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NONLOCAL IN TIME PROBLEMS FOR EVOLUTION EQUATIONS OF SECOND ORDER

G. A. AVALISHVILI

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Abstract. In this paper, nonlocal in time problem for abstract evolution equation of second order is studied and theorem on existence and uniqueness of its solution is proved. Some applications of this theorem for hyperbolic partial differential equations and systems are considered and it is proved, that well-posedness of the mentioned problems depends on algebraic properties of ratios between the dimensions of the spatial boundary and the times appearing in the nonlocal in time initial conditions.

1. Introduction

Nonlocal in time problems are non-classical initial boundary value problems, where instead of classical initial conditions we have a combination of the initial value of the solution and values of the solution for later times. These problems are generalizations of periodical problem, which is a particular case of problem of controllability by the initial conditions, where we seek for such initial conditions that state of the dynamical system at a certain moment of time coincides with its initial state.

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Nonlocal in time problems were formulated and studied in [4] for parabolic equations. Later in [5–7] problems of this type were investigated for some equations of mathematical physics. In the present paper we study nonlocal in time problem for abstract evolution equation of second order and generalize results obtained before for hyperbolic equations in [6, 7]. We investigate dependence of well-posedness of nonlocal in time problems on algebraic properties of expressions, which contain given moments of time and dimensions of the spatial boundary.

The outline of the paper is the following. In Section 2 the variational formulation of nonlocal in time problem for abstract evolution equation of second order is given and theorem on existence and uniqueness of its solution is proved. In Section 3 we consider some applications of the general theorem formulated in Section 2 for hyperbolic partial differential equations and systems. We show that for some nonlocal in time problems for hyperbolic equations in case of a parallelepiped as spatial domain, the existence and uniqueness of a solution depend on algebraic properties of ratios of the moments of time and lengths of the sides of the parallelepiped.

2. An abstract nonlocal in time problem

Let $V \subset H$ be separable Hilbert spaces such that the injection $V \hookrightarrow H$ is continuous and V is dense in H. Denote with $(.,.)$ scalar product in H. Assume that H is identified with its dual space by scalar product in H, then it is possible to construct continuous and dense injection of H in V' (V' is a dual space of V) ([2]). Therefore we obtain

$$V \hookrightarrow H \hookrightarrow V'$$

with continuous and dense injections. Let $\mathfrak{L}(X;Y)$ denotes the space of linear continuous operators from X to Y (X, Y are Banach spaces). $L^p(0,T;X)$ denotes the space of measurable functions $g:(0,T)\to X$ equipped with the norm

$$\|g\|_{L^p(0,T;X)} \equiv \left(\int_0^T \|g(t)\|_X^p dt\right)^{1/p} < +\infty, \quad \text{for } p < +\infty,$$
$$\|g\|_{L^p(0,T;X)} \equiv \sup_{t\in[0,T]} \text{ess}\, \|g(t)\|_X < +\infty, \qquad \text{for } p = +\infty.$$

$C^0([0,T];X)$ is the space of continuous functions of $t \in [0,T]$ with values in X. In the case of $X = \mathbb{R}$, $C^0([0,T];X)$ denotes the space $C^0([0,T])$ of continuous real-valued functions on $[0,T]$. From the definition of Bochner's integral it follows that function $g \in L^p(0,T;X)$ can be identified with distribution in $(0,T)$ with values in X ([9]), and denote with

$$g' = \frac{dg}{dt} \in \mathfrak{D}'((0,T);X) = \mathfrak{L}(\mathfrak{D}(0,T);X)$$

the derivative of g in the sense of distributions ($\mathfrak{D}(0,T)$ stands for the space of infinitely differentiable functions with compact support in (0,T)).

Furthermore, let $A \in \mathfrak{L}(V;V')$ be a linear continuous operator such that the bilinear form $a(\varphi,\psi) = \langle A\varphi,\psi\rangle_{V',V}$ ($\langle .,.\rangle_{V',V}$ denotes duality between V and V') is symmetric and coercive, i.e.

$$\begin{aligned} &a(\varphi,\psi) = a(\psi,\varphi), && \forall\, \varphi,\psi \in V, \\ &a(\varphi,\varphi) \geq \alpha\|\varphi\|_V^2, \quad \alpha = \text{const} > 0, && \forall\, \varphi \in V, \end{aligned} \tag{2.1}$$

and the set of eigenvectors of the operator A is complete in V.

The variational formulation of the nonlocal in time problem for abstract evolution equation of second order

$$\frac{d^2u}{dt^2} + Au = f, \qquad t \in (0,T),$$

is as follows: find a function $u \in C^0([0,T];V)$, $u' \in C^0([0,T];H)$, which satisfies equation

$$\frac{d}{dt}(u'(.),v) + a(u(.),v) = (f(.),v), \quad \forall\, v \in V, \tag{2.2}$$

in the sense of distributions in $(0,T)$ and the following nonlocal initial conditions

$$\begin{aligned} u(0) &= Bu + u_0, \\ u'(0) &= Cu' + u_1, \end{aligned} \tag{2.3}$$

where $u_0 \in V$, $u_1 \in H$, $f \in L^2(0,T;H)$,

$$B \in \mathfrak{L}(C^0([0,T];V);V), \quad C \in \mathfrak{L}(C^0([0,T];H);H).$$

