $\mathcal{L}$ and $\mathbb{L}$ denote the $\sigma$-algebra of all Lebesgue measurable sets in $\mathbb{R}$ and the $\sigma$-ideal of all Lebesgue null sets. The natural topology on $\mathbb{R}$ is denoted by $\mathcal{T}_0$. If $\mathcal{T}$ is a topology on $\mathbb{R}$, then we fix the notation:

$\mathcal{B}(\mathcal{T})$ — the $\sigma$-algebra of all Borel sets with respect to $\mathcal{T}$,
$\mathcal{B}a(\mathcal{T})$ — the $\sigma$-algebra of all sets having the Baire property with respect to $\mathcal{T}$,
$\mathcal{K}(\mathcal{T})$ — the $\sigma$-ideal of all meager sets with respect to $\mathcal{T}$.

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For any set $X \subset \mathbb{R}$, $\operatorname{Int}_{\mathcal{T}} X$ is the interior of $X$ with respect to $\mathcal{T}$, and $\overline{X}^{\mathcal{T}}$ is the closure of $X$ with respect to $\mathcal{T}$. If $\mathcal{T} = \mathcal{T}_0$, then we use shortly the following symbols: $\mathcal{B}$, $\mathcal{B}a$, $\mathbb{K}, \operatorname{Int} X, \overline{X}$. The symmetric difference of sets $X$ and $Y$ we shall denote by $X \triangle Y$, and $S \triangle \mathcal{J}$ denotes the smallest $\sigma$-algebra containing $S$ and $\mathcal{J}$. For any sets $X$ and $Y$ belonging to $S$, the fact that $X \triangle Y \in \mathcal{J}$ will be denoted by $X \sim Y$. For each set $X \subset \mathbb{R}$ and $a, t \in \mathbb{R}$, we denote

$$tX = \{y \in \mathbb{R} : y = tx,\ x \in X\},$$
$$X + a = \{y \in \mathbb{R} : y = x + a,\ x \in X\}.$$

By $\mathcal{J}_0$ we shall denote the ideal consisting of the empty set, and by $\mathcal{J}_\omega$ the $\sigma$-ideal of the countable sets. Only proper $\sigma$-ideals are considered. The cardinality of the continuum is denoted by $\mathfrak{c}$.

## 1. The concept of the density topology

Let $X \in \mathcal{L}$. We say that 0 is a Lebesgue density point of $X$ if $\lim_{h\to 0^+} l(X \cap [-h,h])/(2h) = 1$. It is not difficult to check that the last assertion is equivalent to the statement saying that $\lim_{n\to\infty} l(nX \cap [-1,1]) = 2$. This is equivalent to the fact that the sequence of characteristic function $\{f_n\}_{n\in\mathbb{N}} = \{\chi_{nX\cap[-1,1]} : n \in \mathbb{N}\}$ tends in measure to $\chi_{[-1,1]}$ (see [15]). Using the Riesz theorem, we obtain that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges with respect to the $\sigma$-ideal of the Lebesgue null sets. It means that every subsequence of the sequence $\{f_n\}_{n\in\mathbb{N}}$ contains subsequence convergent to $\chi_{[-1,1]}$ almost everywhere.

The concept of convergence with respect to a $\sigma$-ideal (see [14]) enables one to introduce a density point with respect to the Baire category (see [13], [15], [16]). We extend this concept to consider the density topologies with respect to invariant $\sigma$-ideals.

**Definition 1.1.** We shall say that a family $\mathcal{A}$ of subsets of $\mathbb{R}$ is invariant if for each $X \in \mathcal{A}$ and all $n \in \mathbb{N}$, $a \in \mathbb{R}$, we have that $nX \in \mathcal{A}$ and $X + a \in \mathcal{A}$.

**Definition 1.2.** We shall say that a pair $(\mathcal{S}, \mathcal{J})$, where $\mathcal{S}$ is a $\sigma$-algebra of subsets of $\mathbb{R}$ and $\mathcal{J}$ is a $\sigma$-ideal of subsets of $\mathbb{R}$, is invariant if $\mathcal{J} \subset \mathcal{S}$, and both the $\sigma$-algebra $\mathcal{S}$ and the $\sigma$-ideal $\mathcal{J}$ are invariant.

We consider only invariant pairs $(\mathcal{S}, \mathcal{J})$ such that $\mathcal{B} \subset \mathcal{S}$.

**Remark 1.3.** If $\mathcal{J}$ is an invariant $\sigma$-ideal, then the pair $(\mathcal{B} \triangle \mathcal{J}, \mathcal{J})$ is invariant.

From now, let $(\mathcal{S}, \mathcal{J})$ be an invariant pair.

**Definition 1.4.** We shall say that 0 is a $\mathcal{J}$-density point of an $\mathcal{S}$-measurable set $X$ if and only if the sequence of characteristic functions $\{\chi_{nX\cap[-1,1]} : n \in \mathbb{N}\}$ is convergent with respect to the $\sigma$-ideal $\mathcal{J}$ to the characteristic function $\chi_{[-1,1]}$ (it means that every subsequence of the sequence $\chi_{[-1,1]}$ contains a subsequence convergent to $\chi_{[-1,1]}$ everywhere except for a set belonging to $\mathcal{J}$.

A point $x_0 \in \mathbb{R}$ is a $\mathcal{J}$-density point of a set $X \in \mathcal{S}$ if and only if 0 is a $\mathcal{J}$-density point of the set $X - x_0$.

For each $X \in \mathcal{S}$, we define

$$\Phi_{\mathcal{J}}(X) = \{x \in \mathbb{R} : x \text{ is a } \mathcal{J}\text{-density point of } X\}.$$

The following property is an easy and useful characterization of the fact that 0 is a $\mathcal{J}$-density point of the set $X$.

**Lemma 1.5** (cf. [3], [15])**.** *The number* 0 *is a $\mathcal{J}$-density point of the set $X \in \mathcal{S}$ if and only if, for each increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers, there exists a subsequence $\{n_{k_j}\}_{j\in\mathbb{N}}$ such that*

$$\limsup_{j\to\infty}([-1,1]\backslash n_{k_j}X) \in \mathcal{J}.$$

It is clear that the last condition has the form

$$\bigcap_{i=1}^{\infty}\bigcup_{j=i}^{\infty}([-1,1]\backslash n_{k_j}X) \in \mathcal{J}.$$

Directly from the definition of a $\mathcal{J}$-density point we have

**Proposition 1.6.** *For every $\mathcal{S}$-measurable set $X$, every positive integer $n$ and every real number $a$, if $x \in \Phi_{\mathcal{J}}(X)$, then $nx \in \Phi_{\mathcal{J}}(nX)$ and $(x+a) \in \Phi_{\mathcal{J}}(X+a)$.*

**Proposition 1.7.** *For any $\mathcal{S}$-measurable sets $X$ and $Y$, if $X \subset Y$, then $\Phi_{\mathcal{J}}(X) \subset \Phi_{\mathcal{J}}(Y)$.*

As a consequence of the definition of a $\mathcal{J}$-density point we have for each $\sigma$-ideal $\mathcal{J} \subset S$ the following three propositions:

**Proposition 1.8.** *For any $\mathcal{S}$-measurable sets $X$ and $Y$, the following conditions hold:*

I. *if $X \sim Y$, then $\Phi_{\mathcal{J}}(X) = \Phi_{\mathcal{J}}(Y)$,*
II. *$\Phi_{\mathcal{J}}(X \cap Y) = \Phi_{\mathcal{J}}(X) \cap \Phi_{\mathcal{J}}(Y)$,*
III. *$\Phi_{\mathcal{J}}(\emptyset) = \emptyset$, $\Phi_{\mathcal{J}}(\mathbb{R}) = \mathbb{R}$.*

We define the family $\mathcal{T}_{\mathcal{J}}$ of $\mathcal{S}$-measurable sets by

$$\mathcal{T}_{\mathcal{J}} = \{X \in \mathcal{S} : X \subset \Phi_{\mathcal{J}}(X)\}.$$

Propositions 1.6 and 1.8 imply

**Proposition 1.9.** *The family $\mathcal{T}_{\mathcal{J}}$ has the following properties:*

1. $\emptyset, \mathbb{R} \in \mathcal{T}_{\mathcal{J}}$,
2. $\mathcal{T}_{\mathcal{J}}$ *is closed under finite intersections,*
3. *if* $X \in \mathcal{J}$, *then* $\mathbb{R}\backslash X \in \mathcal{T}_{\mathcal{J}}$,
4. $\mathcal{T}_{\mathcal{J}}$ *is invariant with respect to each operation of the form* $nx+a$ *where* $n \in \mathbb{N}$ *and* $a \in \mathbb{R}$.

We are also pointing out the following

**Proposition 1.10.** $\mathcal{T}_0 \subset \mathcal{T}_{\mathcal{J}}$.

**Proof.** Let $V_0 \in \mathcal{T}_0$. Of course, $V \in \mathcal{S}$. If $V = \emptyset$, then, by condition III of Proposition 1.8, we have $V \in \mathcal{T}_{\mathcal{J}}$. Let $x_0 \in V$. Then $0 \in V - x_0$. Since $V - x_0$ is open, there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subset V - x_0$. It is obvious that, for every increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ of positive integers, $\bigcap_{j=1}^{\infty}\bigcup_{i=j}^{\infty}([-1,1]\backslash n_i(V-x_0)) = \emptyset$. This means that $x_0$ is a $\mathcal{J}$-density point of $V$. Since $x_0$ is an arbitrary point, we conclude that $V \in \mathcal{T}_{\mathcal{J}}$. □

Although the family $\mathcal{T}_{\mathcal{J}}$ containing $\emptyset$ and $\mathbb{R}$ is closed under finite intersections, it need not be a topology on the real line.

**Example 1.11.** Let us consider the pair $(\mathcal{B}, \mathcal{J}_\omega)$. Obviously $(\mathcal{B}, \mathcal{J}_\omega)$ is an invariant pair. However, the family $\mathcal{T}_{\mathcal{J}_\omega} = \{X \in \mathcal{B} : X \subset \Phi_{\mathcal{J}_\omega}(X)\}$ is not a topology.

