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SINGULARLY PERTURBED SYSTEMS OF VOLTERRA EQUATIONS

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Abstract. Singularly perturbed linear Volterra integral equations are solved in this paper. To improve the results which has been published earlier, formal solutions of systems of equations are determined and rigorously proved to be asymptotic to the exact solutions.

1. Introduction

This paper considers singularly perturbed linear systems of Volterra integral equations given by

$$\varepsilon \mathbf{u}(t) = \mathbf{f}(t) + \int_0^t \mathbf{A}(t,s)\mathbf{u}(s)\,ds, \quad 0 \le t \le T, \tag{1.1}$$

where $0 < \varepsilon \ll 1$. The vector-valued function $\mathbf{f}(t)$ is continuous for $0 \le t \le T$ and the matrix-valued kernel $\mathbf{A}(t,s)$ is continuous for $0 \le s \le t \le T$. The interest is in finding asymptotic approximations to the continuous vector-valued solution $t \mapsto \mathbf{u}(t;\varepsilon)$ of (1.1) as $\varepsilon \to 0$. The results here are not presented because they are new, but rather to explain in this simple context how the method of additive decomposition can be applied to integral equations. The results here are easily generalized to the case of $\mathbf{f}$ and $\mathbf{A}$

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depending in a regular way on ε, though here it is assumed that they are independent of ε.

The singular nature of (1.1) is easily seen. For $\varepsilon > 0$, (1.1) is a Volterra equation of the second kind which has a continuous solution $\mathbf{u}(t;\varepsilon)$ satisfying $\varepsilon\mathbf{u}(0;\varepsilon) = \mathbf{f}(0)$. For $\varepsilon = 0$, (1.1) reduces to a Volterra equation the first kind

$$\mathbf{0} = \mathbf{f}(t) + \int_0^t \mathbf{A}(t,s)\mathbf{v}(s)\,ds, \quad 0 \le t \le T, \tag{1.2}$$

which does not have a continuous solution unless $\mathbf{f}(0) = \mathbf{0}$. Even in this case, (1.2) has a continuous solution only if $\mathbf{f}(t)$ is continuously differentiable. So there is a loss of regularity for $\mathbf{v}(t)$ compared to the solution $\mathbf{u}(t;\varepsilon)$ of (1.1) for $\varepsilon > 0$. Indeed, if the solution of (1.2) is such that $\mathbf{v}(0) \neq \lim_{\varepsilon\to 0} \mathbf{u}(0;\varepsilon)$, then $\mathbf{v}(t)$ cannot provide a uniformly valid approximation of the solution $\mathbf{u}(t;\varepsilon)$ of (1.1) on $[0,T]$.

The behaviour of the kernel plays an important role in determining the asymptotic character of the continuous solution $\mathbf{u}(t;\varepsilon)$ of (1.1) for small values of ε. Here we impose the condition that all of the eigenvalues of $\mathbf{A}(t,t)$ have negative real parts. This not only forces an initial layer, but forces the solution $\mathbf{u}(t;\varepsilon)$ of (1.1) to decay exponentially in the initial-layer. The solution $\mathbf{u}(t;\varepsilon)$ is slowly varying for $O(\varepsilon) \le t \le T$ as $\varepsilon \to 0$, but changes exponentially on a small interval $0 \le t \le O(\varepsilon)$. This small interval of rapid change is called the *inner region*, *initial layer* or *layer of rapid transition*, and the region of slow variation of $\mathbf{u}(t;\varepsilon)$ as the *outer region*. The thickness of the initial layer approaches zero as $\varepsilon \to 0$.

The aim of this work is to obtain asymptotic approximations to $\mathbf{u}(t;\varepsilon)$ which are uniformly valid for all $0 \le t \le T$ as $\varepsilon \to 0$. The interest is in problems whose solutions have initial layers; solutions with rapid initial exponential growth will not be discussed here. Exponential decay in the boundary layer of the solution $\mathbf{u}(t;\varepsilon)$ suggests the use of the additive decomposition method, as was employed by Lange and Smith [9] in their study of singularly perturbed linear Fredholm equations.

In Section 2, we introduce some notation and explain basic assumptions. We also explain the fundamental ideas of the additive decomposition method, and how it regularizes the singular perturbation problem (1.1). We derive a formal solution $\sum_{n=-1}^{\infty} \mathbf{u}_n(t;\varepsilon)\varepsilon^n$ in this section. In Section 3 it is shown that this is an asymptotic series and that

$$\varepsilon\mathbf{U}_N(t;\varepsilon) = \mathbf{f}(t) + \int_0^t \mathbf{A}(t,s)\mathbf{U}_N(s;\varepsilon)\,ds + O(\varepsilon^{N+1}),$$

where

$$\mathbf{U}_N(t;\varepsilon) = \sum_{n=0}^{N+1} \mathbf{u}_n(t;\varepsilon)\varepsilon^{n-1}.$$

Then we prove that

$$|\mathbf{u}(t;\varepsilon) - \mathbf{U}_N(t;\varepsilon)| = O(\varepsilon^{N+1}) \tag{1.3}$$

uniformly for $0 \le t \le T$ as $\varepsilon \to 0$. This result is important because the method of additive decomposition can lead to spurious solutions (see for example Lange [8]). The method is illustrated in Section 4 by an example from Angell and Olmstead [1] and another example whose boundary layer stability condition fails.

2. Mathematical preliminaries

2.1. Notation and assumptions.

The n-dimensional space $\mathbb{R}^n$ is given the norm $|\mathbf{x}| = \max_{1\le i\le n} |x_i|$ for each $\mathbf{x}$ in $\mathbb{R}^n$, and the space $\mathbb{R}^{n\times n}$ of $n \times n$ matrices with real entries is given the norm $|\mathbf{M}| = \max_{1\le i,j\le n} |M_{ij}|$ for all $\mathbf{M}$ in $\mathbb{R}^{n\times n}$. The spectrum $\sigma(\mathbf{M})$ of $\mathbf{M}$ is the set of eigenvalues of $\mathbf{M}$. It is well–known (see, for example Hirsch and Smale [6, Chapter 7, Theorem 1]) that, if $\operatorname{Re}\lambda < \alpha \le \alpha_1 < 0$ for all $\lambda \in \sigma(\mathbf{M})$, there is a constant $\kappa > 0$ such that

$$|\mathrm{e}^{\mathbf{M}t}\mathbf{x}| \le \kappa \mathrm{e}^{-\alpha_1 t}|\mathbf{x}|. \tag{2.1}$$

The kernel $\mathbf{A} : \triangle_T \to \mathbb{R}^{n\times n}$ is defined on

$$\triangle_T = \{(t,s) \in \mathbb{R}^2 : 0 \le s \le t \le T\}. \tag{2.2}$$

It is convenient to use the notation

$$\mathbf{B}(t) = \mathbf{A}(t,t). \tag{2.3}$$

Partial derivatives are usually denoted by $\partial_1\mathbf{A}$ and $\partial_2\mathbf{A}$ instead of $\partial\mathbf{A}/\partial t$ and $\partial\mathbf{A}/\partial s$ respectively. Similarly the derivative of $\mathbf{u}$ is usually denoted by $\mathbf{u}'(t)$ rather than $d\mathbf{u}/dt$.

The following assumptions are used throughout. The first is a regularity assumption on the data $\mathbf{f}$ and $\mathbf{A}$; the second is a stability condition for the solution within the boundary layer.

($\mathbf{H_1}$) The functions $\mathbf{f} : [0,T] \to \mathbb{R}^n$ and $\mathbf{A} : \triangle_T \to \mathbb{R}^{n\times n}$ are both C^∞.

($\mathbf{H_2}$) There exists a number $\alpha > 0$ such that

$$\max_{\lambda\in\sigma(\mathbf{B}(t))} \{\operatorname{Re}(\lambda)\} \le -\alpha,$$

for all $0 \le t \le T$.

2.2. Heuristic analysis.

Here we describe how the additive decomposition technique can be applied to integral equations of the type (1.1). The method of additive decomposition, also called the O'Malley and Hoppensteadt method, was initially applied by O'Malley [12], [13] and Hoppensteadt [7] to investigate the behaviour of solutions of singularly perturbed systems of ordinary differential equations. The book Smith [15] contains a clear account of its application to singularly perturbed ordinary differential equations. This method was later employed by Angell and Olmstead in [1] and [2] to get formal solutions of singularly perturbed Volterra integral equations, linear and nonlinear. Lange and Smith in [9], in a very careful study of singularly perturbed linear Fredholm equations applied the method systematically to get a complete formal solution and proved estimates of the type (1.3). The singularly perturbed Fredholm equations investigated in [9] have the additional complication of two boundary layers. It is also indicated there how internal layers can be analysed. The additive decomposition has also been employed by Lange and Smith [10] and Skinner [14]. The presentation is similar to §3 and §6 of Lange and Smith [9].