For the formulated problem we have the following

Theorem 2.1. *Suppose that there exist linear continuous functionals $b, c : C^0([0,T]) \to \mathbb{R}$ such that*

$$B(h(t)v_n) = b(h(t))v_n, \quad C(h(t)v_n) = c(h(t))v_n, \quad \forall\, h \in C^0([0,T]),$$

for all $n \in \mathbb{N}$, where $\{v_n\}_{n\in\mathbb{N}}$ is an orthonormal (in H) system of eigenvectors of the operator A with eigenvalues $\{\lambda_n^2\}_{n\in\mathbb{N}}$. If there exists a real positive number $q > 0$ for which the inequality

$$\begin{aligned} &|(1 - b(\cos(\lambda_n t)))(1 - c(\cos(\lambda_n t))) + b(\sin(\lambda_n t))c(\sin(\lambda_n t))| \\ &> q, \end{aligned} \tag{2.4}$$

holds for all $n \in \mathbb{N}$, then nonlocal problem (2.2), (2.3) *has a unique solution. Moreover, the mapping $\{f, u_0, u_1\} \to \{u, du/dt\}$ is continuous from $L^2(0,T;H) \times V \times H$ to $C^0([0,T];V) \times C^0([0,T];H)$.*

Proof. Let us seek a solution of the formulated problem by the generalized Fourier series

$$u(t) = \sum_{n=1}^{\infty} u_n(t) v_n. \tag{2.5}$$

Inserting (2.5) into equation (2.2) and after some formal transformations, we obtain

$$\begin{aligned} u_n(t) =& \frac{1}{\lambda_n} \int_0^t f_n(\tau) \sin(\lambda_n(t-\tau)) d\tau \\ &+ A_n \cos(\lambda_n t) + B_n \sin(\lambda_n t), \end{aligned} \tag{2.6}$$

where $f(t) = \sum_{n=1}^{\infty} f_n(t) v_n$, $f_n(t) = (f(t), v_n)$. Coefficients A_n, B_n can be calculated from the conditions (2.3), i.e.

$$\begin{aligned} & A_n(1 - b(\cos(\lambda_n t))) - B_n b(\sin(\lambda_n t))) \\ & = u_{0n} + \frac{1}{\lambda_n} b \left(\int_0^t f_n(\tau) \sin(\lambda_n(t-\tau)) d\tau \right), \\ & A_n c(\sin(\lambda_n t)) + B_n(1 - c(\cos(\lambda_n t))) \\ & = \frac{u_{1n}}{\lambda_n} + \frac{1}{\lambda_n} c \left(\int_0^t f_n(\tau) \cos(\lambda_n(t-\tau)) d\tau \right), \end{aligned} \tag{2.7}$$

where $u_{0n} = (u_0, v_n)$, $u_{1n} = (u_1, v_n)$. From (2.7), taking (2.4) into account, we can uniquely determine A_n, B_n for which the following estimate is valid:

$$\begin{aligned} \max(|A_n|,\ |B_n|) \leq C_1 |u_{0n}| + C_2 \left| \frac{u_{1n}}{\lambda_n} \right| \\ + C_3 \frac{1}{|\lambda_n|} \left(\int_0^T f_n^2(\tau) d\tau \right)^{1/2} (\|b\| + \|c\|), \end{aligned} \tag{2.8}$$

where $\|b\|$, $\|c\|$ are norms of the functionals b, c respectively.

Let us consider series (2.5), where $u_n(t)$ is replaced by expression (2.6), and prove that it is a solution of the formulated nonlocal in time problem.

First, we show that series (2.5) converges uniformly with respect to t in the space V. Denote with

$$F_n(t) = \frac{1}{\lambda_n} \int_0^t f_n(\tau) \sin(\lambda_n(t-\tau)) d\tau \equiv \frac{\gamma_n(t)}{\lambda_n}$$

and prove, that the series $\sum_{n=1}^{\infty}\gamma_n^2(t)$ converges uniformly with respect to t, and its sum is less than $T\int_0^T\|f\|_H^2dt$. Indeed,

$$\gamma_n^2(t)\le\int_0^t\sin^2(\lambda_n(t-\tau))d\tau\int_0^t f_n^2(\tau)d\tau\le T\int_0^T f_n^2(\tau)d\tau,$$

i.e., the series $\sum_{n=1}^{\infty}\gamma_n^2(t)$ is dominated by the converging number series $T\sum_{n=1}^{\infty}\int_0^T f_n^2(\tau)d\tau$, the sum of which is equal to $T\|f\|^2_{L^2(0,T;H)}$.

By the coerciveness of the bilinear form $a(.,.)$, we obtain

$$\begin{aligned}0\le a&\left(u_0-\sum_{n=1}^{N}\beta_n\frac{v_n}{\lambda_n},u_0-\sum_{n=1}^{N}\beta_n\frac{v_n}{\lambda_n}\right)=a(u_0,u_0)\\&-2\sum_{n=1}^{N}\beta_n a\left(u_0,\frac{v_n}{\lambda_n}\right)+\sum_{n=1}^{N}\beta_n^2 a\left(\frac{v_n}{\lambda_n},\frac{v_n}{\lambda_n}\right)\\&=a(u_0,u_0)-\sum_{n=1}^{N}\beta_n^2,\end{aligned}\tag{2.9}$$

where $\beta_n=(1/\lambda_n)a(u_0,v_n)$. From the latter inequality it follows that $\sum_{n=1}^{\infty}\beta_n^2\le a(u_0,u_0)$.