To prove this, we use the example given in Lemma 2.18 from [3]. Namely, there exists a perfect set $C \subset \mathbb{R}$ such that each number $x \in C$ is a $\mathcal{J}_\omega$-density point of the set $\mathbb{R}\backslash C$. Simultaneously, by Proposition 1.10, we have that $\mathbb{R}\backslash C \subset \Phi_{\mathcal{J}_\omega}(\mathbb{R}\backslash C)$. Hence $\Phi_{\mathcal{J}_\omega}(\mathbb{R}\backslash C) = \mathbb{R}$. Let $P$ be a non-Borel subset of $C$. If $x \in P$, then $\{x\} \cup (\mathbb{R}\backslash C) \in \mathcal{T}_{\mathcal{J}_\omega}$ because $\{x\} \cup (\mathbb{R}\backslash C) \in \mathcal{B}$ and $\{x\}\cup(\mathbb{R}\backslash C) \subset \Phi_{\mathcal{J}_\omega}(\{x\}\cup(\mathbb{R}\backslash C))$. But $\bigcup_{x\in P}(\{x\}\cup(\mathbb{R}\backslash C)) = P\cup(\mathbb{R}\backslash C) \notin \mathcal{B}$.

Motivated by this example, we introduce the following

**Definition 1.12.** If the family

$$\mathcal{T}_{\mathcal{J}} = \{X \in \mathcal{S} : X \subset \Phi_{\mathcal{J}}(X)\}$$

forms a topology, then $\mathcal{T}_{\mathcal{J}}$ is called the $\mathcal{J}$-density topology associated with the pair $(\mathcal{S}, \mathcal{J})$ or the $\mathcal{J}$-density topology generated by the pair $(\mathcal{S}, \mathcal{J})$.

**Example 1.13.** If $\mathcal{J}$ is an invariant $\sigma$-ideal, then the pair $(2^{\mathbb{R}}, \mathcal{J})$ is invariant and, by Propositions 1.7 and 1.9, we conclude that the family $\mathcal{T}_{\mathcal{J}}$ is a $\mathcal{J}$-density topology associated with the pair $(2^{\mathbb{R}}, \mathcal{J})$.

The whole difficulty to prove that an invariant pair $(\mathcal{S}, \mathcal{J})$ generates a $\mathcal{J}$-density topology lies in the verification whether the family $\mathcal{T}_{\mathcal{J}}$ is closed under an arbitrary union. In Example 1.13 we could avoid this difficulty because of the fact that $\mathcal{S} = 2^{\mathbb{R}}$. In some cases, the following property of the operator $\Phi_{\mathcal{J}}$ is very useful. We denote it by IV along to the properties I–III in Proposition 1.8.

IV. *For every $\mathcal{S}$-measurable set $X$,*

$$X \sim \Phi_{\mathcal{J}}(X).$$

It is an analogue of the classical Lebesgue density theorem in the abstract sense when we consider the density with respect to an invariant $\sigma$-ideal $\mathcal{J}$.

**Proposition 1.14** (cf. [1])**.** *The following conditions are equivalent:*

1. $\forall_{X \in S}\ X \setminus \Phi_{\mathcal{J}}(X) \in \mathcal{J}$,
2. $\forall_{X \in S}\ X \sim \Phi_{\mathcal{J}}(X)$.

By Proposition 1.14, condition IV can be interpreted as: $\mathcal{J}$-almost every point of every $\mathcal{S}$-measurable set is a $\mathcal{J}$-density point of that set.

**Definition 1.15.** We say that an invariant pair $(\mathcal{S}, \mathcal{J})$ has the $\mathcal{J}$-density property if condition IV is satisfied.

The $\mathcal{J}$-density property for a pair $(S, \mathcal{J})$ implies that for every $X \in S$ we have $\Phi_{\mathcal{J}}(X) \in S$.

Operator $\Phi_{\mathcal{J}}$ satisfying conditions I–IV is called, in the lifting theory, the lower density operator on $(\mathbb{R}, S, \mathcal{J})$. Thus in the context of Proposition 6.37 and Theorem 6.39 from [10] we have

**Theorem 1.16.** *Every invariant pair $(\mathcal{S}, \mathcal{J})$ having the $\mathcal{J}$-density property and satisfying countable chain condition (c.c.c.) generates the $\mathcal{J}$-density topology $\mathcal{T}_{\mathcal{J}}$.*

**Theorem 1.17.** *If an invariant pair $(\mathcal{S}, \mathcal{J})$ has the $\mathcal{J}$-density property and generates the $\mathcal{J}$-density topology, then $\mathcal{K}(\mathcal{T}_{\mathcal{J}}) = \mathcal{J}$ and $\mathcal{B}a(\mathcal{T}_{\mathcal{J}}) = \mathcal{S}$.*

There are two fundamental examples in which, by Theorem 1.16, we get the abstract density topologies.

**Example 1.18.** Let $\mathcal{S} = \mathcal{L}$ and $\mathcal{J} = \mathbb{L}$. It is well known that the pair $(\mathcal{S}, \mathcal{J})$ is invariant. Also, $(\mathcal{S}, \mathcal{J})$ satisfies c.c.c. Moreover, for each set $X \in \mathcal{S}$, $\Phi_{\mathcal{J}}(X)$ is the set of density points of $X$. By the Lebesgue density theorem, we have that $X \sim \Phi_{\mathcal{J}}(X)$ and thus, by Theorem 1.16, the family

$$\mathcal{T}_{\mathcal{J}} = \{X \in \mathcal{S} : X \subset \Phi_{\mathcal{J}}(X)\}$$

is a topology known as the density topology, usually labelled by $\mathcal{T}_d$ and called the $d$-topology(see [4], [5]).

**Example 1.19.** Let $\mathcal{S} = Ba$ and $\mathcal{J} = \mathbb{K}$. The pair $(\mathcal{S}, \mathcal{J})$ is invariant and satisfies c.c.c. We easily conclude that, for each set $V \in \mathcal{T}_0$, $V \subset \Phi_{\mathcal{J}}(V) \subset \overline{V}$ (see [15]). Since $\overline{V} \backslash V$ is a meager set, we have that $\Phi_{\mathcal{J}}(V) \sim V$. If $X \in \mathcal{S}$, then $X = V \triangle Z$ where $V \in \mathcal{T}_0$ and $Z \in \mathcal{J}$. Since $X \sim V$, from Proposition 1.8 we have $\Phi_{\mathcal{J}}(X) = \Phi_{\mathcal{J}}(V)$. This implies that $\Phi_{\mathcal{J}}(X) \sim X$. By Theorem 1.16, the family

$$\mathcal{T}_{\mathcal{J}} = \{X \in \mathcal{S} : X \subset \Phi_{\mathcal{J}}(X)\}$$

forms a topology. It is a category analogue of the density topology (see [13], [3]). In the literature on that topic, it is known as the $\mathcal{I}$-density topology. By that reason we shall denote it is the sequel by $\mathcal{T}_{\mathcal{I}}$.

Further examples of the $\mathcal{J}$-density topologies generated by invariant pairs $(\mathcal{S}, \mathcal{J})$ having the $\mathcal{J}$-density property are included in [1]. They concern product $\sigma$-ideals, and $\sigma$-algebras on the plane, related to them.

The $\mathcal{J}$-density property for the pairs $(\mathcal{S}, \mathcal{J})$ in Examples 1.18 and 1.19 plays an important role in deriving the $\mathcal{J}$-density topology by a lower density operator. We consider an example convincing us that the $\mathcal{J}$-density property of the pair $(\mathcal{S}, \mathcal{J})$ is not necessary for the operator $\Phi_{\mathcal{J}}$ to induce the $\mathcal{J}$-density topology.

First, we pay attention to the following

**Lemma 1.20.** *If $(\mathcal{S}_n, \mathcal{J}_n)_{n\in\mathbb{N}}$ is a sequence of invariant pairs such that, for every positive integer $n$, the pair $(\mathcal{S}_n, \mathcal{J}_n)_{n\in\mathbb{N}}$ induces the $\mathcal{J}$-density topology $\mathcal{T}_{\mathcal{J}_n}$, then the pair $(\mathcal{S}, \mathcal{J})$, where $\mathcal{S} = \bigcap_{n=1}^{\infty} \mathcal{S}_n$ and $\mathcal{J} = \bigcap_{n=1}^{\infty} \mathcal{J}_n$, is invariant and yields the $\mathcal{J}$-density topology $\mathcal{T}_{\mathcal{J}}$. Moreover, $\mathcal{T}_{\mathcal{J}} = \bigcap_{n=1}^{\infty} \mathcal{T}_{\mathcal{J}_n}$.*

**Proof.** It is clear that the pair $(\mathcal{S}, \mathcal{J})$ is invariant. To prove that $\mathcal{T}_{\mathcal{J}} = \bigcap_{n=1}^{\infty} \mathcal{T}_{\mathcal{J}_n}$, it is sufficient to observe that, for each $X \in \mathcal{S}$, we have

$$\Phi_{\mathcal{J}}(X) = \bigcap_{n=1}^{\infty} \Phi_{\mathcal{J}_n}(X).$$

For every positive integer $n$, $\mathcal{J} \subset \mathcal{J}_n$. This implies that $\Phi_{\mathcal{J}}(X) \subset \bigcap_{n=1}^{\infty} \Phi_{\mathcal{J}_n}(X)$. Now, let $x \in \bigcap_{n=1}^{\infty} \Phi_{\mathcal{J}_n}(X)$. We show that $x \in \Phi_{\mathcal{J}}(X)$.