The analysis in this and the next section is formal. The forcing function $\mathbf{f}(t)$ and kernel $\mathbf{A}(t,s)$ are assumed to be C^∞. The solution $\mathbf{u}(t;\varepsilon)$ of (1.1) can be represented as

$$\mathbf{u}(t;\varepsilon) = \frac{1}{\varepsilon}\mathbf{f}(t) + \frac{1}{\varepsilon}\int_0^t \mathbf{\Gamma}(t,s;\varepsilon)\mathbf{f}(s)\,ds, \quad 0 \le t \le T, \tag{2.4}$$

where $\mathbf{\Gamma}(t,s;\varepsilon)$ is the resolvent kernel of $\mathbf{A}(t,s)/\varepsilon$, which by definition is the solution of

$$\mathbf{\Gamma}(t,s;\varepsilon) = \frac{1}{\varepsilon}\mathbf{A}(t,s) + \frac{1}{\varepsilon}\int_s^t \mathbf{A}(t,v)\mathbf{\Gamma}(v,s;\varepsilon)\,dv, \quad 0 \le s \le t \le T.$$

The matrix $\mathbf{\Gamma}(t,s;\varepsilon)$ is also C^∞. Detailed accounts of the theory of linear nonconvolution Volterra equations can be found in Miller [11, Chapter IV] and Gripenberg, Londen and Staffans [5, Chapter 9].

To model an initial layer for $\mathbf{u}(t;\varepsilon)$ we introduce a new scaled time scale $\tau = t/\mu(\varepsilon)$. The idea is that if the initial layer region is described with respect to the new time scale no rapid variation in the solution should be exhibited. A solution $\mathbf{u}(t;\varepsilon)$ is sought in the form

$$\mathbf{u}(t;\varepsilon) = \mathbf{y}(t;\varepsilon) + \varphi(\varepsilon)\mathbf{z}(t/\mu(\varepsilon);\varepsilon), \tag{2.5}$$

where $\mathbf{y}(t;\varepsilon)$ represents the outer approximation and $\mathbf{z}(\tau;\varepsilon)$ an initial layer correction function. The function $\mu(\varepsilon)$ describes the width of the layer and $\varphi(\varepsilon)$ describes the magnitude of $\mathbf{u}(t;\varepsilon)$ in the layer. Therefore we require that[1]

$$\mathbf{y}(t;\varepsilon)=\mathrm{ord}(1),\quad \mathbf{z}(\tau;\varepsilon)=\mathrm{ord}(1)\quad \text{as } \varepsilon\to 0.$$

At any fixed $t>0$, the outer approximation, $\mathbf{y}(t;\varepsilon)$ should give a good approximation to $\mathbf{u}(t;\varepsilon)$ as $\varepsilon\to 0$, we impose the condition

$$\mathbf{z}(\tau;\varepsilon)\to\mathbf{0},\quad \text{as } \tau\to\infty. \tag{2.6}$$

The substitution of (2.5) into (1.1) gives

$$\varepsilon\mathbf{y}(t;\varepsilon)+\varepsilon\varphi(\varepsilon)\mathbf{z}(t/\mu(\varepsilon);\varepsilon)=\int_0^t \mathbf{A}(t,s)\mathbf{y}(s;\varepsilon)\,ds + \varphi(\varepsilon)\mu(\varepsilon)\int_0^{t/\mu(\varepsilon)}\mathbf{A}(t,\mu(\varepsilon)\sigma)\mathbf{z}(\sigma;\varepsilon)\,d\sigma+\mathbf{f}(t). \tag{2.7}$$

This is equivalent to

$$\varepsilon\mathbf{y}(\mu(\varepsilon)\tau;\varepsilon)+\varepsilon\varphi(\varepsilon)\mathbf{z}(\tau;\varepsilon)=\int_0^{\mu(\varepsilon)\tau}\mathbf{A}(\mu(\varepsilon)\tau,s)\mathbf{y}(s;\varepsilon)\,ds + \varphi(\varepsilon)\mu(\varepsilon)\int_0^{\tau}\mathbf{A}(\mu(\varepsilon)\tau,\mu(\varepsilon)\sigma)\mathbf{z}(\sigma;\varepsilon)\,d\sigma+\mathbf{f}(\mu(\varepsilon)\tau). \tag{2.8}$$

The width $\mu(\varepsilon)$ and amplitude $\varphi(\varepsilon)$ in the boundary layer can be found by examining the dominant balance. Of course $\mu(\varepsilon)=o(1)$ as $\varepsilon\to 0$. We shall only consider the leading order terms in $\mathbf{y}(t;\varepsilon)$ and $\mathbf{z}(\tau;\varepsilon)$, and therefore write

$$\mathbf{y}(t;\varepsilon)=\mathbf{y}_0(t)+o(1),\quad \mathbf{z}(\tau;\varepsilon)=\mathbf{z}_0(\tau)+o(1)\quad \text{as } \varepsilon\to 0.$$

Of course $\mathbf{z}_0(\tau)\to 0$ as $\tau\to\infty$. Also we assume that there is a real number γ and nontrivial kernels $\mathbf{B}(\tau,\sigma)$ and $\mathbf{C}(t,\sigma)$ such that

$$\mathbf{A}(\varepsilon\tau,\varepsilon\sigma)\sim\varepsilon^{\gamma}\mathbf{B}(\tau,\sigma)$$
$$\mathbf{A}(t,\varepsilon\sigma)\sim\varepsilon^{\gamma}\mathbf{C}(t,\sigma)$$

uniformly as $\varepsilon\to 0$. For simplicity we suppose that $\mathbf{f}(0)\neq\mathbf{0}$. Equations (2.7) and (2.8) imply that as $\varepsilon\to 0$

$$\varepsilon\mathbf{y}_0(t)+\varepsilon\varphi(\varepsilon)\mathbf{z}_0(t/\mu(\varepsilon))\sim\int_0^t\mathbf{A}(t,s)\mathbf{y}_0(s)\,ds + \varphi(\varepsilon)\mu(\varepsilon)\int_0^{t/\mu(\varepsilon)}\mathbf{A}(t,\mu(\varepsilon)\sigma)\mathbf{z}(\sigma;\varepsilon)\,d\sigma+\mathbf{f}(t), \tag{2.9}$$

$$\varepsilon\mathbf{y}_0(\mu(\varepsilon)\tau)+\varepsilon\varphi(\varepsilon)\mathbf{z}_0(\tau)\sim\int_0^{\mu(\varepsilon)\tau}\mathbf{A}(\mu(\varepsilon)\tau,s)\mathbf{y}_0(s)\,ds + \varphi(\varepsilon)\mu(\varepsilon)^{\gamma+1}\int_0^{\tau}\mathbf{B}(\tau,\sigma)\mathbf{z}_0(\sigma)\,d\sigma+\mathbf{f}(0). \tag{2.10}$$

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[1] Two functions $\theta(\varepsilon)$ and $\psi(\varepsilon)$ defined in a neighbourhood $(0,\varepsilon_0)$ satisfy $\theta(\varepsilon)=\mathrm{ord}(\psi(\varepsilon))$ if $\theta(\varepsilon)=O(\psi(\varepsilon))$ but $\theta(\varepsilon)\neq o(\psi(\varepsilon))$ as $\varepsilon\to 0$.

Examining the dominant balance in the second relation, we see that

$$\operatorname{ord}(\varepsilon\varphi(\varepsilon)) = \operatorname{ord}(\mu(\varepsilon)^{\gamma+1}\varphi(\varepsilon)) = \operatorname{ord}(1) \quad \text{as } \varepsilon \to 0.$$

Hence we choose

$$\mu(\varepsilon) = \varepsilon^{1/(1+\gamma)}, \quad \varphi(\varepsilon) = \frac{1}{\varepsilon}.$$

It then follows by letting $\varepsilon \to 0$ with $\tau \geq 0$ fixed in (2.8), that $\mathbf{z}_0$ obeys the equation

$$\mathbf{z}_0(\tau) = \int_0^\tau \mathbf{B}(\tau,\sigma)\mathbf{z}_0(\sigma)\,d\sigma + \mathbf{f}(0).$$

To get an equation for $\mathbf{y}_0$ the order as $\varepsilon \to 0$ of the term

$$\varphi(\varepsilon)\mu(\varepsilon)\int_0^{t/\mu(\varepsilon)} \mathbf{A}(t,\mu(\varepsilon)\sigma)\mathbf{z}(\sigma;\varepsilon)\,d\sigma \tag{2.11}$$

in (2.7) must be calculated. In the standard case of exponential decay in the boundary layer, each of the integrals

$$\int_0^{t/\mu(\varepsilon)} \mathbf{A}(t,\mu(\varepsilon)\sigma)\{\mathbf{z}(\sigma;\varepsilon) - \mathbf{z}_0(\sigma)\}\,d\sigma,$$

$$\int_0^{t/\mu(\varepsilon)} \{\mathbf{A}(t,\mu(\varepsilon)\sigma) - \mu(\varepsilon)^\gamma\mathbf{C}(\mathbf{t},\sigma)\}\mathbf{z_0}(\sigma)\,\mathbf{d}\sigma,$$

$$\mu(\varepsilon)^\gamma\int_{t/\mu(\varepsilon)}^\infty \mathbf{C}(\mathbf{t},\sigma)\mathbf{z_0}(\sigma)\,\mathbf{d}\sigma,$$

can be formally shown to vanish, and hence (2.11) has leading order

$$\int_0^\infty \mathbf{C}(\mathbf{t},\sigma)\mathbf{z_0}(\sigma)\,\mathbf{d}\sigma \tag{2.12}$$

in this case. However finding the order of (2.11) as $\varepsilon \to 0$ in the case of algebraic decay of the solution in the boundary layer is not so straightforward. Indeed in Section 4.2 an example is discussed for which the evaluation of the layer limit in (2.11) requires knowledge of the asymptotic behaviour of higher order terms in $\mathbf{z}(\tau;\varepsilon)$ not just the leading order term $\mathbf{z}_0(\tau)$. For the standard case of exponentially decaying boundary layers, we find by letting $\varepsilon \to 0$ with $0 < t \leq T$ fixed in (2.7) that $\mathbf{y}_0$ obeys

$$0 = \int_0^t \mathbf{A}(t,s)\mathbf{y}_0(s)\,ds + \int_0^\infty \mathbf{C}(\mathbf{t},\sigma)\mathbf{z_0}(\sigma)\,\mathbf{d}\sigma + \mathbf{f}(\mathbf{t}).$$