Taking into account estimate (2.8), $\sum_{n=1}^{\infty}u_{1n}^2=\|u_1\|_H^2$ and $u_{0n}=(u_0,v_n)$ $=(1/\lambda_n^2)a(u_0,v_n)$, we obtain that the series $\sum_{n=1}^{\infty}(\lambda_n u_n(t))^2$ is dominated by the converging number series $\widetilde{C}_4\sum_{n=1}^{\infty}\left(\int_0^T f_n^2(\tau)d\tau+u_{1n}^2+\beta_n^2\right)$ and the following estimate holds

$$\sum_{n=1}^{\infty}(\lambda_n u_n(t))^2\le C_4\left(\|f\|^2_{L^2(0,T;H)}+\|u_1\|_H^2+\|u_0\|_V^2\right).$$

Let us denote $u_{pq}=\sum_{n=p}^{q}u_n(t)v_n$. Since the system $\{v_s\}_{s\in\mathbb{N}}$ is orthogonal in H, we get that it is orthogonal with respect to the form $a(.,.)$, i.e., $a(v_s,v_{s'})=\lambda_n^2(v_s,v_{s'})=0,\ s\ne s',\ s,s'\in\mathbb{N}$. Consequently, for each $\varepsilon>0$ there is a natural number $N(\varepsilon)\in\mathbb{N}$ such that

$$a(u_{pq},u_{pq})=\sum_{n=p}^{q}(\lambda_n u_n(t))^2<\varepsilon,\quad\text{for } p\ge N(\varepsilon),\ q\ge p.$$

From conditions (2.1) we obtain

$$\|u_{pq}\|_V^2\le\frac{1}{\alpha}\sum_{n=p}^{q}(\lambda_n u_n(t))^2<\frac{\varepsilon}{\alpha},\quad\text{for } p\ge N(\varepsilon),\ q\ge p,$$

that implies the uniform (with respect to t) convergence of series (2.5). Therefore $u \in C^0([0,T];V)$ and the following estimate is valid

$$\|u(t)\|_V^2 \le C_5 \left(\|f\|^2_{L^2(0,T;H)} + \|u_1\|_H^2 + \|u_0\|_V^2 \right). \tag{2.10}$$

Similarly it can be checked that the series obtained by differentiation of series (2.5) converges uniformly with respect to t in the space H and consequently $u' \in C^0([0,T];H)$. From the construction of $u(t)$ it is obvious that $u(t)$ satisfies initial conditions (2.3). Therefore, it suffices to prove that $u(t)$ satisfies equation (2.2).

Indeed, as $f \in L^2(0,T;H)$, $u_n(t)$ has second derivative for almost all $t \in (0,T)$, which is equal to $-\lambda_n^2 u_n(t) + f_n(t)$ and is square integrable in $(0,T)$. Hence, for any function $\varphi \in \mathfrak{D}(0,T)$ and $v \in V$ we have

$$-\int_0^T u_n'(\tau)(v_n, v)\varphi'(\tau)d\tau = \int_0^T \left(f_n(\tau) - \lambda_n^2 u_n(\tau)\right)(v_n, v)\varphi(\tau)d\tau$$

and

$$-\int_0^T \sum_{n=1}^N u_n'(\tau)(v_n, v)\varphi'(\tau)d\tau + \int_0^T \sum_{n=1}^N \lambda_n^2 u_n(\tau)(v_n, v)\varphi(\tau)d\tau$$
$$= \int_0^T \sum_{n=1}^N f_n(\tau)(v_n, v)\varphi(\tau)d\tau. \tag{2.11}$$

Letting N tend to ∞ in (2.11), we obtain

$$-\int_0^T (u'(\tau), v)\varphi'(\tau)d\tau + \int_0^T a(u(\tau), v)\varphi(\tau)d\tau = \int_0^T (f(\tau), v)\varphi(\tau)d\tau$$

and, therefore, $u(t)$ satisfies equation (2.2) in the sense of distributions in $(0,T)$.

So $u(t)$ is a solution of nonlocal problem (2.2), (2.3) and, according to the inequalities (2.8), (2.9), (2.10), we have

$$\|u\|^2_{C^0([0,T];V)} \le C_5 \left(\|f\|^2_{L^2(0,T;H)} + \|u_1\|_H^2 + \|u_0\|_V^2 \right),$$
$$\left\| \frac{du}{dt} \right\|^2_{C^0([0,T];H)} \le C_6 \left(\|f\|^2_{L^2(0,T;H)} + \|u_1\|_H^2 + \|u_0\|_V^2 \right),$$

that implies continuity of the mapping $\{f, u_0, u_1\} \to \{u, du/dt\}$ if problem (2.2), (2.3) has a unique solution.