Let $\{n_i\}_{i\in\mathbb{N}}$ be an arbitrary sequence of positive integers. We prove that there exists a subsequence $\{n_{i_k}\}_{k\in\mathbb{N}}$ such that $\chi_{n_{i_k}(X-x)\cap[-1,1]} \xrightarrow[k\to\infty]{} \chi_{[-1,1]}$ $\mathcal{J}$-a.e. Since $x \in \bigcap_{n=1}^{\infty} \Phi_{\mathcal{J}_n}(X)$, we can construct, by induction, a sequence of sequences $\{n_i^{(m)}\}_{i,m\in\mathbb{N}}$ such that, for every $m$, $\{n_i^{(m)}\}_{i,m\in\mathbb{N}} \subset \{n_i^{(m-1)}\}_{i,m\in\mathbb{N}}$, where $\{n_i^{(0)}\} = \{n_i\}_{i\in\mathbb{N}}$, and a sequence of sets $\{A_m\}_{m\in\mathbb{N}}$ such that $A_m \in \mathcal{J}_m$ for each positive integer $m$, and that $\chi_{n_i^{(m)}(X-x)\cap[-1,1]}(x) \xrightarrow[i\to\infty]{} \chi_{[-1,1]}(x)$ for any $x \notin A_m$. This implies that the sequence $\{n_{i_m}\}_{m\in\mathbb{N}}$, where $n_{i_m} = n_m^{(m)}$ for each $m \in \mathbb{N}$ (in other words $\{n_{i_m}\}_{m\in\mathbb{N}}$ is the diagonal sequence for the double sequence $\{n_i^{(m)}\}_{i,m\in\mathbb{N}}$) has the property that $\chi_{n_{i_m}(X-x)\cap[-1,1]}(x) \xrightarrow[i\to\infty]{} \chi_{[-1,1]}(x)$ for any $x \notin \bigcap_{m=1}^{\infty} A_m$. Namely, if $x \notin \bigcap_{m=1}^{\infty} A_m$, there exists $m_0$ such that $x \notin A_{m_0}$. Then $\chi_{n_i^{(m_0)}(X-x)\cap[-1,1]}(x) \xrightarrow[i\to\infty]{} \chi_{[-1,1]}(x)$. Hence the sequence $\{\chi_{n_{i_m}(X-x)\cap[-1,1]}(x)\}_{m\in\mathbb{N}}$ converges to $\chi_{[-1,1]}(x)$. Since $\bigcap_{m=1}^{\infty} A_m \in \mathcal{J}$, we conclude that $x$ is a $\mathcal{J}$-density point of $X$. Hence $x \in \Phi_{\mathcal{J}}(X)$. Now, we have

$$\begin{aligned}\mathcal{T}_{\mathcal{J}} &= \{X \in \mathcal{S} : X \subset \Phi_{\mathcal{J}}(X)\} = \{X \in \bigcap_{n=1}^{\infty} \mathcal{S}_n : X \subset \bigcap_{n=1}^{\infty} \Phi_{\mathcal{J}_n}(X)\} \\ &= \bigcap_{n=1}^{\infty} \{X \in \mathcal{S}_n : X \subset \Phi_{\mathcal{J}_n}(X)\} = \bigcap_{n=1}^{\infty} \mathcal{T}_{\mathcal{J}_n}.\end{aligned}$$

It follows that $\mathcal{T}_{\mathcal{J}}$ is a topology as the intersection of topologies and, at the same time, $\mathcal{T}_{\mathcal{J}} = \bigcap_{n=1}^{\infty} \mathcal{T}_{\mathcal{J}_n}$. □

**Example 1.21.** Let $\mathcal{S} = \mathcal{B}a \cap \mathcal{L}$ and $\mathcal{J} = \mathbb{K} \cap \mathbb{L}$. The pair $(\mathcal{S}, \mathcal{J})$ is invariant. By Examples 1.18, 1.19 and Lemma 1.20 the pair $(\mathcal{S}, \mathcal{J})$ generates the $\mathcal{J}$-density topology $\mathcal{T}_{\mathcal{J}}$ for which $\mathcal{T}_{\mathcal{J}} = \mathcal{T}_d \cap \mathcal{T}_{\mathcal{I}}$. We point out that the pair $(\mathcal{S}, \mathcal{J})$ does not possess the $\mathcal{J}$-density property. Namely, let Borel sets $A$ and $B$ be a decomposition of reals, such that $A \in \mathbb{L}$, $B \in \mathbb{K}$ (see [12]). Then $A \in \mathcal{S}$ and $A \notin \mathcal{J}$. By Lemma 1.20, we have $\Phi_{\mathcal{J}}(A) = \Phi_{\mathbb{L}}(A) \cap \Phi_{\mathbb{K}}(A)$. Since $\Phi_{\mathbb{L}}(A) = \emptyset$, we have that $\Phi_{\mathcal{J}}(A) = \emptyset$. Consequently, $\Phi_{\mathcal{J}}(X) \sim X$ for each $X \in \mathcal{S}$.

It is also true in this example that:

**Lemma 1.22** (cf. [2])**.** $\mathcal{B}a \cap \mathcal{L} = \mathcal{B} \triangle (\mathbb{K} \cap \mathbb{L})$.

This example shows that the $\mathcal{J}$-density property is not necessary to assert that an invariant pair $(\mathcal{S}, \mathcal{J})$ yields the $\mathcal{J}$-density topology. This is a motivation for considering the $\mathcal{J}$-density topology related to an invariant pair $(\mathcal{S}, \mathcal{J})$ without the $\mathcal{J}$-density property.

We have the following

**Observation 1.23.** *For every invariant $\sigma$-ideal $\mathcal{J}$, there exists the smallest $\sigma$-algebra $\mathcal{S}(\mathcal{J})$ such that $(\mathcal{S}(\mathcal{J}), \mathcal{J})$ is an invariant pair generating the $\mathcal{J}$-density topology.*

**Proof.** Let $\{\mathcal{S}_t\}_{t\in T}$ be the family of all invariant $\sigma$-algebras such that, for each $t \in T$, the pair $(\mathcal{S}_t, \mathcal{J})$ is invariant and yields the $\mathcal{J}$-density topology $\mathcal{T}^t_{\mathcal{J}}$. We see that $T \neq \emptyset$ because, by Example 1.13, the pair $(2^{\mathbb{R}}, \mathcal{J})$ is invariant and yields the $\mathcal{J}$-density topology. Putting $\mathcal{S}(\mathcal{J}) = \bigcap_{t\in T} \mathcal{S}_t$, we have that the pair $(\mathcal{S}(\mathcal{J}), \mathcal{J})$ is invariant and

$$\begin{aligned}\mathcal{T}_{\mathcal{J}} &= \{X \in \mathcal{S}(\mathcal{J}) : X \subset \Phi_{\mathcal{J}}(X)\}\\ &= \bigcap_{t\in T}\{X \in \mathcal{S}_t : X \subset \Phi_{\mathcal{J}}(X)\} = \bigcap_{t\in T} \mathcal{T}^t_{\mathcal{J}}.\end{aligned}$$

The last assertion means that the pair $(\mathcal{S}(\mathcal{J}), \mathcal{J})$ induces the $\mathcal{J}$-density topology. □

**Remark 1.24.** By the definition of the invariant pair $(\mathcal{S}(\mathcal{J}), \mathcal{J})$, it is clear that

$$\mathcal{B} \vartriangle \mathcal{J} \subset \mathcal{S}(\mathcal{J}) \subset 2^{\mathbb{R}}.$$

In Examples 1.18 and 1.19 we see that if $\mathcal{J} = \mathbb{L}$ or $\mathcal{J} = \mathbb{K}$, then $\mathcal{S}(\mathcal{J}) = \mathcal{B} \vartriangle \mathcal{J}$. Also, for $\mathcal{J} = \mathbb{K} \cap \mathbb{L}$, from Example 1.21 and Lemma 1.22 we have $\mathcal{S}(\mathcal{J}) = \mathcal{B} \vartriangle \mathcal{J}$. However, Example 1.11 says that if $\mathcal{J}$ is the $\sigma$-ideal of countable sets, then $\mathcal{S}(\mathcal{J}) \neq \mathcal{B} = \mathcal{B} \vartriangle \mathcal{J}$. Simultaneously, $\mathcal{S}(\mathcal{J}) \subset \mathcal{B} \vartriangle (\mathbb{K} \cap \mathbb{L})$. Thus $\mathcal{S}(\mathcal{J}) \neq 2^{\mathbb{R}}$.

**Problem 1.25.** *Does there exist an invariant $\sigma$-ideal $\mathcal{J}$ such that $\mathcal{S}(\mathcal{J}) = 2^{\mathbb{R}}$?*

## 2. Properties of the density topologies

In the definition of the $\mathcal{J}$-density topology $\mathcal{T}_{\mathcal{J}}$ generated by an invariant pair $(\mathcal{S}, \mathcal{J})$, only some $\mathcal{S}$-measurable sets are taken under consideration: namely, an $\mathcal{S}$-measurable set $X$ is $\mathcal{T}_{\mathcal{J}}$-open if $X \subset \Phi_{\mathcal{J}}(X)$. Other $\mathcal{S}$-measurable sets are not members of the family $\mathcal{T}_{\mathcal{J}}$. In this context, the natural question arises:

How can we decrease the $\sigma$-algebra $\mathcal{S}$ in the sense of inclusion to another $\sigma$-algebra $\mathcal{S}' \subset \mathcal{S}$ such that the pair $(\mathcal{S}', \mathcal{J})$ is invariant and yields the $\mathcal{J}$-density topology $\mathcal{T}'_{\mathcal{J}}$ which is identical with the $\mathcal{J}$-density topology $\mathcal{T}_{\mathcal{J}}$?