It is easy to see that if $(\mathbf{H_2})$ holds then

$$\mathbf{A}(\varepsilon\tau,\varepsilon\sigma) \sim \mathbf{A}(0,0) \tag{2.13}$$

$$\mathbf{A}(t,\varepsilon\sigma) \sim \mathbf{A}(t,0) \tag{2.14}$$

as $\varepsilon \to 0$, where $\mathbf{A}(0,0)$ and $\mathbf{A}(t,0)$ are non-zero. Then the width and amplitude of the boundary become

$$\mu(\varepsilon) = \varepsilon, \quad \varphi(\varepsilon) = \frac{1}{\varepsilon}. \tag{2.15}$$

In the standard case $\mathbf{y}_0$ and $\mathbf{z}_0$ then satisfy

$$0 = \int_0^t \mathbf{A}(t,s)\mathbf{y}_0(s)\,ds + \mathbf{A}(t,0)\int_0^\infty \mathbf{z}_0(\sigma)\,d\sigma + \mathbf{f}(t) \tag{2.16}$$

$$\mathbf{z}_0(\tau) = \int_0^\tau \mathbf{A}(0,0)\mathbf{z}_0(\sigma)\,d\sigma + \mathbf{f}(0). \tag{2.17}$$

A consequence of the magnitude $O(\varepsilon^{-1})$ of the boundary layer is that the term $\varepsilon\mathbf{u}(t)$ on the right of (1.1) contributes to equation (2.17) for the inner correction term. It also follows from (2.9) that (2.12) is the contribution to the integral in (1.1) from narrow initial layer $0 \le t \le O(\varepsilon)$ is $O(1)$ as $\varepsilon \to 0$ with $t > 0$ fixed. Also note that the integral equation (2.16) is not the reduced equation (1.2), unless the second integral on the right side is zero. In the special case where $\mathbf{f}(0) = \mathbf{0}$, the boundary layer has $O(1)$ magnitude and the leading order term $\mathbf{z}_0$ obeys a different equation.

The solution of (2.17) is $\mathbf{z}_0(\tau) = e^{A(0,0)\tau}\mathbf{f}(0)$. If $(\mathbf{H_2})$ holds,

$$\int_0^\infty \mathbf{z}_0(\tau)\,d\tau = -\mathbf{A}(0,0)^{-1}\mathbf{f}(0),$$

and (2.16) becomes

$$0 = \int_0^t \mathbf{A}(t,s)\mathbf{y}_0(s)\,ds + \mathbf{f}(t) - \mathbf{A}(t,0)\mathbf{A}(0,0)^{-1}\mathbf{f}(0),$$

which has a smooth solution.

2.3. Derivation of the formal solution.

In this section we assume that (2.13), (2.14) and (2.15) hold, so that we seek a formal solution in the form

$$\mathbf{u}(t;\varepsilon) = \mathbf{y}(t;\varepsilon) + \frac{1}{\varepsilon}\mathbf{z}(t/\varepsilon;\varepsilon). \tag{2.18}$$

The vector functions $\mathbf{y}(t;\varepsilon)$ and $\mathbf{z}(\tau;\varepsilon)$ are given asymptotically by

$$\mathbf{y}(t;\varepsilon) \sim \sum_{j=0}^\infty \varepsilon^j \mathbf{y}_j(t), \tag{2.19}$$

$$\mathbf{z}(\tau;\varepsilon) \sim \sum_{j=0}^\infty \varepsilon^j \mathbf{z}_j(\tau), \tag{2.20}$$

as $\varepsilon \to 0$. To ensure that (2.6) holds we assume that

$$\lim_{\tau\to\infty} \mathbf{z}_j(\tau) = 0, \quad j = 0, 1, 2, \dots$$

Putting $\mathbf{y}_{-1}(t) = \mathbf{0}$, it follows from (2.7) that

$$\sum_{j=0}^{\infty} \varepsilon^j \mathbf{y}_{j-1}(t) + \sum_{j=0}^{\infty} \varepsilon^j \mathbf{z}_j(t/\varepsilon) \sim \mathbf{f}(t) + \sum_{j=0}^{\infty} \varepsilon^j \int_0^t \mathbf{A}(t,s)\mathbf{y}_j(s)\, ds + \sum_{j=0}^{\infty} \varepsilon^j \int_0^{t/\varepsilon} \mathbf{A}(t,\varepsilon\sigma)\mathbf{z}_j(\sigma)\, d\sigma. \tag{2.21}$$

The orders of the terms in

$$\sum_{j=0}^{\infty} \varepsilon^j \int_0^{t/\varepsilon} \mathbf{A}(t,\varepsilon\sigma)\mathbf{z}_j(\sigma)\, d\sigma \tag{2.22}$$

in (2.21) as $\varepsilon \to 0$ depend on the decay rate of the layer term $\mathbf{z}_j(\tau)$. We assume that $\mathbf{z}_j(\tau)$ decays exponentially so that, for each integer $j \geq 0$, there are positive constants β_j and c_j such that

$$|\mathbf{z}_j(\tau)| \leq c_j \mathrm{e}^{-\beta_j \tau}, \quad \tau \geq 0. \tag{2.23}$$

By writing out the Taylor expansion of $\mathbf{A}(t, \varepsilon\sigma)$ we find that

$$\mathbf{A}(t,\varepsilon\sigma) \sim \sum_{i=0}^{\infty} \varepsilon^i \mathbf{E}_i(t,\sigma) \quad \text{as } \varepsilon \to 0,$$

where

$$\mathbf{E}_i(t,\sigma) = \frac{1}{i!}\sigma^i \left[\partial_2^i \mathbf{A}\right](t,0). \tag{2.24}$$

Hence, noting that $\mathbf{E}_i(t,\sigma)$ is defined for all (t,σ) in $\mathbb{R}^+ \times \mathbb{R}^+$, (2.22) has the asymptotic expansion

$$\begin{aligned}\sum_{j=0}^{\infty} \varepsilon^j \sum_{i=0}^{\infty} \varepsilon^i \int_0^{t/\varepsilon} \mathbf{E}_i(t,\sigma)\mathbf{z}_j(\sigma)\, d\sigma &= \sum_{j=0}^{\infty} \varepsilon^j \sum_{i=0}^{\infty} \varepsilon^i \int_0^{\infty} \mathbf{E}_i(t,\sigma)\mathbf{z}_j(\sigma)\, d\sigma \\ &\quad - \sum_{j=0}^{\infty} \varepsilon^j \sum_{i=0}^{\infty} \varepsilon^i \int_{t/\varepsilon}^{\infty} \mathbf{E}_i(t,\sigma)\mathbf{z}_j(\sigma)\, d\sigma \\ &\sim \sum_{j=0}^{\infty} \varepsilon^j \sum_{i=0}^{j} \int_0^{\infty} \mathbf{E}_i(t,\sigma)\mathbf{z}_{j-i}(\sigma)\, d\sigma - \mathbf{J}(t/\varepsilon, \varepsilon)\end{aligned}$$

as $\varepsilon \to 0$, where

$$\mathbf{J}(\tau,\varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j \sum_{i=0}^{\infty} \varepsilon^i \int_{\tau}^{\infty} \mathbf{E}_i(\varepsilon\tau,\sigma)\mathbf{z}_j(\sigma)\, d\sigma.$$

We introduce the homogeneous polynomial of degree i

$$\mathbf{F}_i(\tau,\sigma) = \frac{1}{i!}\left[(\tau\partial_1 + \sigma\partial_2)^i \mathbf{A}\right](0,0), \tag{2.25}$$

which has the property that

$$\sum_{i=0}^{\infty} \varepsilon^i \mathbf{E}(\varepsilon\tau,\sigma) \sim \sum_{i=0}^{\infty} \varepsilon^i \mathbf{F}(\tau,\sigma), \quad \text{as } \varepsilon \to 0.$$

It follows that

$$\mathbf{J}(\tau,\varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j \mathbf{J}_j(\tau) \quad \text{as } \varepsilon \to 0,$$

where

$$\mathbf{J}_j(\tau) = \sum_{i=0}^{j} \int_{\tau}^{\infty} \mathbf{F}_i(\tau,\sigma)\mathbf{z}_{j-i}(\sigma)\,d\sigma.$$

However it follows from (2.23) that for any $0 \le l \le i$

$$\tau^l \left| \int_{\tau}^{\infty} \sigma^{i-l}\mathbf{z}_{j-i}(\sigma)\,d\sigma \right| \le \tau^l c_{i-j} \int_{\tau}^{\infty} \sigma^{i-l} e^{-\tau\beta_{i-j}}\,d\sigma \to 0 \quad \text{as } \tau \to 0,$$

and hence from (2.25) that

$$\mathbf{J}_j(\tau) \to 0 \quad \text{as } \tau \to 0.$$

Equation (2.21) can be decomposed into functions of t and functions of t/ε which decay to zero. The following Lemma is used to derive the coefficients $\mathbf{y}_j(t)$ and $\mathbf{z}_j(\tau)$ of (2.19) and (2.20).