Let us prove that the formulated nonlocal problem has at most one solution. Indeed, suppose that there exist two solutions $u(t)$ and $v(t)$ of the problem. Then their difference $w(t) = u(t) - v(t)$ is a solution of the homogeneous nonlocal in time problem, i.e., for $f \equiv 0$, $u_0 = 0$, $u_1 = 0$. Furthermore,

$w(t)$ is the solution of classical evolution problem for the equation (2.2) with initial conditions $w(0)$ and $w'(0)$, which has a unique solution and

$$w(t) = \sum_{n=1}^{\infty} (\widetilde{A}_n \cos(\lambda_n t) + \widetilde{B}_n \sin(\lambda_n t)) v_n.$$

Since $w(t)$ satisfies homogeneous conditions (2.3), then $(\widetilde{A}_n, \widetilde{B}_n)$ is a solution of homogeneous system (2.7). According to the conditions (2.4), the determinant of the system is different from zero and, therefore, $\widetilde{A}_n = \widetilde{B}_n = 0$, $\forall n \in \mathbb{N}$ and $w(t) \equiv 0$. Thus, $u(t) \equiv v(t)$ and solution is unique. □

Remark 2.1. From the proof of the uniqueness of solution for nonlocal in time problem (2.2), (2.3) it follows, that if in the formulation of Theorem 2.1 we have $q = 0$, then problem (2.2), (2.3) has at most one solution.

3. Nonlocal in time problems for hyperbolic equations and systems

In the present section we consider some applications of the Theorem 2.1 for the nonlocal in time problems for hyperbolic equations and systems.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Denote with $H^s(\Omega) = W^{2,s}(\Omega)$ the Sobolev space of order $s \in \mathbb{N}$, $H_0^s(\Omega) = W_0^{2,s}(\Omega)$ is the closure of the set $\mathfrak{D}(\Omega)$ of infinitely differentiable functions with compact support in Ω, in the space $H^s(\Omega)$. Assume that a_{ij}, ρ $(i, j = \overline{1, n})$ are functions defined in Ω such that

$$a_{ij}, \rho \in L^{\infty}(\Omega),\ \rho(x) \geq 0,\ a_{ij}(x) = a_{ji}(x),\ i, j = \overline{1, n},$$

$$\exists \alpha > 0,\ \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \alpha(\xi_1^2 + \ldots + \xi_n^2), \quad \forall (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n,$$

almost everywhere in Ω. Let $V = H_0^1(\Omega)$, $H = L^2(\Omega)$,

$$Au = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + \rho(x) u, \tag{3.1}$$

and as the initial conditions (2.3) take

$$\begin{aligned} u(x, 0) &= \sum_{i=1}^{k} \alpha_i^1 u(x, T_i) + \sum_{i,j=1}^{k} \int_{T_i}^{T_j} \rho_{ij}^1(\tau) u(x, \tau) d\tau + u_0(x), \\ u_t(x, 0) &= \sum_{i=1}^{k} \alpha_i^2 u_t(x, T_i) + \sum_{i,j=1}^{k} \int_{T_i}^{T_j} \rho_{ij}^2(\tau) u_t(x, \tau) d\tau + u_1(x), \end{aligned} \tag{3.2}$$

where $x \in \Omega$, ρ^1_{ij}, ρ^2_{ij} are measurable bounded real-valued functions, α^1_i, α^2_i are real constants, $T_i \in (0,T]$ $(i,j=\overline{1,k})$. Note that under these conditions the bilinear form

$$a(\varphi,\psi)=\sum_{i,j=1}^{n}\int_\Omega a_{ij}\frac{\partial\varphi}{\partial x_j}\frac{\partial\psi}{\partial x_i}dx+\int_\Omega \rho\varphi\psi dx,\quad \forall\varphi,\psi\in H^1_0(\Omega),$$

defined by the operator A, fulfills conditions (2.1) and the set of eigenfunctions of the operator A is complete in $H^1_0(\Omega)$. Hence, from Theorem 2.1 it immediately follows validity of the similar theorem in the case of nonlocal in time problem for multidimensional hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2}+Au=f,\quad \text{in } Q=\Omega\times(0,T). \tag{3.3}$$

Theorem 3.1. *If there exists a real constant $q>0$ such that*

$$\begin{aligned}&\Bigg|\prod_{r=1}^{2}\left(1-\sum_{i=1}^{k}\alpha_i^r\cos(\lambda_n T_i)-\sum_{i,j=1}^{k}\int_{T_i}^{T_j}\rho^r_{ij}(\tau)\cos(\lambda_n\tau)d\tau\right)\\&+\prod_{r=1}^{2}\left(\sum_{i=1}^{k}\alpha_i^r\sin(\lambda_n T_i)+\sum_{i,j=1}^{k}\int_{T_i}^{T_j}\rho^r_{ij}(\tau)\sin(\lambda_n\tau)d\tau\right)\Bigg|\\&>q,\end{aligned} \tag{3.4}$$

for all $n\in\mathbb{N}$, $u_0\in H^1_0(\Omega)$, $u_1\in L^2(\Omega)$, $f\in L^2(Q)$, then nonlocal in time problem (3.2), (3.3) *with homogeneous boundary conditions has a unique solution $u\in C^0([0,T];H^1_0(\Omega))$, $u'\in C^0([0,T];L^2(\Omega))$.*

Corollary 3.1. *If in nonlocal conditions* (3.2) $\rho^1_{ij}\equiv\rho^2_{ij}\equiv 0$ *and coefficients* α^1_i,α^2_i $(i,j=\overline{1,k})$ *satisfy the inequality*

$$\sum_{i=1}^{k}(|\alpha^1_i|+|\alpha^2_i|)<1, \tag{3.5}$$

then the nonlocal problem has a unique solution.