**Theorem 2.1.** *Let $(\mathcal{S}, \mathcal{J})$ be an invariant pair generating the $\mathcal{J}$-density topology $\mathcal{T}_{\mathcal{J}}$. The family $\mathcal{K}(\mathcal{T}_{\mathcal{J}})$ of meager sets with respect to the topology $\mathcal{T}_{\mathcal{J}}$ is identical with $\mathcal{J}$ if and only if there exists a $\sigma$-algebra $\mathcal{S}'$ such that*

1. $\mathcal{J} \subset \mathcal{S}' \subset \mathcal{S}$,
2. $(\mathcal{S}', \mathcal{J})$ *is invariant,*
3. $(\mathcal{S}', \mathcal{J})$ *has the $\mathcal{J}$-density property,*
4. $\mathcal{T}'_{\mathcal{J}} = \{X \in \mathcal{S}' : X \subset \Phi_{\mathcal{J}}(X)\}$ *is the $\mathcal{J}$-density topology associated with the pair $(\mathcal{S}', \mathcal{J})$, and $\mathcal{T}'_{\mathcal{J}} = \mathcal{T}_{\mathcal{J}}$.*

**Proof.** *Necessity.* Let $\mathcal{S}' = \mathcal{T}_{\mathcal{J}} \vartriangle \mathcal{J}$. Since $\mathcal{J} = \mathcal{K}(\mathcal{T}_{\mathcal{J}})$, we have that $\mathcal{S}'$ is the $\sigma$-algebra of all sets having the Baire property with respect to the topology $\mathcal{T}_{\mathcal{J}}$. Because $\mathcal{J} \subset \mathcal{S}$ and $\mathcal{T}_{\mathcal{J}} \subset \mathcal{S}$, we see that condition 1 is satisfied. By Proposition 1.9, we see that the family $\mathcal{T}_{\mathcal{J}}$ is invariant with respect to every linear operation of the form $nx + a$ where $n$ is a positive integer and $a$ is an arbitrary real number. It implies that the pair $(\mathcal{S}', \mathcal{J})$ is invariant. Now, we prove that the pair $(\mathcal{S}', \mathcal{J})$ has the $\mathcal{J}$-density property. Let $X \in \mathcal{S}'$. Then $X = V \vartriangle Y$ where $V \in \mathcal{T}_{\mathcal{J}}$ and $Y \in \mathcal{J}$. Thus $\Phi_{\mathcal{J}}(X) = \Phi_{\mathcal{J}}(V \vartriangle Y) = \Phi_{\mathcal{J}}(V) \supset V$. Hence $X \setminus \Phi_{\mathcal{J}}(X) \subset (V \vartriangle Y) \setminus V \subset Y \in \mathcal{J}$. Since $\mathcal{S}'$ is a $\sigma$-algebra, we conclude, by Proposition 1.14 that $X \sim \Phi_{\mathcal{J}}(X)$ for any $X \in \mathcal{S}'$. Hence the pair $(\mathcal{S}', \mathcal{J})$ has the $\mathcal{J}$-density property. Further, we prove condition 4. It is sufficient to establish that $\mathcal{T}'_{\mathcal{J}} = \mathcal{T}_{\mathcal{J}}$. Since $\mathcal{S}' \subset \mathcal{S}$, we have that $\mathcal{T}'_{\mathcal{J}} \subset \mathcal{T}_{\mathcal{J}}$. The inclusion $\mathcal{T}_{\mathcal{J}} \subset \mathcal{S}'$ implies $\mathcal{T}_{\mathcal{J}} \subset \mathcal{T}'_{\mathcal{J}}$. Thus we conclude that $\mathcal{T}'_{\mathcal{J}}$ is a topology and, by the definition of the family $\mathcal{T}'_{\mathcal{J}}$, we see that it is the $\mathcal{J}$-density topology associated with the pair $(\mathcal{S}', \mathcal{J})$.

*Sufficiency.* Let us consider the pair $(\mathcal{S}', \mathcal{J})$ satisfying conditions 1–4. By condition 2, we can define the family $\mathcal{T}'_{\mathcal{J}}$ with respect to the pair $(\mathcal{S}', \mathcal{J})$. Condition 4 guarantees that $\mathcal{T}'_{\mathcal{J}}$ is the $\mathcal{J}$-density topology associated with the pair $(\mathcal{S}', \mathcal{J})$. Condition 3 implies that the topology $\mathcal{T}'_{\mathcal{J}}$ is induced by the lower operator $\Phi_{\mathcal{J}}$ and thus, by Theorem 1.17 the family $\mathcal{K}(\mathcal{T}'_{\mathcal{J}})$ of meager sets with respect to the topology $\mathcal{T}'_{\mathcal{J}}$ is identical with the $\sigma$-ideal $\mathcal{J}$. The equality $\mathcal{T}'_{\mathcal{J}} = \mathcal{T}_{\mathcal{J}}$ implies that $\mathcal{K}(\mathcal{T}'_{\mathcal{J}}) = \mathcal{J}$. □

**Remark 2.2.** There exists an example of an invariant pair $(\mathcal{S}, \mathcal{J})$ without the $\mathcal{J}$-density property for which there exists a $\sigma$-algebra $\mathcal{S}' \subset \mathcal{S}$ such that the pair $(\mathcal{S}', \mathcal{J})$ is invariant and has the $\mathcal{J}$-density property. This example is based on an extension of Lebesgue measure (see [6], [8]).

**Proposition 2.3.** *If $(\mathcal{S}, \mathcal{J})$ is an invariant pair generating the $\mathcal{J}$-density topology $\mathcal{T}_{\mathcal{J}}$, such that $\mathcal{K}(\mathcal{T}_{\mathcal{J}}) = \mathcal{J}$, then the smallest $\sigma$-algebra $\mathcal{S}(\mathcal{J})$ such that the invariant pair $(\mathcal{S}(\mathcal{J}), \mathcal{J})$ generates the $\mathcal{J}$-density topology identical with $\mathcal{T}_{\mathcal{J}}$ is equal to $\mathcal{B}a(\mathcal{T}_{\mathcal{J}})$.*

**Proof.** By the proof of Theorem 2.1, we conclude that $\mathcal{S}(\mathcal{J}) \subset \mathcal{T}_{\mathcal{J}} \vartriangle \mathcal{J}$. Since $\mathcal{T}_{\mathcal{J}} \subset \mathcal{S}(\mathcal{J})$ and $\mathcal{J} \subset \mathcal{S}(\mathcal{J})$, we have that $\mathcal{T}_{\mathcal{J}} \vartriangle \mathcal{J} \subset \mathcal{S}(\mathcal{J})$. Thus $\mathcal{S}(\mathcal{J}) = \mathcal{T}_{\mathcal{J}} \vartriangle \mathcal{J} = \mathcal{T}_{\mathcal{J}} \vartriangle \mathcal{K}(\mathcal{T}_{\mathcal{J}}) = \mathcal{B}a(\mathcal{T}_{\mathcal{J}})$. □

**Proposition 2.4.** *If $(\mathcal{S}, \mathcal{J})$ is an invariant pair generating the $\mathcal{J}$-density topology $\mathcal{T}_{\mathcal{J}}$, then*

1. $\mathcal{B} \vartriangle \mathcal{J} \subset \mathcal{T}_{\mathcal{J}} \vartriangle \mathcal{K}(\mathcal{T}_{\mathcal{J}})$,
2. $\mathcal{B} \vartriangle \mathcal{J} \subset \mathcal{S}(\mathcal{J}) \subset \mathcal{S}$.

*Moreover, if the pair $(\mathcal{S}, \mathcal{J})$ has the $\mathcal{J}$-density property, then $\mathcal{T}_{\mathcal{J}} \vartriangle \mathcal{K}(\mathcal{T}_{\mathcal{J}}) = \mathcal{S}(\mathcal{J}) = \mathcal{S}$.*

**Proof.** The above inclusions are obvious. If the pair $(\mathcal{S}, \mathcal{J})$ has the $\mathcal{J}$-density property, then, by Theorem 1.17, we have $\mathcal{K}(\mathcal{T}_{\mathcal{J}}) = \mathcal{J}$ and $\mathcal{S} = \mathcal{T}_{\mathcal{J}} \vartriangle \mathcal{K}(\mathcal{T}_{\mathcal{J}})$. Thus, by the previous proposition, the equality holds. □

**Corollary 2.5.** *If $(\mathcal{B} \vartriangle \mathcal{J}, \mathcal{J})$ is an invariant pair generating the $\mathcal{J}$-density topology $\mathcal{T}_{\mathcal{J}}$ and $(\mathcal{B} \vartriangle \mathcal{J}, \mathcal{J})$ has the $\mathcal{J}$-density property, then $\mathcal{B}a(\mathcal{T}_{\mathcal{J}}) = \mathcal{S}(\mathcal{J}) = \mathcal{B} \vartriangle \mathcal{J}$.*

Now, we estimate the cardinality of $S(\mathcal{J})$. We need the following lemmas:

**Lemma 2.6.** *For each $X \subset \mathbb{R}$, we have $\Phi_{\mathcal{J}_0}(X) \subset X$.*

**Proof.** Let $x \in \Phi_{\mathcal{J}_0}(X)$. Thus 0 is a $\mathcal{J}_0$-density point of the set $X - x$. From Lemma 1.5 we easily conclude that $0 \in X - x$. Thus $x \in X$. □

**Lemma 2.7.** *There exists a nonempty perfect set $F \subset \mathbb{R}$ such that $\Phi_{\mathcal{J}_0}((\mathbb{R}\backslash F) \cup \{x\}) = (\mathbb{R}\backslash F) \cup \{x\}$ for each $x \in F$.*

**Proof.** Let $H$ be any Hamel basis of the space of reals over the field of rational numbers, containing a nonempty perfect set $F$ (see [9]). Since $\mathbb{R}\backslash F \in \mathcal{T}_0$, Proposition 1.10 gives that $\mathbb{R}\backslash F \subset \Phi_{\mathcal{J}_0}(\mathbb{R}\backslash F)$. We have to prove that each point $x \in F$ is a $\mathcal{J}_0$-density point of $(\mathbb{R}\backslash F) \cup \{x\}$. Let $\{n_k\}_{k\in\mathbb{N}}$ be any increasing subsequence of positive integers. We show that

$$[-1, 1] \subset \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} n_k(((\mathbb{R}\backslash F) \cup \{x\}) - x).$$