Lemma 2.1. *For each integer $j \ge 0$, let $\mathbf{p}_j(t)$ be a continuous function on $[0,T]$ and $\mathbf{q}_j(\tau)$ a continuous function on $[0,\infty)$ such that $\mathbf{q}_j(\tau) \to 0$ as $\tau \to \infty$. Suppose that for every integer $N \ge 1$,*

$$\sum_{j=0}^{N-1} \left\{\mathbf{p}_j(t) + \mathbf{q}_j(t/\varepsilon)\right\}\varepsilon^j = O(\varepsilon^N), \tag{2.26}$$

uniformly as $\varepsilon \to 0$. Then $\mathbf{p}_j = \mathbf{0}$ and $\mathbf{q}_j = \mathbf{0}$ for every $j \ge 0$.

Proof. There is a uniformly bounded function $\mathbf{r}_0$, defined for all $0 \le t \le T, \tau \ge 0$ and $0 < \varepsilon \le \varepsilon_0$, such that

$$\mathbf{p}_0(t) + \mathbf{q}_0(t/\varepsilon) = \varepsilon\mathbf{r}_0(t, t/\varepsilon, \varepsilon).$$

By letting $\varepsilon \to 0$ for each fixed $t \in (0,T]$, it follows that $\mathbf{p}_0(t) = \mathbf{0}$. The continuity of $\mathbf{p}_0$ then implies $\mathbf{p}_0 = \mathbf{0}$ on $[0,T]$. Therefore substituting $t = \varepsilon\tau$, we have

$$\mathbf{q}_0(\tau) = \varepsilon\mathbf{r}_0(\varepsilon\tau, \tau, \varepsilon).$$

Hence, on taking the limit as $\varepsilon \to 0$ for each fixed $\tau > 0$, we deduce that $\mathbf{q}_0 = \mathbf{0}$. An obvious induction argument completes the proof. □

It has been shown that (2.21) can be expressed in the form (2.26) with $\mathbf{p}_j$ and $\mathbf{q}_j$ given by

$$\mathbf{p}_j(t) = \mathbf{y}_{j-1}(t) - \mathbf{f}(t) - \int_0^t \mathbf{A}(t,s)\mathbf{y}_j(s)\,ds - \sum_{i=0}^{j} \int_0^\infty \mathbf{E}_{j-i}(t,\sigma)\mathbf{z}_j(\sigma)\,d\sigma,$$
$$\mathbf{q}_j(\tau) = \mathbf{z}_j(\tau) + \mathbf{J}_j(\tau).$$

It is convenient to introduce

$$\boldsymbol{\psi}_j(\tau) = \begin{cases} \mathbf{0}, & j = 0, \\ \displaystyle\sum_{i=0}^{j-1} \int_\tau^\infty \mathbf{F}_{j-i}(\tau,\sigma)\mathbf{z}_i(\sigma)\,d\sigma, & j \geq 1, \end{cases} \tag{2.27}$$

$$\boldsymbol{\phi}_j(t) = \begin{cases} \mathbf{f}(t) + \int_0^\infty \mathbf{A}_0(t,0)\mathbf{z}_0(\sigma)\,d\sigma, & j = 0, \\ \displaystyle\sum_{i=0}^{j} \int_0^\infty \mathbf{E}_{j-i}(t,\sigma)\mathbf{z}_i(\sigma)\,d\sigma, & j \geq 1. \end{cases} \tag{2.28}$$

It is important to note that $\boldsymbol{\psi}_j$ and $\boldsymbol{\phi}_{j-1}$ are determined by $\mathbf{z}_0, \dots, \mathbf{z}_{j-1}$. Later we use the identity

$$\boldsymbol{\phi}_j(0) = \boldsymbol{\psi}_j(0) + \int_0^\infty \mathbf{A}(0,0)\mathbf{z}_j(\sigma)\,d\sigma. \tag{2.29}$$

From (2.27) and (2.28)

$$\mathbf{p}_j(t) = \mathbf{y}_{j-1}(t) - \int_0^t \mathbf{A}(t,s)\mathbf{y}_j(s)\,ds - \boldsymbol{\phi}_j(t),$$
$$\mathbf{q}_j(\tau) = \mathbf{z}_j(\tau) + \int_\tau^\infty \mathbf{A}(0,0)\mathbf{z}_j(\sigma)\,d\sigma + \boldsymbol{\psi}_j(t).$$

By applying Lemma 2.1 we obtain the following equations for $\mathbf{y}_j(t)$ and $\mathbf{z}_j(\tau)$:

$$\mathbf{y}_{j-1}(t) = \int_0^t \mathbf{A}(t,s)\mathbf{y}_j(s)\,ds + \boldsymbol{\phi}_j(t), \tag{2.30}$$

$$\mathbf{z}_j(\tau) = -\int_\tau^\infty \mathbf{A}(0,0)\mathbf{z}_j(\sigma)\,d\sigma - \boldsymbol{\psi}_j(\tau). \tag{2.31}$$

The integral equations are augmented by initial conditions. Since

$$\mathbf{u}(0;\varepsilon) = \frac{\mathbf{f}(0)}{\varepsilon} \sim \sum_{j=0}^{\infty} \varepsilon^j \Big(\mathbf{y}_j(0) + \frac{1}{\varepsilon}\mathbf{z}_j(0) \Big),$$

we impose the conditions

$$\mathbf{z}_j(0) = \begin{cases} \mathbf{f}(0), & j = 0, \\ -\mathbf{y}_{j-1}(0), & j \geq 1. \end{cases} \tag{2.32}$$

3. Proof of asymptotic solution

3.1. Properties of the formal solution.

In this section, we first show in Proposition 3.1 that there exists solutions $\mathbf{y}_j$ and $\mathbf{z}_j$ to equations (2.30) and (2.31) satisfying the initial condition (2.32). Moreover $\mathbf{z}_j(\tau) \to \mathbf{0}$ exponentially as $\tau \to \infty$. Therefore

$$\mathbf{u}_n(t;\varepsilon) = \mathbf{y}_{n-1}(t) + \mathbf{z}_n(t/\varepsilon),$$

can be defined for $n \geq 0$. Then

$$\mathbf{U}(t;\varepsilon) = \sum_{n=0}^{\infty} \mathbf{u}_n(t;\varepsilon)\varepsilon^{n-1} \tag{3.1}$$

is an asymptotic series as $\varepsilon \to 0$. If we define the truncated sum

$$\mathbf{U}_N(t;\varepsilon) = \sum_{n=0}^{N+1} \mathbf{u}_n(t;\varepsilon)\varepsilon^{n-1}, \tag{3.2}$$

then we can consider the residual $\boldsymbol{\rho}_N(t;\varepsilon)$ given by

$$\varepsilon\mathbf{U}_N(t;\varepsilon) = \mathbf{f}(t) + \int_0^t \mathbf{A}(t,s)\mathbf{U}_N(s;\varepsilon)\,ds - \boldsymbol{\rho}_N(t;\varepsilon). \tag{3.3}$$

Thus $\mathbf{U}_N(t;\varepsilon)$ satisfies the original equation (1.1) approximately with a residual $\boldsymbol{\rho}_N(t;\varepsilon)$. We express $\boldsymbol{\rho}'_N(t;\varepsilon)$ as the sum of a function of t and a function of t/ε. In the same manner as in the construction of the formal solution, functions of t/ε contribute only in the initial layer region; away from the layer, functions of t dominate. In Proposition 3.3 various results are given which demonstrate that $\boldsymbol{\rho}_N(t;\varepsilon)$ is small for $0 \leq t \leq T$ as $\varepsilon \to 0$. Similar results are given in Chapter 5 of [15] for a linear overdamped initial-value problems. The estimates in Lemma 3.3 are stronger than those of Section 7 of Smith and Lange [10].

Proposition 3.1. *Suppose that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold, and let* $0 < \beta < \alpha$. *Then for every integer* $j \geq 0$ *there exist solutions* $\mathbf{y}_j \in C^\infty([0,T];\mathbb{R}^n)$ *of* (2.30) *and solutions* $\mathbf{z}_j(\tau) \in C^\infty([0,\infty);\mathbb{R}^n)$ *of* (2.31) *and* (2.32). *Moreover there are positive constants* c_j *such that*

$$|\mathbf{z}_j(\tau)| \leq c_j \mathrm{e}^{-\beta\tau}, \quad \tau \geq 0. \tag{3.4}$$

Proof. We choose α_1 such that $\beta < \alpha_1 < \alpha$. Consider the hypothesis that for some integer $N \geq 0$ there are solutions $\mathbf{y}_j(t)$ of (2.30) for all $0 \leq j \leq N-1$ and solutions $\mathbf{z}_j(\tau)$ (2.31) for all $0 \leq j \leq N$ satisfying

$$|\mathbf{z}_j(\tau)| \leq \mathrm{e}^{-\alpha_1\tau} p_j(\tau), \quad \tau \geq 0, \tag{3.5}$$

where $p_j(\tau)$ is a polynomial of degree j with positive coefficients. Once this hypothesis has been established for all $N \geq 0$, Proposition 3.1 follows immediately.