Proof. According to Theorem 3.1, it suffices to check validity of condition (3.4). From (3.5) we obtain

$$\left(1-\sum_{i=1}^{k}\alpha_i^1\cos(\lambda_n T_i)\right)\left(1-\sum_{i=1}^{k}\alpha_i^2\cos(\lambda_n T_i)\right) + \left(\sum_{i=1}^{k}\alpha_i^1\sin(\lambda_n T_i)\right)\left(\sum_{i=1}^{k}\alpha_i^2\sin(\lambda_n T_i)\right) \geq 1-\sum_{i=1}^{k}|\alpha_i^1|-\sum_{i=1}^{k}|\alpha_i^2|>0$$

and, instead of q, we can take any positive real number less than $1-\sum_{i=1}^{k}(|\alpha_i^1|+|\alpha_i^2|)$. □

Now, we consider some particular cases of the nonlocal in time problem for hyperbolic equation and show an essential difference between classical and nonlocal problems.

Let $\Omega=(0,l)$, $A\equiv-\partial^2/\partial x^2$, $f\equiv 0$ and conditions (3.2) are of the following form:

$$\begin{aligned} u(x,0) &= \sum_{i=1}^{k}\alpha_i u(x,T_i)+u_0(x), \\ u_t(x,0) &= \sum_{i=1}^{k}\alpha_i u_t(x,T_i)+u_1(x), \end{aligned} \qquad x\in\Omega, \tag{3.6}$$

where $\alpha_i\neq 0$ is a real constant, $T_i\in(0,T]$ $(i=\overline{1,k})$.

Therefore we obtain nonlocal in time problem for the string oscillation equation, which, according to Theorem 3.1, has a unique solution if $\sum_{i=1}^{k}|\alpha_i| < 1$. In the case of $\sum_{i=1}^{k}|\alpha_i|=1$, from inequality (3.4) and Remark 2.1, we obtain that if among the points $\{T_i\}_{i=1}^{k}$ at least one is such that the ratio T_i/l is irrational, then the nonlocal problem has at most one solution. It must be pointed out that if $k=1$, then condition of irrationality of the ratio T_1/l is a necessary and sufficient condition for the uniqueness of solution of the nonlocal problem. Moreover, for the special case of T_1/l, we have the theorem of the existence of a solution.

Theorem 3.2. *Suppose that T_1/l is an irrational algebraic number of degree $r>1$. If $u_0\in H^r(\Omega)$, $u_1\in H^{r-1}(\Omega)$, $u_0, Au_0,\dots, A^{[(r-1)/2]}u_0, u_1,\dots, A^{[r/2]-1}u_1\in H_0^1(\Omega)$, then nonlocal in time problem* (3.3), (3.6) *with homogeneous boundary conditions has a unique solution for any $\alpha_1\in\mathbb{R}$ ($[y]$ denotes the integer part of the real number y).*

Proof. According to Liouville's Theorem ([3]), since T_1/l is an algebraic irrational number of degree $r > 1$, there exists $c > 0$ such that

$$\left|\frac{T_1}{l} - \frac{p}{n}\right| \geq \frac{c}{n^r}, \qquad \forall\, n, p \in \mathbb{N},$$

and, hence,

$$\left|n\frac{T_1}{l} - p\right| \geq \frac{c}{n^{r-1}}, \qquad \forall\, n, p \in \mathbb{N}.$$

From the proof of Theorem 2.1

$$A_n = \frac{u_{0n}\left(1 - \alpha_1 \cos\left(\frac{\pi n}{l}T_1\right)\right) + \frac{u_{1n}l}{\pi n}\alpha_1 \sin\left(\frac{\pi n}{l}T_1\right)}{(1+\alpha_1^2) - 2\alpha_1 \cos\left(\frac{\pi n}{l}T_1\right)},$$

$$B_n = \frac{\frac{u_{1n}l}{\pi n}\left(1 - \alpha_1 \cos\left(\frac{\pi n}{l}T_1\right)\right) - u_{0n}\alpha_1 \sin\left(\frac{\pi n}{l}T_1\right)}{(1+\alpha_1^2) - 2\alpha_1 \cos\left(\frac{\pi n}{l}T_1\right)},$$

where

$$u_{0n} = \sqrt{\frac{2}{l}}\int_0^l u_0(x)\sin\left(\frac{\pi n}{l}x\right)dx,\ u_{1n} = \sqrt{\frac{2}{l}}\int_0^l u_1(x)\sin\left(\frac{\pi n}{l}x\right)dx.$$

Hence, if $|\alpha_1| \neq 1$, then the nonlocal problem has a unique solution.