Let $\alpha \in [-1, 1]$. Clearly, we may assume that $\alpha \neq 0$. There exists at most one positive integer $k$ such that $\alpha \notin n_k((\mathbb{R}\backslash F) - x)$. Indeed, let us suppose that we have $k_1$ and $k_2$ such that $k_1 \neq k_2$ and $\alpha \notin n_{k_1}((\mathbb{R}\backslash F) - x)$, $\alpha \notin n_{k_2}((\mathbb{R}\backslash F) - x)$. Consequently,

$$\frac{\alpha}{n_{k_1}} + x = z_1 \quad \text{and} \quad \frac{\alpha}{n_{k_2}} + x = z_2, \qquad \text{where } z_1, z_2 \in F.$$

Since $\alpha \neq 0$, we have $z_1 \neq z_2 \neq x$ and

$$(n_{k_1} - n_{k_2})x + n_{k_1} z_1 + n_{k_2} z_2 = 0.$$

Since $H$ is a Hamel basis, $n_{k_1} = n_{k_2} = 0$, contrary to the fact that $n_{k_1} \neq n_{k_2}$ and, consequently, $\alpha \in \bigcup_{l=1}^{\infty} \bigcap_{k=l}^{\infty} n_k(((\mathbb{R}\backslash F) \cup \{x\}) - x)$. Therefore $(\mathbb{R}\backslash F) \cup \{x\} \subset \Phi_{\mathcal{J}_0}((\mathbb{R}\backslash F) \cup \{x\})$. By the previous lemma, we have $\Phi_{\mathcal{J}_0}((\mathbb{R}\backslash F) \cup \{x\}) = (\mathbb{R}\backslash F) \cup \{x\}$. □

**Theorem 2.8.** *If $\mathcal{T}_{\mathcal{J}}$ is the family associated with the invariant pair $(\mathcal{S}, \mathcal{J})$, then $\mathcal{T}_{\mathcal{J}} \backslash \mathcal{T}_0 \neq \emptyset$.*

**Proof.** By Lemma 2.7, there exists a nonempty perfect set $F \subset \mathbb{R}$ such that $\Phi_{\mathcal{J}_0}((\mathbb{R}\backslash F) \cup \{x\}) = (\mathbb{R}\backslash F) \cup \{x\}$ for each $x \in F$. Let $x \in F$ and $Y = (\mathbb{R}\backslash F) \cup \{x\}$, then $Y \in \mathcal{B}$. Thus $Y \in \mathcal{S}$ and $Y = \Phi_{\mathcal{J}_0}(Y) \subset \Phi_{\mathcal{J}}(Y)$. Hence $Y \in \mathcal{T}_{\mathcal{J}} \backslash \mathcal{T}_0$. □

**Theorem 2.9.** *For every invariant pair $(\mathcal{S}, \mathcal{J})$ generating the $\mathcal{J}$-density topology $\mathcal{T}_{\mathcal{J}}$,* $\operatorname{card} \mathcal{S} = 2^{\mathfrak{c}}$.

**Proof.** By Lemma 2.7 there exists a nonempty perfect set $F \subset \mathbb{R}$ such that, for each $x \in F$, we have $\Phi_{\mathcal{J}_0}((\mathbb{R}\backslash F) \cup \{x\}) = (\mathbb{R}\backslash F) \cup \{x\}$. It is clear that $\Phi_{\mathcal{J}_0}((\mathbb{R}\backslash F) \cup \{x\}) \subset \Phi_{\mathcal{J}}((\mathbb{R}\backslash F) \cup \{x\})$. Since $(\mathbb{R}\backslash F) \cup \{x\} \in \mathcal{S}$, we conclude that $(\mathbb{R}\backslash F) \cup \{x\} \in \mathcal{T}_{\mathcal{J}}$ for each $x \in F$. Let us suppose that $\operatorname{card} \mathcal{S} < 2^{\mathfrak{c}}$. Then there exists a set $X \subset F$ such that $(\mathbb{R}\backslash F) \cup X \notin \mathcal{S}$. At the same time, $(\mathbb{R}\backslash F) \cup X = \bigcup_{x \in X}((\mathbb{R}\backslash F) \cup \{x\}) \in \mathcal{T}_{\mathcal{J}}$ and, by the definition of the $\mathcal{J}$-density topology, it should be a member of $\mathcal{S}$. This contradiction proves that $\operatorname{card} \mathcal{S} = 2^{\mathfrak{c}}$. □

**Corollary 2.10.** *For every invariant $\sigma$-ideal $\mathcal{J}$,* $\operatorname{card} \mathcal{S}(\mathcal{J}) = 2^{\mathfrak{c}}$.

Now we present some properties of the density topologies with respect to $\sigma$-ideals having some connections with measure and category.

**Definition 2.11.** We shall say that a $\sigma$-ideal $\mathcal{J} \subset 2^{\mathbb{R}}$ is controlled by measure if $\mathcal{J} \subset \mathbb{L}$ or $\mathbb{L} \subset \mathcal{J}$.

**Definition 2.12.** We shall say that a $\sigma$-ideal $\mathcal{J} \subset 2^{\mathbb{R}}$ is controlled by category if $\mathcal{J} \subset \mathbb{K}$ or $\mathbb{K} \subset \mathcal{J}$.

The following lemma will be useful in further considerations.

**Lemma 2.13.** *If* $(\mathcal{S}_1, \mathcal{J}_1)$ *and* $(\mathcal{S}_2, \mathcal{J}_2)$ *are invariant pairs generating the* $\mathcal{J}_1$*-density topology* $\mathcal{T}_{\mathcal{J}_1}$ *and the* $\mathcal{J}_2$*-density topology* $\mathcal{T}_{\mathcal{J}_2}$*, respectively, and* $\mathcal{S}_1 \subset \mathcal{S}_2$*,* $\mathcal{J}_1 \subset \mathcal{J}_2$*, then the pair* $(\mathcal{S}_2, \mathcal{J}_1)$ *is invariant and generates the* $\mathcal{J}_1$*-density topology* $\mathcal{T}^2_{\mathcal{J}_1}$ *for which* $\mathcal{T}_{\mathcal{J}_1} \subset \mathcal{T}^2_{\mathcal{J}_1} \subset \mathcal{T}_{\mathcal{J}_2}$*.*

**Proof.** It is obvious that the pair $(\mathcal{S}_2, \mathcal{J}_1)$ is invariant. Let $\mathcal{T}^2_{\mathcal{J}_1} = \{X \in \mathcal{S}_2 : X \subset \Phi_{\mathcal{J}_\infty}(X)\}$. By Proposition 1.9, it is sufficient to show that the union of any subfamily of sets belonging to the family $\mathcal{T}^2_{\mathcal{J}_1}$ is a member of $\mathcal{T}^2_{\mathcal{J}_1}$. Since $\mathcal{J}_1 \subset \mathcal{J}_2$, therefore $\mathcal{T}^2_{\mathcal{J}_1} \subset \mathcal{T}_{\mathcal{J}_2}$. Hence the union of any subfamily of subsets of the family $\mathcal{T}^2_{\mathcal{J}_1}$ is a $\mathcal{T}_{\mathcal{J}_2}$-open set. Thus it is an $\mathcal{S}_2$-measurable set and, in that way, belongs to the family $\mathcal{T}^2_{\mathcal{J}_1}$. Since $\mathcal{S}_1 \subset \mathcal{S}_2$, we have $\mathcal{T}_{\mathcal{J}_1} \subset \mathcal{T}^2_{\mathcal{J}_1}$. □

**Theorem 2.14.** *If* $\mathcal{J}$ *is an invariant* $\sigma$*-ideal such that* $\mathcal{J} \subset \mathbb{K}$ *, then the* $\mathcal{J}$*-density topology* $\mathcal{T}_\mathcal{J}$ *generated by the pair* $(\mathcal{S}(\mathcal{J}), \mathcal{J})$ *has the property that* $\mathcal{K}(\mathcal{T}_\mathcal{J}) = \mathbb{K}$ *and* $\mathcal{B}a(\mathcal{T}_\mathcal{J}) = \mathcal{B}a$*.*

**Proof.** We show that $\mathcal{K}(\mathcal{T}_\mathcal{J}) \subset \mathbb{K}$. Let $X \in \mathcal{K}(\mathcal{T}_\mathcal{J})$. It suffices to assume that a $X$ is a $\mathcal{T}_\mathcal{J}$-nowhere dense closed set. It is clear that $X \in \mathcal{S}(\mathcal{J})$. It is obvious that the pair $(\mathcal{B}a, \mathcal{J})$ is invariant. From Example 1.21 and Lemma 2.13 we conclude that this pair generates the $\mathcal{J}$-density topology $\mathcal{T}'_\mathcal{J}$, and $\mathcal{T}_\mathcal{J} \subset \mathcal{T}'_\mathcal{J} \subset \mathcal{T}_\mathcal{I}$. This implies that $\mathbb{R}\setminus X \in \mathcal{T}_\mathcal{I}$ and then $X \in \mathcal{B}a$. The set $X$ having the Baire property has the form $X = V \vartriangle Z$, where $V \in \mathcal{T}_0$ and $Z \in \mathbb{K}$. We show that $V = \emptyset$. Let us suppose that $V \neq \emptyset$. Of course, $V \in \mathcal{T}_\mathcal{J}$. Since $X$ is $\mathcal{T}_\mathcal{J}$-nowhere dense, there exists a nonempty $\mathcal{T}_\mathcal{J}$-open set $V_1$ such that $V_1 \subset V$ and $V_1 \cap X = \emptyset$. Since $\mathcal{T}_\mathcal{J} \subset \mathcal{T}_\mathcal{I}$, we have $V_1 \in \mathcal{T}_\mathcal{I}$. As $V_1 \neq \emptyset$, we infer that $V_1 \notin \mathbb{K}$. Since $Z = X \vartriangle V = X \vartriangle [(V \setminus V_1) \cup V_1] \supset V_1$, we get a contradiction with the fact that $Z \in \mathbb{K}$ and $V_1 \notin \mathbb{K}$. Finally, $V = \emptyset$ and $X = Z$. Therefore $X \in \mathbb{K}$. Now, we show that $\mathbb{K} \subset \mathcal{K}(\mathcal{T}_\mathcal{J})$. Let $X$ be a nowhere dense set with respect to the natural topology. Assume that $X$ is closed. It is clear that $X$ has the Baire property with respect to $\mathcal{T}_\mathcal{J}$. Thus $X = V \vartriangle Z$, where $V \in \mathcal{T}_\mathcal{J} \subset \mathcal{T}_\mathcal{I}$ and $Z \in \mathcal{K}(\mathcal{T}_\mathcal{J}) \subset \mathbb{K}$. We have $V = X \vartriangle Z$, hence $V \in \mathbb{K}$. So, the set $V$ as $\mathcal{T}_\mathcal{I}$-open must be empty. This implies that $X = Z$. Consequently, $X \in \mathcal{K}(\mathcal{T}_\mathcal{J})$. We show that $\mathcal{B}a(\mathcal{T}_\mathcal{J}) = \mathcal{B}a$. By Proposition 1.10, we have that $\mathcal{T}_0 \subset \mathcal{T}_\mathcal{J}$ and by the first part of the proof that $\mathcal{K}(\mathcal{T}_\mathcal{J}) = \mathbb{K}$, we infer that $\mathcal{B}a \subset \mathcal{B}a(\mathcal{T}_\mathcal{J})$. We have observed that $\mathcal{S}(\mathcal{J}) \subset \mathcal{B}a$, then $\mathcal{T}_\mathcal{J} \subset \mathcal{B}a$. Including the fact that $\mathcal{K}(\mathcal{T}_\mathcal{J}) = \mathbb{K}$ we get that $\mathcal{B}a(\mathcal{T}_\mathcal{J}) \subset \mathcal{B}a$. Finally, $\mathcal{B}a(\mathcal{T}_\mathcal{J}) = \mathcal{B}a$. □