The solution of

$$\mathbf{z}_0(\tau) = -\int_\tau^\infty \mathbf{A}(0,0)\mathbf{z}_0(\sigma)\,d\sigma, \quad \mathbf{z}_0(0) = \mathbf{f}(0),$$

is $\mathbf{z}_0(\tau) = \mathrm{e}^{\mathbf{A}(0,0)\tau}\mathbf{f}(0)$. Hence by (2.1),

$$|\mathbf{z}_0(\tau)| \leq \kappa e^{-\alpha_1\tau}|\mathbf{f}(0)|, \quad \tau \geq 0.$$

Also $\mathbf{y}_{-1}(t) = \mathbf{0}$. Hence the induction hypothesis is true for $N = 0$.

Suppose now that it holds for some $N \geq 0$. Then $\boldsymbol{\phi}_N(t)$ is well-defined and smooth. The equation

$$\begin{aligned}\mathbf{A}(t,t)^{-1}&[\mathbf{y}'_{N-1}(t) - \boldsymbol{\phi}'_N(t)] \\ &= \mathbf{y}_N(t) + \int_0^t \mathbf{A}(t,t)^{-1}\partial_1\mathbf{A}(t,s)\mathbf{y}_N(s)\,ds,\end{aligned} \tag{3.6}$$

which is obtained by differentiating (2.30), is a Volterra equation of the second kind. Since the kernel and forcing function are C^∞, so is the unique solution $\mathbf{y}_N(t)$. It follows that

$$\mathbf{y}_{N-1}(t) = \int_0^t \mathbf{A}(t,s)\mathbf{y}_N(s)\,ds + \boldsymbol{\phi}_N(t) + \text{const.}$$

However the constant is zero because (2.29) and (2.31) give $-\mathbf{z}_N(0) = \boldsymbol{\phi}_N(0)$, and the induction hypothesis implies that the initial condition $\mathbf{z}_N(0) = -\mathbf{y}_{N-1}(0)$ holds.

The induction hypothesis also implies that $\boldsymbol{\psi}_{N+1}(\tau)$ is well-defined. Moreover a tedious calculation using (2.1) and (3.4) establishes that

$$|\boldsymbol{\psi}_{N+1}(\tau)| \leq e^{-\alpha_1\tau} P_{N+1}(\tau),$$

where $P_{N+1}(\tau)$ is a polynomial of degree N. $\mathbf{z}_{N+1}$ satisfies the ordinary differential equation

$$\mathbf{z}'_{N+1}(\tau) = \mathbf{A}(0,0)\mathbf{z}_{N+1}(\tau) - \boldsymbol{\psi}'_{N+1}(\tau), \quad \mathbf{z}_{N+1}(0) = \mathbf{y}_N(0).$$

The solution of this can be found using variation of parameters and written as

$$\begin{aligned}\mathbf{z}_{N+1}(\tau) =& e^{\mathbf{A}(0,0)\tau}[\mathbf{y}_N(0) - \boldsymbol{\psi}_{N+1}(0)] + \boldsymbol{\psi}_{N+1}(\tau)\\ &+ \mathbf{A}(0,0)\int_0^\tau e^{\mathbf{A}(0,0)(\tau-\sigma)}\boldsymbol{\psi}_{N+1}(\sigma)\,d\sigma. \end{aligned}\tag{3.7}$$

The norm of the last integral is easily bounded by

$$\begin{aligned}\left|\mathbf{A}(0,0)\int_0^\tau e^{\mathbf{A}(\tau-\sigma)}\boldsymbol{\psi}_{N+1}(\sigma)\,d\sigma\right| &\le |\mathbf{A}(0,0)|\int_0^\tau |e^{\mathbf{A}(0,0)(\tau-\sigma)}\boldsymbol{\psi}_{N+1}(\sigma)|\,d\sigma\\ &\le \kappa|\mathbf{A}(0,0)|\int_0^\tau e^{-\alpha_1(\tau-\sigma)}P_{N+1}(\sigma)\,d\sigma,\end{aligned}$$

and it can be shown from (3.7) that $\mathbf{z}_{N+1}(\tau)$ satisfies an estimate of the form

$$|\mathbf{z}_{N+1}(\tau)| \le e^{-\alpha_1\tau}p_{N+1}(\tau), \quad \tau \ge 0,$$

where $p_{N+1}(\tau)$ is a polynomial of degree $N+1$. This proves the induction hypothesis. □

Remark 3.2. The formal series (3.1) is a uniform asymptotic series, because

$$\frac{|\mathbf{u}_{n+1}(t;\varepsilon)|}{|\mathbf{u}_n(t;\varepsilon)|} \to \frac{|\mathbf{y}_{n+1}(t)|}{|\mathbf{y}_n(t)|} \quad \text{as } \varepsilon \to 0,$$

implying that $\mathbf{u}_{n+1}(t;\varepsilon)\varepsilon^{n+1} = o(\mathbf{u}_n(t;\varepsilon)\varepsilon^n)$ uniformly for $0 \le t \le T$ as $\varepsilon \to 0$.

Proposition 3.3. *Suppose that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold. Then for each* $N \ge 0$,

$$|\boldsymbol{\rho}_N(t;\varepsilon)| = O(\varepsilon^{N+1})$$

uniformly for $0 \le t \le T$ *as* $\varepsilon \to 0$, *and there are positive constants* d_N *and* e_N *such that*

$$|\boldsymbol{\rho}'_N(t;\varepsilon)| \le e_N\varepsilon^{N+1}, \qquad \int_0^t |\boldsymbol{\rho}'_N(s;\varepsilon)|\,ds \le d_N\varepsilon^{N+1}, \tag{3.8}$$

for all $0 < \varepsilon \le \varepsilon_0$ *and for all* t *in* $[0,T]$, *for some* $\varepsilon_0 > 0$.

Proof. Later we shall use the estimates in (3.8), and therefore only prove these in detail. To demonstrate the other result an almost identical argument is used.

Since $\boldsymbol{\rho}_N(0;\varepsilon) = \mathbf{0}$, differentiation of (3.3) gives

$$\begin{aligned}\boldsymbol{\rho}'_N(t;\varepsilon) =& -\varepsilon\mathbf{U}'_N(t;\varepsilon) + \mathbf{f}'(t) + \mathbf{A}(t,t)\mathbf{U}_N(t;\varepsilon)\\ &+ \int_0^t \partial_1\mathbf{A}(t,s)\mathbf{U}_N(s;\varepsilon)\,ds.\end{aligned}\tag{3.9}$$

The substitution of (3.2) and the differentiated version of (2.30) into this yields

$$\begin{aligned}\boldsymbol{\rho}_N'(t;\varepsilon) = &-\varepsilon^{N+1}\mathbf{y}_N'(t) - \sum_{i=0}^{N+1}\varepsilon^{i-1}\mathbf{z}_i'(t/\varepsilon) + \sum_{i=0}^{N+1}\varepsilon^{i-1}\mathbf{A}(t,t)\mathbf{z}_i(t/\varepsilon)\\ &-\sum_{i=0}^{N}\varepsilon^i\sum_{k=0}^{i}\int_0^\infty \partial_1\mathbf{E}_{i-k}(t,\sigma)\mathbf{z}_k(\sigma)\,d\sigma\\ &+\sum_{i=0}^{N}\varepsilon^i\int_0^{t/\varepsilon}\partial_1\mathbf{A}(t,\varepsilon\sigma)\mathbf{z}_i(\sigma)\,d\sigma. \qquad (3.10)\end{aligned}$$

Using the Taylor expansion of $\mathbf{A}(t,\varepsilon\sigma)$ we can derive

$$\begin{aligned}\sum_{i=0}^{N}\varepsilon^i\sum_{k=0}^{\infty}\varepsilon^k\int_0^{t/\varepsilon}\partial_1\mathbf{E}_k(t,\sigma)\mathbf{z}_i(\sigma)\,d\sigma &= \sum_{i=0}^{N}\varepsilon^i\sum_{k=0}^{\infty}\varepsilon^k\int_0^{\infty}\partial_1\mathbf{E}_k(t,\sigma)\mathbf{z}_i(\sigma)\,d\sigma\\ &-\sum_{i=0}^{N}\varepsilon^i\sum_{k=0}^{\infty}\varepsilon^k\int_{t/\varepsilon}^{\infty}\partial_1\mathbf{E}_k(t,\sigma)\mathbf{z}_i(\sigma)\,d\sigma.\end{aligned}$$

By substituting this into (3.10), we get

$$\begin{aligned}\boldsymbol{\rho}_N'(t;\varepsilon) = &-\varepsilon^{N+1}\mathbf{y}_N'(t) + \sum_{i=0}^{N+1}\varepsilon^{i-1}\mathbf{z}_i'(t/\varepsilon)\\ &+\sum_{i=0}^{N+1}\varepsilon^{i-1}\sum_{k=0}^{\infty}\varepsilon^k\mathbf{F}_k(t/\varepsilon,t/\varepsilon)\mathbf{z}_i(t/\varepsilon)\\ &+\sum_{i=N+1}^{\infty}\varepsilon^i\sum_{k=0}^{i}\int_0^\infty\partial_1\mathbf{E}_{i-k}(t,\sigma)\mathbf{z}_k(\sigma)\,d\sigma\\ &-\sum_{i=0}^{\infty}\varepsilon^i\sum_{k=0}^{i}\int_{t/\varepsilon}^{\infty}\mathbf{F}_{i-k}'(t/\varepsilon,\sigma)\mathbf{z}_k(\sigma)\,d\sigma, \qquad (3.11)\end{aligned}$$

where

$$\mathbf{F}_i'(\tau,\sigma) = \frac{1}{i!}[(\tau\partial_1+\sigma\partial_2)^i\partial_1\mathbf{A}](0,0).$$