Now consider the case when $|\alpha_1| = 1$. We have:
for $\alpha_1 = 1$,

$$A_n = \frac{u_{0n}}{2} + \frac{u_{1n}l}{2\pi n}\cot\left(\frac{\pi n T_1}{2l}\right),\ B_n = \frac{u_{1n}l}{2\pi n} - \frac{u_{0n}}{2}\cot\left(\frac{\pi n T_1}{2l}\right);$$

for $\alpha_1 = -1$,

$$A_n = \frac{u_{0n}}{2} - \frac{u_{1n}l}{2\pi n}\tan\left(\frac{\pi n T_1}{2l}\right),\ B_n = \frac{u_{1n}l}{2\pi n} + \frac{u_{0n}}{2}\tan\left(\frac{\pi n T_1}{2l}\right).$$

Denote with $m(x) = \min\{(2/\pi)x, 2-(2/\pi)x\}$, $0 \leq x \leq \pi$. It is easily checked that for all $n \in \mathbb{N}$,

$$\left|\sin\left(\frac{\pi n T_1}{2l}\right)\right| = \left|\sin\left(\frac{\pi n T_1}{2l} - \pi\left[\frac{nT_1}{2l}\right]\right)\right| > m\left(\frac{\pi n T_1}{2l} - \pi\left[\frac{nT_1}{2l}\right]\right),$$

$$\left|\cos\left(\frac{\pi n T_1}{2l}\right)\right| = \left|\cos\left(\frac{\pi n T_1}{2l} - \pi\left[\frac{nT_1}{2l}\right]\right)\right| > 1 + 2\left[\frac{nT_1}{2l}\right] - \frac{nT_1}{l},$$

whence

$$\left|\cot\left(\frac{\pi n T_1}{2l}\right)\right| \leq \widetilde{c}n^{r-1},\ \left|\tan\left(\frac{\pi n T_1}{2l}\right)\right| \leq \widetilde{c}n^{r-1}.$$

Therefore,

$$\max\{|A_n|,\ |B_n|\} \leq C_1\left(|u_{0n}| + \frac{|u_{1n}|}{n}\right)n^{r-1}.$$

Since $A(\sin(\lambda_n x)) = \lambda_n^2 \sin(\lambda_n x)$, $\lambda_n = \pi n/l$, $n \in \mathbb{N}$ and u_0, $Au_0, \dots$, $A^{[(r-1)/2]}u_0 \in H_0^1(\Omega)$, we obtain:
if r is even ($r = 2r_0$), then

$$u_{0n} = \sqrt{\frac{2}{l}}\frac{1}{\lambda_n^2}\int_0^l u_0(x)A(\sin(\lambda_n x))dx = \sqrt{\frac{2}{l}}\frac{1}{\lambda_n^2}\int_0^l (Au_0)(x)\sin(\lambda_n x)dx$$
$$= \dots = \sqrt{\frac{2}{l}}\frac{1}{\lambda_n^{2r_0}}\int_0^l (A^{r_0}u_0)(x)\sin(\lambda_n x)dx = \frac{\overline{u}_{0n}}{\lambda_n^{2r_0}},$$

if r is odd ($r = 2r_0 + 1$), then

$$u_{0n} = \sqrt{\frac{2}{l}}\frac{1}{\lambda_n^{2r_0}}\int_0^l (A^{r_0}u_0)(x)\sin(\lambda_n x)dx$$
$$= \sqrt{\frac{2}{l}}\frac{1}{\lambda_n^{2r_0+1}}\int_0^l \frac{d}{dx}\left((A^{r_0}u_0)(x)\right)\frac{d}{dx}\left(\frac{1}{\lambda_n}\sin(\lambda_n x)\right)dx = \frac{\widetilde{u}_{0n}}{\lambda_n^{2r_0+1}},$$

where $\sum_{n=1}^{\infty}(\overline{u}_{0n})^2 \le c_2\|u_0\|^2_{H^r(\Omega)}$,

$$0 \le \int_0^l \left(\frac{d}{dx}\left((A^{r_0}u_0)(x) - \sum_{n=1}^{N}\frac{\widetilde{u}_{0n}}{\lambda_n}\sqrt{\frac{2}{l}}\sin(\lambda_n x)\right)\right)^2 dx$$
$$= \int_0^l \left(\frac{d}{dx}\left((A^{r_0}u_0)(x)\right)\right)^2 dx - \sum_{n=1}^{N}(\widetilde{u}_{0n})^2,$$

whence $\sum_{n=1}^{\infty}(\widetilde{u}_{0n})^2 \le c_3\|u_0\|^2_{H^r(\Omega)}$. Thus,

$$u_{0n} = \frac{\widehat{u}_{0n}l^r}{\pi^r n^r}, \quad \sum_{n=1}^{\infty}(\widehat{u}_{0n})^2 \le \widehat{c}\|u_0\|^2_{H^r(\Omega)},$$

and similarly

$$u_{1n} = \frac{\widehat{u}_{1n}l^{r-1}}{\pi^{r-1}n^{r-1}}, \quad \sum_{n=1}^{\infty}(\widehat{u}_{1n})^2 \le \widehat{c}\|u_1\|^2_{H^{r-1}(\Omega)}.$$

From the latter estimates for u_{0n}, u_{1n} we deduce, that the series

$$u(x,t) = \sum_{n=1}^{\infty}\sqrt{\frac{2}{l}}\left(A_n\cos\left(\frac{\pi n}{l}t\right) + B_n\sin\left(\frac{\pi n}{l}t\right)\right)\sin\left(\frac{\pi n}{l}x\right) \qquad (3.7)$$

converges uniformly (with respect to t) in the space $H^1(\Omega)$ and the series obtained by differentiation of (3.7) converges uniformly in $L^2(\Omega)$. Thus $u \in C^0([0,T]; H_0^1(\Omega))$ and $u' \in C^0([0,T]; L^2(\Omega))$. □

Note, that similar results in the case of Dirichlet problem for hyperbolic equations are obtained in [1, 8]. More precisely, a solution of the Dirichlet boundary value problem for the string oscillation equation is uniquely determined if and only if the ratio $\xi = T/l$ of the sides of the rectangle $[0,l]\times[0,T]$ is irrational; the solution exists for all boundary values, which are differentiable sufficiently many times, if ξ cannot "too rapidly" be approximated by rationals.