**Corollary 2.15.** *If $\mathcal{S} = \mathcal{B}a \cap \mathcal{L}$ and $\mathcal{J} = \mathbb{K} \cap \mathbb{L}$, then $\mathcal{K}(\mathcal{T}_{\mathcal{J}}) = \mathbb{K}$ and $\mathcal{B}a(\mathcal{T}_{\mathcal{J}}) = \mathcal{B}a$.*

**Proof.** By Lemma 1.22 and Remark 1.24, $\mathcal{S}(\mathcal{J}) = \mathcal{B}a \cap \mathcal{L}$. Thus, by Theorem 2.14, $\mathcal{K}(\mathcal{T}_{\mathcal{J}}) = \mathbb{K}$ and $\mathcal{B}a(\mathcal{T}_{\mathcal{J}}) = \mathcal{B}a$. □

**Property 2.16.** *No invariant pair $(\mathcal{S}, \mathcal{J})$ generating the $\mathcal{J}$-density topology $\mathcal{T}_{\mathcal{J}}$ and such that $\mathcal{J} \subsetneqq \mathbb{K}$ possesses the $\mathcal{J}$-density property.*

**Proof.** By Theorem 2.14, the $\mathcal{J}$-density topology $\mathcal{T}_{\mathcal{J}}$ generated by the pair $(\mathcal{S}(\mathcal{J}), \mathcal{J})$ does not possess the $\mathcal{J}$-density property since, otherwise, by Theorem 1.17, we would have that $\mathcal{K}(\mathcal{T}_{\mathcal{J}}) = \mathcal{J}$, contrary to the fact that $\mathcal{J} \neq \mathbb{K}$. Since $\mathcal{S}(\mathcal{J}) \subset \mathcal{S}$, we deduce that $(\mathcal{S}, \mathcal{J})$ does not possess the $\mathcal{J}$-density property. □

It is worth observing that the property described in Theorem 2.14 does not hold in the case of the $\sigma$-ideal $\mathbb{L}$ considered instead of $\mathbb{K}$. Indeed, let $\mathcal{S} = \mathcal{B}a \cap \mathcal{L}$ and $\mathcal{J} = \mathbb{K} \cap \mathbb{L}$. Then, by Corollary 2.15, we have that $\mathcal{K}(\mathcal{T}_{\mathcal{J}}) = \mathbb{K}$. Hence $\mathcal{K}(\mathcal{T}_{\mathcal{J}}) \setminus \mathbb{L} \neq \emptyset$ and $\mathbb{L} \setminus \mathcal{K}(\mathcal{T}_{\mathcal{J}}) \neq \emptyset$.

For invariant $\sigma$-ideals containing $\mathbb{L}$ or $\mathbb{K}$, we have the following

**Theorem 2.17.** *If $\mathcal{J}$ is an invariant $\sigma$-ideal such that $\mathcal{J} \supset \mathbb{K}$ ($\mathcal{J} \supset \mathbb{L}$), then*

1. $\mathcal{S}(\mathcal{J}) = \mathcal{B} \vartriangle \mathcal{J}$,
2. $(\mathcal{S}(\mathcal{J}), \mathcal{J})$ *has the $\mathcal{J}$-density property,*
3. $\mathcal{J} = \mathbb{K}$ ($\mathcal{J} = \mathbb{L}$) *if and only if* $\mathcal{T}_{\mathcal{J}} = \mathcal{T}_{\mathcal{I}}$ ($\mathcal{T}_{\mathcal{J}} = \mathcal{T}_d$),

*where $\mathcal{T}_{\mathcal{J}}$ is the topology generated by the invariant pair $(\mathcal{S}(\mathcal{J}), \mathcal{J})$.*

**Proof.** Let us suppose that $\mathcal{J} \supset \mathbb{K}$. In the case of condition 1, it is sufficient to prove that the invariant pair $(\mathcal{B} \vartriangle \mathcal{J}, \mathcal{J})$ yields the $\mathcal{J}$-density topology. First of all, we notice that the pair $(\mathcal{B} \vartriangle \mathcal{J}, \mathcal{J})$ has the $\mathcal{J}$-density property. Namely, let $X \in \mathcal{B} \vartriangle \mathcal{J}$; then $X = Y \vartriangle Z$, where $Y \in \mathcal{B}$ and $Z \in \mathcal{J}$. Thus

$$\begin{aligned} X \setminus \Phi_{\mathcal{J}}(X) &= (Y \vartriangle Z) \setminus \Phi_{\mathcal{J}}(Y \vartriangle Z) \\ &= (Y \vartriangle Z) \setminus \Phi_{\mathcal{J}}(Y) \subset (Y \vartriangle Z) \setminus \Phi_{\mathbb{K}}(Y) \subset (Y \setminus \Phi_{\mathbb{K}}(Y)) \cup Z \in \mathcal{J}. \end{aligned}$$

Hence, by Proposition 1.14, for each $X \in \mathcal{B} \vartriangle \mathcal{J}$, we have $X \sim \Phi_{\mathcal{J}}(X)$. Thus, by Proposition 1.8, the operator $\Phi_{\mathcal{J}}$ is a lower density operator. Moreover, we prove that the pair $(\mathcal{B} \vartriangle \mathcal{J}, \mathcal{J})$ satisfies countable chain condition (c.c.c.). In fact, it is clear that the pair $(\mathcal{B}, \mathbb{K})$ satisfies c.c.c. Let us suppose that the pair $(\mathcal{B} \vartriangle \mathcal{J}, \mathcal{J})$ does not satisfy c.c.c. Then there exists a sequence $\{X_\alpha\}_{\alpha<\omega_1}$ of pairwise disjoint sets such that, for each $\alpha < \omega_1$, $X_\alpha = Y_\alpha \vartriangle Z_\alpha$, where $Y_\alpha \in \mathcal{B}$, $Z_\alpha \in \mathcal{J}$ and $X_\alpha \in (\mathcal{B} \vartriangle \mathcal{J}) \setminus \mathcal{J}$. We put $W_0 = Y_0$ and $W_\alpha = Y_\alpha \setminus \bigcup_{\beta<\alpha} W_\beta$ for any $0 < \alpha < \omega_1$. If $\alpha_1, \alpha_2 < \omega_1$, and $\alpha_1 \neq \alpha_2$, then $W_{\alpha_1} \cap W_{\alpha_2} = \emptyset$. Since $W_\alpha \in \mathcal{B} \setminus \mathcal{J}$ for $0 \leq \alpha < \omega_1$, this contradicts the fact that the pair $(\mathcal{B}, \mathbb{K})$ satisfies c.c.c. Now, by Theorem 1.16, we deduce that the pair $(\mathcal{B} \vartriangle \mathcal{J}, \mathcal{J})$ yields the $\mathcal{J}$-density topology. In that way, $\mathcal{S}(\mathcal{J}) = \mathcal{B} \vartriangle \mathcal{J}$. The proof of condition 1 is completed. We see that it contains a proof of the fact that the pair $(\mathcal{S}(\mathcal{J}), \mathcal{J})$ has the $\mathcal{J}$-density property.

Now, we prove condition 3. Necessity is obvious. Let us show sufficiency. We only need to prove that $\mathcal{J} \subset \mathbb{K}$. Suppose that $\mathcal{J} \setminus \mathbb{K} \neq \emptyset$. Let $X \in \mathcal{J} \setminus \mathbb{K}$. We consider two cases: $X \in \mathcal{B} \vartriangle \mathbb{K}$ and $X \notin \mathcal{B} \vartriangle \mathbb{K}$. If $X \in \mathcal{B} \vartriangle \mathbb{K}$, then $\Phi_{\mathbb{K}}(X) \cap X \in \mathcal{T}_{\mathcal{I}}$ and $\Phi_{\mathbb{K}}(X) \cap X \neq \emptyset$ because $X \notin \mathbb{K}$. According to the assumption, we have that $\Phi_{\mathbb{K}}(X) \cap X \subset \Phi_{\mathcal{J}}(\Phi_{\mathbb{K}}(X) \cap X)$. The last assertion is not true because $\Phi_{\mathcal{J}}(\Phi_{\mathbb{K}}(X) \cap X) = \emptyset$. Let $X \notin \mathcal{B} \vartriangle \mathbb{K}$. Since $X \in \mathcal{J}$, then $\mathbb{R} \setminus X \in \mathcal{T}_{\mathcal{J}}$. Thus $\mathbb{R} \setminus X \in \mathcal{T}_{\mathcal{I}}$. It follows that $X \in \mathcal{B} \vartriangle \mathbb{K}$, which contradicts the fact that $X \notin \mathcal{B} \vartriangle \mathbb{K}$. The proof of the case that $\mathcal{J} \supset \mathbb{L}$ runs in the same way. □

The following theorem gives us another property of invariant pairs having the density property.