By putting the differentiated version of (2.31) into (3.11), we obtain

$$\begin{aligned}\boldsymbol{\rho}'_N(t;\varepsilon) = &- \varepsilon^{N+1}\mathbf{y}'_N(t) + \sum_{i=N+1}^{\infty} \varepsilon^i \sum_{k=0}^{i} \int_0^\infty \partial_1 \mathbf{E}_{i-k}(t,\sigma)\mathbf{z}_k(\sigma)\,d\sigma \\ &+ \sum_{i=N+1}^{\infty} \varepsilon^i \sum_{k=0}^{i} \mathbf{F}_{i-k+1}(t/\varepsilon, t/\varepsilon)\mathbf{z}_k(t/\varepsilon) \\ &+ \sum_{i=N+1}^{\infty} \varepsilon^i \sum_{k=0}^{i} \int_{t/\varepsilon}^\infty \mathbf{F}'_{i-k}(t/\varepsilon,\sigma)\mathbf{z}_k(\sigma)\,d\sigma,\end{aligned}$$

where the following relation has been used

$$\partial_1 \mathbf{F}_i(\tau,\sigma) = \mathbf{F}'_{i-1}(\tau,\sigma).$$

To summarise it has been shown that

$$\boldsymbol{\rho}^1_N(t;\varepsilon) = \boldsymbol{\rho}^1_N(t;\varepsilon) + \boldsymbol{\rho}^2_N(t/\varepsilon;\varepsilon) + O(\varepsilon^{N+2}),$$

where

$$\begin{aligned}\boldsymbol{\rho}^1_N(t;\varepsilon) =& \varepsilon^{N+1}\Bigg\{ - \mathbf{y}_N(t) \\ &+ \sum_{k=0}^{N+1} \int_0^\infty \partial_1 \mathbf{E}_{N+1-k}(t,\sigma)\mathbf{z}_k(\sigma)\,d\sigma \Bigg\} \qquad (3.12) \\ \boldsymbol{\rho}^2_N(\tau;\varepsilon) =& \sum_{i=N+1}^{\infty} \varepsilon^i \sum_{k=0}^{i} \Bigg\{ \mathbf{F}_{i-k+1}(\tau,\tau)\mathbf{z}_k(\tau) \\ &- \int_\tau^\infty \mathbf{F}'_{i-k}(\tau,\sigma)\mathbf{z}_k(\sigma)\,d\sigma \Bigg\}. \qquad (3.13)\end{aligned}$$

By (3.12)

$$|\boldsymbol{\rho}^1_N(t;\varepsilon)| \le \gamma^1_N \varepsilon^{N+1}, \quad \int_0^t |\boldsymbol{\rho}^1_N(s;\varepsilon)|\,ds \le \gamma^2_N \varepsilon^{N+1}, \qquad (3.14)$$

uniformly for all $0 \le t \le T$, where γ^1_N and γ^2_N are positive constants. Using (3.5) the function $\boldsymbol{\rho}^2_N(\tau;\varepsilon)$ satisfies

$$|\boldsymbol{\rho}^2_N(\tau;\varepsilon)| \le \varepsilon^{N+1} Q_N(\tau) \mathrm{e}^{-\alpha_1 \tau} \le \varepsilon^{N+1}\gamma^3_N e^{-\beta\tau},$$

where Q_N is a polynomial with positive coefficients, and $\beta < \alpha_1 < \alpha$. Hence there is a positive γ^4_N such that

$$\frac{1}{\varepsilon}\int_0^t |\boldsymbol{\rho}_{N^2}(s/\varepsilon;\varepsilon)|\,ds \le \gamma^4_N \varepsilon^{N+1},$$

uniformly for $0 \le t \le T$ as $\varepsilon \to 0$. The conclusions of the proposition now follow. □

3.2. Main theorem.

In this section we state and prove our main result. It says that for $N \ge 0$,

$$|\mathbf{u}(t;\varepsilon) - \mathbf{U}_N(t;\varepsilon)| = O(\varepsilon^{N+1}) \quad \text{as } \varepsilon \to 0,$$

and hence that $\mathbf{U}(t;\varepsilon)$ given by (3.1) is an *asymptotic solution.*

Theorem 3.4. *Suppose* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *in Section* 2 *are satisfied. Let* $\mathbf{u}(t;\varepsilon)$ *be the solution of* (1.1) *and* $\mathbf{U}_N(t;\varepsilon)$ *the partial sum given in* (3.2). *Then for each integer* $N \ge 0$, *there are positive constants* C_{N+1} *and* ε_0, *independent of* ε, *such that*

$$|\mathbf{u}(t;\varepsilon) - \mathbf{U}_N(t;\varepsilon)| \le C_{N+1}\varepsilon^{N+1}, \tag{3.15}$$

uniformly for $0 \le t \le T$ *and* $0 < \varepsilon \le \varepsilon_0$.

Proof. It is convenient to fix $N \ge 0$ and define

$$\mathbf{r}_N(t;\varepsilon) = \mathbf{u}(t;\varepsilon) - \mathbf{U}_N(t;\varepsilon).$$

By subtracting (3.3) from (1.1) we get

$$\varepsilon\mathbf{r}_N(t;\varepsilon) = \boldsymbol{\rho}_N(t;\varepsilon) + \int_0^t \mathbf{A}(t,s)\mathbf{r}_N(s;\varepsilon)\,ds.$$

Differentiation yields

$$\mathbf{r}'_N(t;\varepsilon) = \frac{1}{\varepsilon}\mathbf{B}(t)\mathbf{r}_N(t;\varepsilon) + \frac{1}{\varepsilon}\boldsymbol{\rho}'_N(t;\varepsilon) + \frac{1}{\varepsilon}\int_0^t \partial_1\mathbf{A}(t,s)\mathbf{r}_N(s;\varepsilon)\,ds,$$
$$\mathbf{r}_N(0;\varepsilon) = \mathbf{0}, \tag{3.16}$$

where $\mathbf{B}(t)$ is given by (2.3).

The solution of the ordinary differential equation

$$\mathbf{r}'_N(t;\varepsilon) = \frac{1}{\varepsilon}\mathbf{B}(t)\mathbf{r}_N(t;\varepsilon) + \mathbf{g}(t),$$

can be represented using variation of parameters as

$$\mathbf{r}_N(t;\varepsilon) = \mathbf{\Phi}(t,0;\varepsilon)\mathbf{r}_N(0;\varepsilon) + \int_0^t \mathbf{\Phi}(t,s;\varepsilon)\mathbf{g}(s)\,ds \tag{3.17}$$

where

$$\mathbf{\Phi}(t,s;\varepsilon) = \mathbf{R}(t;\varepsilon)\mathbf{R}(s;\varepsilon)^{-1}, \tag{3.18}$$

and $\mathbf{R}(t;\varepsilon)$ is the fundamental matrix solution satisfying

$$\mathbf{R}'(t;\varepsilon) = \frac{1}{\varepsilon}\mathbf{B}(t)\mathbf{R}(t;\varepsilon).$$

It is a result of Flatto and Levinson [4] that there are constants $\kappa_1 > 0$ and $0 < \alpha_2 < \alpha$ such that

$$|\mathbf{\Phi}(t,s;\varepsilon)| \le \kappa_1 e^{-\alpha_2(t-s)/\varepsilon}, \tag{3.19}$$

since $(\mathbf{H_2})$ holds.

Application of the representation (3.17) to (3.16) yields

$$\begin{aligned}\mathbf{r}_N(t;\varepsilon) =& \frac{1}{\varepsilon}\int_0^t \mathbf{\Phi}(t,s;\varepsilon)\boldsymbol{\rho}'_N(s;\varepsilon)\,ds\\ &+\frac{1}{\varepsilon}\int_0^t \left(\int_v^t \mathbf{\Phi}(t,s;\varepsilon)\partial_1\mathbf{A}(s,v)\,ds\right)\mathbf{r}_N(v;\varepsilon)\,dv\end{aligned} \tag{3.20}$$

However it follows from (3.19) that

$$\begin{aligned}\frac{1}{\varepsilon}\left|\int_v^t \mathbf{\Phi}(t,s;\varepsilon)\partial_1\mathbf{A}(s,v)\,ds\right| &\le \frac{\kappa_1}{\varepsilon}\int_v^t \mathrm{e}^{-\alpha_2(t-s)/\varepsilon}|\partial_1\mathbf{A}(s,v)|\,ds\\ &\le \frac{\kappa_1}{\alpha_2}\max_{(t,s)\in\triangle_T}|\partial_1\mathbf{A}(t,s)| := \kappa_2.\end{aligned}$$

Similarly we see from (3.8) and (3.19) that

$$\frac{1}{\varepsilon}\left|\int_0^t \mathbf{\Phi}(t,s;\varepsilon)\boldsymbol{\rho}'_N(s;\varepsilon)\,ds\right| \le e_N\varepsilon^N\int_s^t \mathrm{e}^{-\alpha_2(t-s)/\varepsilon}\,ds \le \frac{e_N}{\alpha_2}\varepsilon^{N+1}.$$