Similarly we may consider two-dimensional and multidimensional nonlocal in time problem (3.3), (3.6) with homogeneous boundary conditions. Particularly, let $\Omega = (0,l_1)\times ... \times(0,l_s)$, $A \equiv -\sum_{i=1}^{s}\partial^2/\partial x_i^2$, $f\equiv 0$. Before formulation of the analogue to Theorem 3.2 let us introduce the following definition.

Definition 3.1. Let $D_s = \{d_{n_1n_2...n_s};\ n_1,n_2,...,n_s \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}\}$ be an infinite set of non-negative real numbers. We say that the rate of approximation of the real number d on the set D_s is less than r, $r\in\mathbb{N}$, if there exists a constant $c>0$ such that for all $n_1,n_2,...,n_s\in\mathbb{N}_0$,

$$\left|d - \frac{p}{d_{n_1n_2...n_s}}\right| \geq \frac{c}{d^r_{n_1n_2...n_s}}, \quad \forall p\in\mathbb{N},\ d_{n_1n_2...n_s}\in D_s.$$

In the case of multidimensional nonlocal in time problem, if we have $\sum_{i=1}^{k}|\alpha_i|<1$, then problem (3.3), (3.6) has a unique solution. In the case of $\sum_{i=1}^{k}|\alpha_i|=1$, the nonlocal in time problem has at most one solution if there exists T_i for which the equation

$$\left(\frac{n_1}{l_1}\right)^2 + \left(\frac{n_2}{l_2}\right)^2 + ... + \left(\frac{n_s}{l_s}\right)^2 = \left(\frac{p}{T_i}\right)^2 \tag{3.8}$$

is unsolvable in integers. Let us consider the case of $l_1 = l_2 = ... = l_s = l$ and $k=1$. According to the equation (3.8), the uniqueness of a solution of the nonlocal in time problem depends on algebraic properties of T_1/l. More precisely, the following theorem is valid.

Theorem 3.3. *If T_1/l is such, that $n_1,...,n_s,pl/T_1$ are not the generalized Pythagorean numbers, i.e., don't satisfy equation* (3.8) *for any integers $n_1,...,n_s,p$ and the rate of approximation of T_1/l on the set $D_s = \{d_{n_1...n_s} = \sqrt{n_1^2+\ldots+n_s^2};\ n_1,\ldots,n_s\in\mathbb{N}_0\}$ is less than $r>1$, $u_0\in H^r(\Omega)$, $u_1\in H^{r-1}(\Omega)$, $A^{k_\gamma}u_\gamma \in H_0^1(\Omega)$, $0\leq k_\gamma \leq [(r-1+\gamma)/2]-\gamma$, $\gamma=0,1$, then nonlocal problem* (3.3), (3.6) *with homogeneous boundary conditions has a unique solution for any $\alpha_1\in\mathbb{R}$.*

Proof. We only sketch the proof, because it is similar to the proof of Theorem 3.2. Note, that if $|\alpha_1|\neq 1$, then the formulated theorem follows from

Theorem 3.1. Let us consider the case of $|\alpha_1| = 1$. As in the proof of Theorem 2.1 solution of the nonlocal problem we seek by Fourier series

$$u(x_1, ..., x_s, t) = \sum_{n_1=1}^{\infty} \dots \sum_{n_s=1}^{\infty} (A_{n_1\dots n_s} \cos(\lambda_{n_1\dots n_s} t) + B_{n_1\dots n_s} \sin(\lambda_{n_1\dots n_s} t))\, v_{n_1\dots n_s}(x_1, \dots, x_s), \tag{3.9}$$

where

$$v_{n_1\dots n_s}(x_1, \dots, x_s) = \left(\frac{2}{l}\right)^{s/2} \sin\left(\frac{\pi n_1}{l} x_1\right) \dots \sin\left(\frac{\pi n_s}{l} x_s\right), \quad \lambda_{n_1\dots n_s} = \frac{\pi\sqrt{n_1^2 + \dots + n_s^2}}{l}, \quad n_1, \dots, n_s \in \mathbb{N}.$$

Coefficients $A_{n_1\dots n_s}, B_{n_1\dots n_s}$ determined from initial conditions (3.6), are of the following form:

$$A_{n_1\dots n_s} = \frac{u_{0n_1\dots n_s}}{2} + \frac{u_{1n_1\dots n_s}}{2\lambda_{n_1\dots n_s}} \left(\frac{\alpha_1 + 1}{2} \cot + \frac{\alpha_1 - 1}{2} \tan\right) \left(\frac{\lambda_{n_1\dots n_s} T_1}{2}\right),$$

$$B_{n_1\dots n_s} = \frac{u_{1n_1\dots n_s}}{2\lambda_{n_1\dots n_s}} - \frac{u_{0n_1\dots n_s}}{2} \left(\frac{\alpha_1 + 1}{2} \cot + \frac{\alpha_1 - 1}{2} \tan\right) \left(\frac{\lambda_{n_1\dots n_s} T_1}{2}\right).$$