**Theorem 2.18.** *If invariant pairs $(\mathcal{S}_1, \mathcal{J})$, $(\mathcal{S}_2, \mathcal{J})$, having the density property generate the $\mathcal{J}$-density topologies $\mathcal{T}^1_{\mathcal{J}}$ and $\mathcal{T}^2_{\mathcal{J}}$, respectively, then*

$$\mathcal{T}^1_{\mathcal{J}} = \mathcal{T}^2_{\mathcal{J}} \iff \mathcal{S}_1 = \mathcal{S}_2.$$

**Proof.** *Sufficiency* is obvious.

*Necessity.* If $X \in \mathcal{S}_1$, then $\Phi_{\mathcal{J}}(X) \in \mathcal{T}^1_{\mathcal{J}}$ because, by the $\mathcal{J}$-density property, we have that $\Phi_{\mathcal{J}}(X) \in \mathcal{S}_1$ and $\Phi_{\mathcal{J}}(X) \subset \Phi_{\mathcal{J}}(\Phi_{\mathcal{J}}(X))$. Since $\mathcal{T}^1_{\mathcal{J}} = \mathcal{T}^2_{\mathcal{J}}$, therefore $\Phi_{\mathcal{J}}(X) \in \mathcal{T}^2_{\mathcal{J}}$. Simultaneously, $\Phi_{\mathcal{J}}(X) \vartriangle X \in \mathcal{J}$. Therefore $X \in \mathcal{S}_2$ . The proof of the case when $\mathcal{S}_2 \subset \mathcal{S}_1$ runs in the same way. □

**Corollary 2.19.** *In the family of invariant $\sigma$-algebras over $\mathbb{R}$ the unique $\sigma$-algebra $\mathcal{S}$ such that the invariant pair $(\mathcal{S}, \mathbb{K})$ has the $\mathbb{K}$-density property and yields the $\mathbb{K}$-density topology identical with $\mathcal{T}_{\mathcal{I}}$ is the family of sets having the Baire property.*

**Corollary 2.20.** *In the family of invariant $\sigma$-algebras over $\mathbb{R}$ the unique $\sigma$-algebra $\mathcal{S}$ such that the invariant pair $(\mathcal{S}, \mathbb{L})$ has the $\mathbb{L}$-density property and yields the $\mathbb{L}$-density topology identical with $\mathcal{T}_d$ is the family of Lebesgue measurable sets.*

## 3. The separation axioms of the density topologies

We are going to present some properties of the $\mathcal{J}$-density topology $\mathcal{T}_{\mathcal{J}}$ in the aspect of separation axioms. Our results will mostly concern the $\sigma$-ideals controlled by measure and category.

**Property 3.1.** *The space $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$, where $\mathcal{T}_{\mathcal{J}}$ is the $\mathcal{J}$-density topology generated by the invariant pair $(\mathcal{S}, \mathcal{J})$, is Hausdorff.*

**Proof.** By Proposition 1.10, $\mathcal{T}_0 \subset \mathcal{T}_{\mathcal{J}}$. Hence $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$ is Hausdorff. □

**Property 3.2.** *If a $\sigma$-ideal $\mathcal{J}$ is controlled by category, then the topological space $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$ where $\mathcal{T}_{\mathcal{J}}$ is the $\mathcal{J}$-density topology generated by the pair $(\mathcal{S}(\mathcal{J}), \mathcal{J})$ is not regular.*

**Proof.** *Case I.* Let us suppose that $\mathcal{J} \subset \mathbb{K}$. Let us observe that the set $Q$ of rational numbers is $\mathcal{T}_{\mathcal{J}}$-closed. If $\mathcal{J} \supset \mathcal{J}_w$, then it is clear that $\Phi_{\mathcal{J}}(\mathbb{R}\backslash Q) = \mathbb{R}$. Hence $\mathbb{R}\backslash Q$ is $\mathcal{T}_{\mathcal{J}}$-open and $Q$ is $\mathcal{T}_{\mathcal{J}}$-closed. Let $\mathcal{J} = \mathcal{J}_0$. We show that $\Phi_{\mathcal{J}_0}(\mathbb{R}\backslash Q) = \mathbb{R}\backslash Q$. By Lemma 2.6, we have that $\Phi_{\mathcal{J}_0}(R\backslash Q) \subset \mathbb{R}\backslash Q$. Let $x \in \mathbb{R}\backslash Q$. We prove that $x \in \Phi_{\mathcal{J}_0}(\mathbb{R}\backslash Q)$. It suffices to show that, for an arbitrary sequence $\{n_i\}_{i\in\mathbb{N}}$ of positive integers, we have

$$[-1,1] \subset n_i\big((\mathbb{R}\backslash Q) - x\big). \qquad (*)$$

For any $i \in \mathbb{N}$ and $\alpha \in [-1,1] \cap Q$, it follows that

$$\frac{\alpha}{n_i} + x \in \mathbb{R}\backslash Q.$$

Let us notice that, for each $\alpha \in [-1,1]\backslash Q$, the set

$$A_\alpha = \Big\{ i \in \mathbb{N} : \frac{\alpha}{n_i} + x \notin \mathbb{R}\backslash Q \Big\}.$$

is at most a singleton. Indeed, suppose that there are $i_1, i_2 \in \mathbb{N}$, $i_1 \neq i_2$, and $\alpha/n_{i_1} + x = q_1$ and $\alpha/n_{i_2} + x = q_2$, $q_1, q_2 \in Q$. Hence $\alpha(1/n_{i_1} - 1/n_{i_2}) = q_1 - q_2$, contrary to the fact that $\alpha$ is an irrational number. Thus there exists a positive integer $k \in A_\alpha$ such that, for $i \geq k$, $\alpha/n_i + x \in \mathbb{R}\backslash Q$. Therefore

$$\alpha \in n_i\big((\mathbb{R}\backslash Q) - x\big)$$

and the condition $(*)$ is satisfied. We have obtained that $Q$ is closed in an arbitrary topology $\mathcal{T}_{\mathcal{J}}$.

Further, we prove that, for any $x \notin Q$, the sets $\{x\}$ and $Q$ cannot be separated by $\mathcal{T}_{\mathcal{J}}$-open sets. Let us suppose that there exist $x \notin Q$ and $\mathcal{T}_{\mathcal{J}}$-open sets $V_x \ni x$ and $V \supset Q$, such that $V_x \cap V = \emptyset$. It is clear that $\mathcal{S}(\mathcal{J}) \subset \mathcal{B}a$, because the pair $(\mathcal{B}a, \mathcal{J})$ is invariant and yields the $\mathcal{J}$-density topology. Since $\mathcal{T}_{\mathcal{J}} \subset \mathcal{S}(\mathcal{J}) \subset \mathcal{B}a$, the sets $V_x$, $V$ have the Baire property. Also,

$$V_x \subset \Phi_{\mathcal{J}}(V_x) \subset \Phi_{\mathbb{K}}(V_x)$$

and

$$V \subset \Phi_{\mathcal{J}}(V) \subset \Phi_{\mathbb{K}}(V).$$

Hence the nonempty sets $V_x$ and $V$ are open in the $\mathcal{I}$-density topology. This implies that $V_x \notin \mathbb{K}$ and $V \notin \mathbb{K}$. Now, we prove that each open set $V$ in the $\mathcal{I}$-density topology and containing a dense set $D$ is residual. First, we show that, for every nonempty open set $W$, $W \cap V \notin \mathbb{K}$. Since $W \cap D \neq \emptyset$, there exist $x \in V$ and a positive number $\delta$, such that $(x-\delta, x+\delta) \subset W$. Hence $V \cap (x-\delta, x+\delta) \notin \mathbb{K}$. Therefore $V \cap W \notin \mathbb{K}$. The set $V$ having the Baire property has the form $V = A \cup B$, where $A \in G_\delta$ and $B \in \mathbb{K}$. Since $V \cap W \notin \mathbb{K}$, therefore $A \cap W \neq \emptyset$. This means that $A$ is residual and thus $V$ is residual. So, $V \cap V_x \neq \emptyset$, contrary to the fact that $V \cap V_x = \emptyset$.