Hence (3.20) implies that

$$|\mathbf{r}_N(t;\varepsilon)| \le \frac{e_N}{\alpha_2}\varepsilon^{N+1} + \kappa_2\int_0^t |\mathbf{r}_N(v;\varepsilon)|\,dv.$$

By Gronwall's inequality,

$$|\mathbf{r}_N(t;\varepsilon)| \le \frac{e_N}{\alpha_2}\varepsilon^{N+1}e^{\kappa_2 t},$$

and the theorem is proved. □

4. Examples

4.1. Example of boundary layer stability condition.

To illustrate the method, let us consider the following example from [1]

$$\varepsilon u(t) = f(t) - \int_0^t \{(t-s)\omega(s)+\theta(s)\}u(s)\,ds, \qquad t \ge 0. \tag{4.1}$$

where $\theta(t) > 0$. Equation (4.1) is equivalent to "over-damped" initial value second-order ordinary differential equation

$$\varepsilon u''(t) + \theta(t)u'(t) + \{\omega(t)+\theta'(t)\}u(t) = f''(t),\ t > 0, \tag{4.2}$$

with initial conditions

$$u(0) = \frac{1}{\varepsilon} f(0), \quad u'(0) = -\frac{1}{\varepsilon^2}\theta(0) f(0) + \frac{1}{\varepsilon} f'(0).$$

For simplicity we take

$$\omega(t) = 1, \quad \theta(t) = 1,\ f(t) = t + t^2 + \frac{1}{6}t^3$$

because the exact solution of (4.1) can be obtained using Laplace transforms as

$$\begin{aligned} u(t;\varepsilon) =& t + 1 + \frac{1}{\gamma_1 - \gamma_2}\left[\left(\gamma_2 - 1 + \frac{1}{\varepsilon}\right)\mathrm{e}^{\gamma_1 t}\right. \\ & \left. - \left(\gamma_1 - 1 + \frac{1}{\varepsilon}\right)\mathrm{e}^{\gamma_2 t}\right], \end{aligned} \tag{4.3}$$

where

$$\gamma_1, \gamma_2 = \frac{1}{2\varepsilon}(-1 \pm \sqrt{1 - 4\varepsilon}).$$

In this example $f(0) = 0$ and we should use an asymptotic representation other than (2.18). However we find that $z_0(\tau) = 0$ and our representation agrees with the correct one. Note that in this example $a(t, s) = -t + s - 1$ and the boundary layer stability condition holds. For $j \geq 0$, the inner correction solution $z_j(\tau)$ is given by

$$z_j(\tau) = \mathrm{e}^{-\tau} z_j(0) - \int_0^\tau \mathrm{e}^{-(\tau-\sigma)} \psi_j'(\sigma)\, d\sigma, \tag{4.4}$$

where

$$\psi_j(\tau) = \sum_{i=0}^{j-1} \int_\tau^\infty F_{j-i}(\tau, \sigma) z_i(\sigma)\, d\sigma.$$

Since in this example

$$F_i(\tau, \sigma) = \begin{cases} -1, & i = 0, \\ -(\tau - \sigma), & i = 1, \\ 0, & i \geq 2. \end{cases}$$

it follows that

$$\psi_j(\tau) = -\int_\tau^\infty (\tau - \sigma) z_{j-1}(\sigma)\, d\sigma.$$

Therefore we get

$$z_j(\tau) = \mathrm{e}^{-\tau} z_j(0) - \int_0^\tau \mathrm{e}^{-(\tau-\sigma)} \int_\sigma^\infty z_{j-1}(v)\, dv\, d\sigma,$$

where

$$z_0(0) = f(0), \quad z_j(0) = y_{j-1}(0), \quad j \geq 1.$$

By (2.30) the outer solution $y_j(t)$ satisfies

$$y_{j-1}(t) = -\int_0^t (t-s+1)y_j(s)\,ds + \phi_j(t),$$

or

$$y''_{j-1}(t) - \phi''_j(t) = -y'_j(t) - y_j(t),$$

Since

$$E_i(t,\sigma) = \begin{cases} -(1+t), & i = 0, \\ \sigma, & i = 1, \\ 0, & i \geq 2, \end{cases}$$

we find that

$$\phi_j(t) = \sum_{i=j-1}^{j} \int_0^\infty E_{j-1}(t,\sigma) z_i(\sigma)\,d\sigma.$$

Therefore

$$\phi''_j(t) = \begin{cases} 2+t, & j = 0, \\ 0, & j \geq 1, \end{cases}$$

and

$$y_j(t) = -z_{j+1}(0)\mathrm{e}^{-t} - \int_0^t \mathrm{e}^{-(t-s)}(y''_{j-1}(s) - \phi''_j(s))\,ds.$$

From the above equations we see that

$$z_0(\tau) = 0, \quad z_1(\tau) = -\mathrm{e}^{-\tau}, \quad z_2(\tau) = -\tau\mathrm{e}^{-\tau},$$
$$y_0(t) = 1+t, \quad y_1(t) = 0,$$

from which we calculate the first two partial sums of the asymptotic solution to be

$$U_0(t;\varepsilon) = 1 + t - \mathrm{e}^{-t/\varepsilon}, \quad U_1(t;\varepsilon) = 1 + t - \mathrm{e}^{-t/\varepsilon} - t\mathrm{e}^{-t/\varepsilon}.$$

To verify that $U_0(t;\varepsilon)$ is a uniformly valid asymptotic approximation, note that

$$u(t;\varepsilon) - U_0(t;\varepsilon) = \mathrm{e}^{-t/\varepsilon} - \mathrm{e}^{(1-1/\varepsilon)t} + O(\varepsilon^2) = -t\mathrm{e}^{-t/\varepsilon} + O(\varepsilon^2),$$

implying that $|u(t;\varepsilon) - U_0(t;\varepsilon)| \leq C_0\varepsilon$. Similarly

$$u(t;\varepsilon) - U_1(t;\varepsilon) = -\frac{t^2}{2}\mathrm{e}^{-t/\varepsilon} + \varepsilon^2(\mathrm{e}^{-t} - \mathrm{e}^{-t/\varepsilon}) + O(\varepsilon^3),$$

so that $|u(t;\varepsilon) - U_1(t;\varepsilon)| \leq C_1\varepsilon^2$. Therefore the terms $U_0(t;\varepsilon)$ and $U_1(t;\varepsilon)$ found by additive decomposition method are uniformly valid asymptotic approximations to $u(t;\varepsilon)$ for all $0 \leq t \leq T$ as $\varepsilon \to 0$.

Having established a uniformly valid asymptotic expansion using the method of additive decomposition we developed, we now form a composite expansion from the exact solution (4.3). The outer expansion is found by fixing $t > 0$ and letting $\varepsilon \to 0$ in (4.3), obtaining

$$V(t;\varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j v_j(t),$$

where

$$v_0(t) = 1 + t, \quad v_1(t) = 0, \quad v_2(t) = \mathrm{e}^{-t}, \quad \dots .$$

Similarly expressing (4.3) in terms of the inner variable, τ and then taking the inner limit by fixing $\tau > 0$ and letting $\varepsilon \to 0$, the inner expansion takes the form

$$u(\varepsilon\tau;\varepsilon) = W(\tau;\varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j w_j(\tau),$$

where

$$w_0(\tau) = 1 - \mathrm{e}^{-\tau}, \;\; w_1(\tau) = \tau(1 - \mathrm{e}^{-\tau}), \;\; w_2(\tau) = 1 - (1 + \tau^2/2)\mathrm{e}^{-\tau}, \;\; \dots .$$

To obtain these expansion we have used

$$\begin{gathered}\gamma_1 = -1 + O(\varepsilon), \quad \gamma_2 = -\frac{1}{\varepsilon} + 1 + O(\varepsilon), \quad \varepsilon \to 0, \\ \frac{1}{\gamma_1 - \gamma_2}\left(\gamma_2 - 1 + \frac{1}{\varepsilon}\right) = \varepsilon^2 + 4\varepsilon^3 + O(\varepsilon^4), \\ \frac{1}{\gamma_1 - \gamma_2}\left(\gamma_1 - 1 + \frac{1}{\varepsilon}\right) = 1 + \varepsilon^2 + 4\varepsilon^3 + O(\varepsilon^4),\end{gathered}$$

all as $\varepsilon \to 0$. Using a standard procedure we can obtain a uniform approximation to $u(t;\varepsilon)$ by forming a composite expansions from the inner and outer expansions. In fact, we find that $U_0(t;\varepsilon)$ and $U_1(t;\varepsilon)$ are first two composite expansions.

4.2. Example of boundary layer stability condition failing.

Both Lange and Smith [9] and Angell and Olmstead [3] study the integral equation

$$\varepsilon^2 u(t) = f(t) - \int_0^t s u(s)\, ds. \tag{4.5}$$

To avoid fractional powers, ε^2 replaces ε. The exact solution of equation is found, after differentiating, to be

$$u(t;\varepsilon) = \frac{\mathrm{e}^{-t^2/(2\varepsilon^2)}}{\varepsilon^2}\left\{f(0) + \int_0^t \mathrm{e}^{s^2/(2\varepsilon^2)} f'(s)\, ds\right\}. \tag{4.6}$$

For this example $a(t,s) = -s$ and $a(t,t) < 0$ only for $t > 0$. Hence the analysis in Section 3 is no longer applicable.