Taking into account property of T_1/l, we obtain

$$\max\{|A_{n_1\dots n_s}|, |B_{n_1\dots n_s}|\} \le C \left(|u_{0n_1\dots n_s}| + \frac{u_{1n_1\dots n_s}}{\sqrt{n_1^2 + \dots + n_s^2}}\right) \sqrt{n_1^2 + \dots + n_s^2}^{\,r-1},$$

and from the latter estimate, as in the proof of Theorem 3.2, we deduce, that the function $u(x_1, \dots, x_s, t)$ defined by series (3.9) is a solution of the nonlocal in time problem (3.3), (3.6). □

It must be pointed out that Theorem 2.1 allows to investigate nonlocal in time problem for hyperbolic system. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with regular boundary, $V = \mathfrak{H}_0^m(\Omega) = [H_0^m(\Omega)]^N$, $H = \mathfrak{L}^2(\Omega) = [L^2(\Omega)]^N$ and let A be an elliptic operator of order $2m$:

$$A = \sum_{k=0}^{m} (-1)^k \sum \frac{\partial^k}{\partial x_{i_1} \dots \partial x_{i_k}} \left(A_{j_1\dots j_k}^{i_1\dots i_k}(x) \frac{\partial^k}{\partial x_{j_1} \dots \partial x_{j_k}}\right), \tag{3.10}$$

where in the inner sum each index $i_1, \dots, i_k, j_1, \dots, j_k$ independently range over the set $\{1, \dots, n\}$. u is N-component vector-function, $A_{j_1\dots j_k}^{i_1\dots i_k}(x)$ — square matrix of order N, which does not change for any transposition of upper or lower indices and turns into transpose of the matrix if all upper indices are transposed with all lower indices. Furthermore, assume that the elements of the matrices belong to $L^\infty(\Omega)$ and for almost all $x \in \Omega$,

$$\Big(\Big(\sum A^{i_1\dots i_m}_{j_1\dots j_m}(x)t_{j_1\dots j_m},\ t_{i_1\dots i_m}\Big)\Big) \geq \alpha \sum \|t_{i_1\dots i_m}\|^2,\ \alpha>0,$$
$$\Big(\Big(\sum A^{i_1\dots i_k}_{j_1\dots j_k}(x)t_{j_1\dots j_k},\ t_{i_1\dots i_k}\Big)\Big) \geq 0,\ k=0,1,\dots,m-1. \tag{3.11}$$

In (3.11) $t_{i_1\dots i_k}$ is N-component vector, which does not change for any transposition of indices $i_1,\dots,i_k$ $(k=\overline{1,m})$, $((.,.))$, $\|.\|$ denote the scalar product and norm in the N-dimensional Euclidean space, respectively.

Let us consider now nonlocal in time problem for hyperbolic system

$$\frac{\partial^2 u}{\partial t^2} + Au = f, \quad \text{in } Q = \Omega\times(0,T), \tag{3.12}$$

with nonlocal initial conditions of the form (3.2) and homogeneous boundary conditions

$$u = \frac{\partial u}{\partial \nu} = \dots = \frac{\partial^{m-1} u}{\partial \nu^{m-1}} = 0, \quad \text{on } \Gamma = \partial\Omega\times[0,T],$$

where ν is the unit outward normal to $\partial\Omega$. In this case bilinear form $a(.,.)$ defined by the operator A is of the following form

$$a(u,v) = \int_\Omega \sum_{k=0}^{m} \sum \Big(\Big(A^{i_1\dots i_k}_{j_1\dots j_k}(x)\frac{\partial^k u}{\partial x_{j_1}\dots\partial x_{j_k}}, \frac{\partial^k v}{\partial x_{i_1}\dots\partial x_{i_k}}\Big)\Big)\,dx.$$

Taking into account conditions (3.11) it is not difficult to check that bilinear form $a(.,.)$ is symmetric and coercive on $V\times V$. Moreover, it is well known, that $\mathfrak{H}^m_0(\Omega)$ is dense in $\mathfrak{L}^2(\Omega)$ and the set of eigenfunctions of the operator A is complete in V.

So, applying Theorem 2.1, we obtain the following statement.

Theorem 3.4. *If $u_0 \in \mathfrak{H}^m_0(\Omega)$, $u_1 \in \mathfrak{L}^2(\Omega)$, $f \in \mathfrak{L}^2(Q)$ and condition* (3.4) *is fulfilled, then the formulated nonlocal in time problem for the hyperbolic system has a unique solution $u \in C^0([0,T];\mathfrak{H}^m_0(\Omega))$, $u' \in C^0([0,T];\mathfrak{L}^2(\Omega))$.*

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Gia A. Avalishvili
Institute of Applied Mathematics
Tbilisi State University
University str. 2
380043 Tbilisi
Georgia
email: gavalish@viam.hepi.edu.ge