*Case II.* $\mathbb{K} \subset \mathcal{J}$. By Theorem 2.17, $\mathcal{S}(\mathcal{J}) = \mathcal{B} \triangle \mathcal{J}$. Similarly as in the previous case, we prove that, for any $x \notin Q$, the sets $\{x\}$ and $Q$ cannot be separated by $\mathcal{T}_{\mathcal{J}}$-open sets. Let us suppose that there exist $x \notin Q$ and $\mathcal{T}_{\mathcal{J}}$-open sets $V_x \ni x$ and $V \supset Q$, such that $V_x \cap V = \emptyset$. Since $\mathcal{T}_{\mathcal{J}} \subset \mathcal{B} \triangle \mathcal{J}$, therefore $V_x, V \in \mathcal{B} \triangle \mathcal{J}$. It is clear that $V_x \notin \mathcal{J}$. Hence $V_x \notin \mathbb{K}$. Also, $Q \subset V \subset \Phi_{\mathcal{J}}(V)$. Note that

$$\mathcal{B} \triangle \mathcal{J} = \{X \subset \mathbb{R} : X = W \triangle Z, \quad W \in \mathcal{T}_0, \quad Z \in \mathcal{J}\}.$$

Hence $V = W \triangle Z$, where $W \in \mathcal{T}_0$ and $Z \in \mathcal{J}$. Thus $\Phi_{\mathcal{J}}(V) = \Phi_{\mathcal{J}}(W)$. By Proposition 1.8 and 1.10, we have that $W \subset \Phi_{\mathcal{J}}(W) \subset \overline{W}$. Theorefore $\Phi_{\mathcal{J}}(W) = W \cup K$, where $K \in \mathbb{K}$. This implies that $Q \subset V \subset \Phi_{\mathcal{J}}(V) = W \cup K$. We see that the set $\Phi_{\mathcal{J}}(V)$ has the Baire property. For every nonempty open set $U$, $U \cap V \notin \mathcal{J}$ since, otherwise,

$$\emptyset \neq U \cap Q \subset U \cap V \subset \Phi_{\mathcal{J}}(U) \cap \Phi_{\mathcal{J}}(V) = \Phi_{\mathcal{J}}(U \cap V) = \emptyset.$$

So, $U \cap W \notin \mathcal{J}$. Then $U \cap W \neq \emptyset$. Hence $W$ is dense and open. Thus $\Phi_{\mathcal{J}}(V)$ is residual. Then

$$\emptyset \neq V_x \cap \Phi_{\mathcal{J}}(V) \subset \Phi_{\mathcal{J}}(V_x) \cap \Phi_{\mathcal{J}}(V) = \Phi_{\mathcal{J}}(V_x \cap V).$$

Hence $V_x \cap V \neq \emptyset$. □

**Property 3.3.** *The space* $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$*, where* $\mathcal{T}_{\mathcal{J}}$ *is the* $\mathcal{J}$*-density topology generated by an invariant pair* $(\mathcal{S}, \mathcal{J})$ *does not possess the Lindelöf property.*

**Proof.** According to Lemma 2.7 there exists a nonempty perfect set $F$ such that, for each $x \in F$, we have $V_x = (\mathbb{R} \setminus F) \cup \{x\} \in \mathcal{T}_{\mathcal{J}}$. Hence the family $\{V_x\}_{x \in F}$ is a covering of $\mathbb{R}$, but it has no countable subcovering of $\mathbb{R}$. □

**Property 3.4.** *Let $\mathcal{T}_{\mathcal{J}}$ be the $\mathcal{J}$-density topology generated by an invariant pair $(\mathcal{S}, \mathcal{J})$. Then the space $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$ is not separable.*

**Proof.** Let $\mathcal{T}_{\mathcal{J}}$ be the $\mathcal{J}$-density topology generated by an invariant pair $(\mathcal{S}, \mathcal{J})$ and let $\mathcal{T}_{\mathcal{J}_0}$ be the $\mathcal{J}_0$-density topology generated by the invariant pair $(\mathcal{S}(\mathcal{J}), \mathcal{J})$. It is clear that $\mathcal{S}(\mathcal{J}_0) \subset \mathcal{S}$. Lemma 2.13 implies that $\mathcal{T}_{\mathcal{J}_0} \subset \mathcal{T}_{\mathcal{J}}$. Therefore it is sufficient to prove that the space $(\mathbb{R}, \mathcal{T}_{\mathcal{J}_0})$ is not separable. Let $X \subset \mathbb{R}$ be a countable set. We show that there exists a nonempty set $W \in \mathcal{T}_{\mathcal{J}_0}$ such that $W \cap X = \emptyset$. Of course, we may assume that $X$ is infinite. Let $X = \{x_1, x_2, \ldots, x_n, \ldots\}$. Let us consider $\mathbb{R}$ as a vector space $\mathbb{E}$ over the field $Q$ of all rational numbers. Let $B$ be a Hamel basis of $\mathbb{E}$. For any element $x \in \mathbb{E}$ we have the unique representation $x = q_1 b_1 + q_2 b_2 + \ldots + q_m b_m$, where $m \in \mathbb{N}$ and $q_i \in Q \setminus \{0\}$, $b_i \in B$ for $1 \le i \le m$. Let $B(x) = \bigcup_{i=1}^{m} \{b_i\}$ and $B(X) = \bigcup_{i=1}^{\infty} B(x_i)$. Putting $W = \mathbb{E} \setminus \mathrm{lin}(B(X))$, where $\mathrm{lin}(B(X))$ denotes the vector space over $Q$ generated by the set $B(X)$, we have that $W \cap X = \emptyset$. We prove that $W \in \mathcal{T}_{\mathcal{J}_0}$. Firstly we see that $W$ is the complement of a countable set. Thus $X \in \mathcal{S}(\mathcal{J}_0)$ as a Borel set. Further we prove that $W \subset \Phi_{\mathcal{J}_0}(W)$. Let $x \in W$. Of course, $x \neq 0$. According to Lemma 1.5, we have to prove that

$$[-1, 1] \subset \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} n_k (W - x),$$

where $\{n_k\}_{k \in \mathbb{N}}$ is an increasing sequence of positive integers. Let $\alpha \in [-1, 1]$. The case, where $\alpha = 0$ is obvious. Suppose that $\alpha \neq 0$. Let us observe that a set $A_\alpha = \left\{k \in \mathbb{N} : \alpha / n_k + x \notin W\right\}$ is at most a singleton. Suppose to the contrary that there are $n_{k_1}, n_{k_2} \subset A_\alpha$ and $n_{k_1} \neq n_{k_2}$. By definition of the set $W$, we have that

$$\frac{\alpha}{n_{k_1}} + x \in \mathrm{lin}(B(X))$$

and

$$\frac{\alpha}{n_{k_2}} + x \in \mathrm{lin}(B(X)).$$

Hence

$$(n_{k_1} - n_{k_2}) x \in \mathrm{lin}(B(X)).$$

Thus $x \in \mathrm{lin}(B(X))$, contrary to the fact that $x \notin \mathrm{lin}(B(X))$. Finally, there exists a positive integer $j \in A_\alpha$ such that for $k \ge j$, $\alpha / n_k + x \in W$. It implies that

$$\alpha \in \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} n_k (W - x).$$

□

**Property 3.5.** *Assume that $\mathcal{J}$ is an invariant $\sigma$-ideal such that $\mathbb{L} \subset \mathcal{J}$, and $\mathcal{T}_{\mathcal{J}}$ is the $\mathcal{J}$-density topology generated by an invariant pair $(\mathcal{S}(\mathcal{J}), \mathcal{J})$. The space $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$ is regular if and only if $\mathcal{J} = \mathbb{L}$.*

**Proof.** *Sufficiency.* If $\mathcal{J} = \mathbb{L}$, then $\mathcal{S}(\mathcal{J}) = \mathcal{B} \triangle \mathbb{L} = \mathcal{L}$ and the $\mathcal{J}$-density topology $\mathcal{T}_{\mathcal{J}}$ is the density topology $\mathcal{T}_d$ which is regular (see [4]).

*Necessity.* Let $\mathbb{L} \subset \mathcal{J}$. Then, by Theorem 2.17, $\mathcal{S}(\mathcal{J}) = \mathcal{B} \triangle \mathcal{J}$. Since $\mathbb{L} \subset \mathcal{J}$, it is clear that $\mathcal{B} \triangle \mathcal{J} = \mathcal{L} \triangle \mathcal{J}$. For any $X \in \mathcal{J}$, the inner Lebesgue measure, $l_*(X) = 0$. Using the Marczewski method (see [11]), we can define a measure $\mu$ on the $\sigma$-algebra $\mathcal{L} \triangle \mathcal{J}$ in the following manner. Let $X \in \mathcal{L} \triangle \mathcal{J}$. Then $X = Y \triangle Z$, where $Y \in \mathcal{L}$ and $Z \in \mathcal{J}$. Putting $\mu(X) = l(Y)$, we get that $\mu$ is a correctly defined measure on $\mathcal{S}(\mathcal{J})$. Let us notice that, for the measure $\mu$ so defined, the $\sigma$-ideal $\mathcal{I}_\mu$ of $\mu$-null sets is of the form

$$\mathcal{I}_\mu = \{X \in \mathcal{S}(\mathcal{J}) : X = A \cup B, \quad A \in \mathbb{L}, \quad B \in \mathcal{J}\}.$$

Hence $\mathcal{I}_\mu = \mathcal{J}$. At the same time, $\mu$ is an extension of Lebesgue measure $l$ and the pair $(\mathcal{S}(\mathcal{J}), \mathcal{J})$ is invariant. Moreover, for any $n \in \mathbb{N}$, $a \in \mathbb{R}$ and $X \in \mathcal{S}(\mathcal{J})$, we have $\mu(n\,X) = n\,\mu(X)$ and $\mu(X + a) = \mu(X)$. According to the above properties, we claim that a point $x \in \mathbb{R}$ is a $\mu$-density point of a set $X \in \mathcal{S}(\mathcal{J})$ if and only if it is a $\mathcal{J}$-density point of $X$. Thus

$$\mathcal{T}_{\mathcal{J}} = \{X \in \mathcal{S}(\mathcal{J}) : X \subset \Phi_{\mathcal{J}}(X)\} = \{X \in \mathcal{S}(\mathcal{J}) : X \subset \Phi_\mu(X)\},$$

where

$$\Phi_\mu(X) = \{x \in \mathbb{R} : x \text{ is a density point of } X \text{ with respect to measure } \mu\}.$$

By Theorem 2 from [6], we have that $\mathcal{T}_{\mathcal{J}} = \{X : X = A \backslash B,\ A \in \mathcal{T}_d,\ \mu(B) = 0\}$. By Property 7 from [7], $\mathcal{T}_{\mathcal{J}}$ is regular if $\mathcal{T}_{\mathcal{J}} = \mathcal{T}_d$. We show that $\mathcal{J} = \mathbb{L}$. It is sufficient to show that $\mathcal{J} \subset \mathbb{L}$. Let $X \in \mathcal{J}$. Then $\mathbb{R} \backslash X \in \mathcal{T}_{\mathcal{J}}$. Thus $\mathbb{R} \backslash X \in \mathcal{T}_d$, which implies $\mathbb{R} \backslash X \in \mathcal{L}$ and $X \in \mathcal{L}$. It is clear that $0 = \mu(X) = l(X)$. Hence $X \in \mathbb{L}$. □

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