Smith and Lange [9] observe that (4.5) has a number of interesting features. Firstly expansions for the inner and outer solutions can be calculated from (4.6). We see that

$$u(t;\varepsilon) \sim V(t;\varepsilon) = \sum_{j=0}^{\infty} \varepsilon^{2j} v_j(t), \tag{4.7}$$

where

$$v_0(t) = -\frac{f'(t)}{t}, \quad v_1(t) = \frac{1}{t}\left(\frac{f'(t)}{t}\right)'. \tag{4.8}$$

Notice that the integrals

$$\int_0^t s v_j(s)\, ds$$

do not exist for $j \geq 1$. Similarly

$$u(\varepsilon\tau;\varepsilon) \sim W(\tau;\varepsilon) = \frac{1}{\varepsilon^2}\sum_{j=0}^{\infty} \varepsilon^j w_j(\tau), \tag{4.9}$$

where

$$\begin{aligned} w_0(t) = f(0)e^{-\tau^2/2}, \quad w_1(\tau) = f'(0)\int_0^\tau e^{-(\tau^2-\sigma^2)/2}\, d\sigma, \\ w_2(\tau) = f''(0)(1 - e^{-\tau^2/2}). \end{aligned} \tag{4.10}$$

From (4.7), (4.8), (4.9) and (4.10) the composite expansion can be computed such that

$$\begin{aligned} u(t;\varepsilon) =& \frac{f(0)}{\varepsilon^2}e^{-t^2/(2\varepsilon^2)} + \frac{f'(0)}{\varepsilon}\int_0^{t/\varepsilon} e^{-(t^2/\varepsilon^2-\sigma^2)/2}\, d\sigma \\ & - f''(0)e^{-t^2/(2\varepsilon^2)} + \frac{f'(t) - f'(0)}{t} + O(\varepsilon) \end{aligned} \tag{4.11}$$

as $\varepsilon \to 0$ uniformly for $0 \leq t \leq T$.

The analysis of Section 2.2 holds for (4.5) even though $(\mathbf{H_2})$ does not. In fact it shows that the initial layer should have magnitude $O(\varepsilon^{-2})$ and width $O(\varepsilon)$. However Smith and Lange point out that the ansatz

$$u(t;\varepsilon) = y_0(t) + \frac{1}{\varepsilon^2} z_0(t/\varepsilon) + o(1)$$

and exponential decay for all the inner correction terms produces a false leading order approximate solution

$$-\frac{f(0)}{\varepsilon^2}\mathrm{e}^{-t^2/(2\varepsilon^2)} + \frac{f'(t)}{t}, \tag{4.12}$$

which is not uniformly valid for all $0 \leq t \leq T$.

We look for an asymptotic solution of the form

$$u(t;\varepsilon)=\sum_{j=0}^{\infty}\varepsilon^{2j}y_j(t)+\frac{1}{\varepsilon^2}\sum_{j=0}^{\infty}\varepsilon^{j}z_j(t/\varepsilon). \tag{4.13}$$

Since $\varepsilon^2 u(0;\varepsilon)=f(0)$, $y_j(t)$ and $z_j(\tau)$ satisfy the initial conditions

$$z_0(0)=f(0),\quad z_1(0)=0,\quad z_{2j}(0)=-y_{j-1}(0),\quad z_{2j+1}(0)=0$$

for $j\geq 1$. Substituting (4.13) into (4.5) gives

$$\begin{aligned}\varepsilon^2 y_0(t)+z_0(t/\varepsilon)+\varepsilon z_1(t/\varepsilon)+\varepsilon^2 z_2(t/\varepsilon)&=f(t)-\int_0^t s y_0(s)\,ds\\ &-\frac{1}{\varepsilon^2}\int_0^t s z_0(s/\varepsilon)\,ds-\frac{1}{\varepsilon^1}\int_0^t s z_1(s/\varepsilon)\,ds\\ &-\int_0^t s z_2(s/\varepsilon)\,ds+O(\varepsilon^2).\end{aligned} \tag{4.14}$$

This is equivalent to

$$\begin{aligned}\varepsilon^2 y_0(\varepsilon\tau)+z_0(\tau)+\varepsilon z_1(\tau)+\varepsilon^2 z_2(\tau)&=f(\varepsilon\tau)-\int_0^{\varepsilon\tau} s y_0(s)\,ds\\ -\int_0^{\tau}\sigma z_0(\sigma)\,d\sigma-\varepsilon\int_0^{\tau}\sigma z_1(\sigma)\,d\sigma&-\varepsilon^2\int_0^{\tau}\sigma z_2(\sigma)\,d\sigma+O(\varepsilon^2).\end{aligned}$$

It follows that

$$z_j(\tau)=\psi_j(\tau)-\int_0^{\tau}\sigma z_j(\sigma)\,d\sigma. \tag{4.15}$$

where

$$\psi_0(\tau)=f(0),\quad \psi_1(\tau)=f'(0)\tau,\quad \psi_2(\tau)=\frac{1}{2}(f''(0)-y_0(0))\tau^2-y_0(0).$$

Therefore,

$$z_0(\tau)=f_0(0)\mathrm{e}^{-\tau^2/2},\quad z_1(\tau)=f'(0)\int_0^{\tau}\mathrm{e}^{-(\tau^2-\sigma^2)/2}\,d\sigma,$$
$$z_2(\tau)=f''(0)(1-\mathrm{e}^{-\tau^2/2})-y_0(0).$$

In order to calculate the outer solution, we express all terms in (4.15) in terms of the outer variable t and substitute them into (4.14), giving

$$\varepsilon^2 y_0(t)=f(t)-f(0)-f'(0)t-\frac{1}{2}(f''(0)-y_0(0))t^2-\int_0^t s y_0(s)\,ds+O(\varepsilon^2).$$

By letting $\varepsilon\to 0$ an equation for $y_0(t)$ is obtained with solution

$$y_0(t)=\frac{f'(t)-f'(0)}{t}-f''(0)+y_0(0).$$

Since $\lim_{t\to 0} y_0(t) = y_0(0)$, $z_2(\tau) \to f''(0) - y_0(0)$ as $\tau \to \infty$ and we choose $y_0(0) = f''(0)$ so that $z_2(\tau) \to 0$ as $\tau \to \infty$ as required. Also by integrating by parts it can be shown that

$$z_1(\tau) = \frac{f'(0)}{\tau}\left[1 + \sum_{n=0}^{\infty} \frac{(1)(3)(5)\dots(2n-1)}{\tau^{2n}}\right] \quad \text{as } \tau \to \infty,$$

so there is only algebraic decay.

The candidate leading order solution is given by

$$u_0(t;\varepsilon) = y_0(t) + \frac{1}{\varepsilon^2} z_0(t/\varepsilon) + \frac{1}{\varepsilon} z_1(t/\varepsilon) + z_2(t/\varepsilon),$$

which agrees with (4.11). It is not hard to directly show this is a uniformly valid asymptotic solution. Also there is nontrivial contribution to the outer solution from $\lim_{\varepsilon\to 0} \varepsilon^{-1} z_1(t/\varepsilon) = f'(0)t^{-1}$ with $t > 0$ fixed, which would not be the case if $z_1(\tau)$ decayed exponentially.

Our calculations suggest that the method of additive decomposition can also be applied to problems where there is no exponential decay in the boundary layer.

References

[1] Angell, J. S., Olmstead, W. E., *Singularly perturbed Volterra integral equations*, SIAM J. Appl. Math. **47**(1) (1987), 1–14.

[2] Angell, J. S., Olmstead, W. E., *Singularly perturbed Volterra integral equations*, SIAM J. Appl. Math. **47**(6) (1987), 1150–1162.

[3] Angell, J. S., Olmstead, W. E., *Singularly perturbed integral equations with end point boundary layers*, SIAM J. Appl. Math. **49** (1989), 1567–1584.

[4] Flatto, L., Levinson, N., *Periodic solutions of singularly perturbed systems*, J. Rational Mech. Anal. **4** (1955), 943–950.

[5] Gripenberg, G. Londen, S-O., Staffans, O., *Volterra Integral and Functional Equations*, in “Encyclopedia of Mathematics and Its Application”, Cambridge University Press, Cambridge, New York, 1990.

[6] Hirsch, W. M., Smale, S., *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic Press, New York, 1974.

[7] Hoppensteadt, F. C., *Singular perturbations on the infinite interval*, Trans. Amer. Math. Soc. **123** (1966), 521-535.

[8] Lange, C. G., *On spurious solutions of singular perturbation problems*, Stud. in Appl. Math. **68** (1983), 227–257.

[9] Lange, C. G., Smith, D.R., *Singular perturbation analysis of integral equations*, Stud. Appl. Math. **79** (1988), 1–63.

[10] Lange, C. G., Smith, D.R., *Singular perturbation analysis of integral equations*, Stud. Appl. Math. **90** (1993), 1–74.

[11] Miller, R. K., *Nonlinear Volterra Integral Equations*, W. A. Benjamin, California, 1971.

[12] O’Malley, R. E., *Introduction to Singular Perturbation*, Academic Press, New York, London, 1974.

[13] O'Malley, R. E., *Singular Perturbation Methods for Ordinary Differential Equations*, Springer-Verlag, New York, 1991.

[14] Skinner, L. A., *Asymptotic solution to a class of singularly perturbed Volterra integral equations*, Methods Appl. Anal. **2** (1995), 212–222.

[15] Smith, D. R., *Singular Perturbation theory; An Introduction with Applications*, Cambridge University Press, Cambridge, New York, 1985.

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