NONEXISTENCE OF GLOBAL SOLUTIONS TO A CLASS OF NONLINEAR DIFFERENTIAL INEQUALITIES AND APPLICATION TO HYPERBOLIC–ELLIPTIC PROBLEMS

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Abstract. We consider the problem

$$u_{tt} + \delta u_t + \varepsilon a \Delta u + \varphi(\int_\Omega |\nabla u|^2 dx)\Delta u \geq f(x,t),$$

posed in $\Omega \times (0, +\infty)$. Here $\Omega \subset \mathbb{R}^N$ is a an open smooth bounded domain and φ is like $\varphi(s) = bs^\gamma$, $\gamma > 0$, $a > 0$ and $\varepsilon = \pm 1$. We prove, in certain conditions on f and φ that there is absence of global solutions. The method of proof relies on a simple analysis of the ordinary inequality of the type

$$w'' + \delta w' \geq \alpha w + \beta w^p.$$

It is also shown that a global positive solution, when it exists, must decay at least exponentially.

1. Introduction

Our initial interest in this paper is the nonexistence of global positive solution to

$$
\begin{cases}
u_{tt} + \varepsilon a \Delta u + \varphi \left(\int_\Omega |\nabla u|^2 dx \right) \Delta u + \delta u_t \geq f(x,t), \\
\qquad\qquad\qquad\qquad\qquad \text{on } \Omega \times (0,+\infty), \\
u = 0, \qquad\qquad\qquad\quad \text{in } \partial\Omega \times (0,+\infty),
\end{cases}
\tag{1.1}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $a > 0$, $\varepsilon = \pm 1$, f is given function and $\varphi(s)$ is locally Lipschitz function for $s \geq 0$ satisfying

$$
\varphi(s) \geq bs^\gamma,
\tag{1.2}
$$

where b and γ are positive and δ is a real parameter.

When $\varepsilon = 1$ Problem (1.1) is of elliptic type inequality and it is posed in the infinite cylindrical domain $\Omega \times \mathbb{R}_+$. If $\varepsilon = -1$ we have, under certain condition, a damped quasilinear hyperbolic inequality.

There is a vast literature for the quasilinear hyperbolic equations with $-\varphi$ instead of φ. The following model

$$
u_{tt} - \varphi \left(\int_{(0,L)} |u_x|^2 dx \right) u_{xx} = 0,
$$

describes, in one dimension, the non-linear vibrations of elastic string. The natural generalization is given by

$$
u_{tt} - \varphi \left(\int_\Omega |\nabla u|^2 dx \right) \Delta u = 0,
$$

and was studied by Pohozaev [17] in the case where φ is a real C^1 function defined for nonnegative real satisfying $\varphi \geq a_0 > 0$. The author obtained existence and global solutions for analytic initial data. Later Lions [13] formulated the Pohozaev result in abstract context. The work of Ono [16] deals with

$$
u_{tt} + \varphi(\|A^{1/2}u\|_2)Au + \delta u_t = |u|^p u,
\tag{1.3}
$$

where $A = -\Delta$, $\varphi(s) = a + bs^\gamma$, $a \geq 0$, $b \geq 0$, $a + b > 0$ and $\gamma > 0$. He proved that for $p > 2\gamma$ the local solution is not global if

$$
E(0) < 0 \quad (\text{resp. } \mathrm{E}(0) \geq 0 \text{ and } \int_\Omega \mathrm{u}_0 \mathrm{u}_1 > 0),
$$

where $E(0)$ is the initial energy associated to (1.3).

In [6] Callegari and Manfrin studied the Klein–Gordon equation with the nonlocal nonlinearities

$$u_{tt} - a\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u + \varphi\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)u = 0, \quad x \in \mathbb{R}^N, t > 0,$$

where $a, \varphi \geq \eta > 0$ are $C^1(-\theta, \theta)$ functions. The authors proved that if the initial data is sufficiently small, the problem has a unique global regular solution.

We here intend to discuss conditions which assure that the perturbed elliptic-hyperbolic problem (1.1) has no global solution. In particular we obtain a sufficient condition, on the initial data, such that the mean $w(t) = \int_\Omega u(x,t)\Phi_1(x)dx$ is not global. Here Φ_1 is a positive eigenfunction of $-\Delta$ in $H_0^1(\Omega)$. The key of the proof is to analysis the ordinary differential inequality of the type

$$w'' + \delta w' \geq \beta w + \alpha w^p,$$

where $\delta, \beta \in \mathbb{R}$, $p > 1$ and $\alpha > 0$. First we study in Section 2 the global nonexistence of positive solutions to the differential inequalities of the type

$$w'' + g(w') \geq h(w).$$

As a consequence we prove, in Section 3, the non-global character of solutions to (1.1). In the same manner we investigate the elliptic inequality. We prove that if u is a global nonnegative solution such that $\delta > 0$ and $\delta + \lambda_1 a > 1$ then the limit

$$\lim_{t \to +\infty} e^{\sqrt{\delta + \lambda_1 a}t} \int_\omega u(x,t)\Phi_1 dx,$$

is finite. Here λ_1 is the first eigenvalue associated to Φ_1. This result is obtained without assuming that u tends to 0 as $t \to \infty$.

2. Nonexistence results for differential inequalities

In this section we collect auxiliary results which will be used later. We investigate here a more general problem of the form

$$w'' + g(w') \geq h(w), \tag{2.1}$$

with regard to $w(0)$ and $w'(0)$. With the help of methods of ordinary differential inequality, we prove in a certain circumstance that there is absence of global positive solutions to (2.1). The functions g and h are assumed to

be continuous and satisfy, throughout the paper,

$$h(r) > 0, \quad \text{for all } r > r_0, \tag{2.2}$$

$$\liminf_{r \to +\infty} \frac{h(r)}{r^p} > 0, \quad p > 1, \tag{2.3}$$

$$g(r) \le Kr, \quad \text{for all } r \in \mathbb{R}, \tag{2.4}$$

where $r_0 > 0$, $p > 1$ and K is a real constant.

In [15], [18] the authors established the nonexistence of global positive solution to a class of differential inequalities of the type $w'' + A|w'|^q + B|w'| \ge Cw^p$, $p > q \ge 1$. Here we study a different model. First we have the following

Proposition 2.1. *There is no global solution, $w \in C^2$, to* (2.1) *such that*

$$w(0) > r_0, \quad w'(0) \ge 0. \tag{2.5}$$

Proof. Suppose that a global smooth solution w to (2.1), (2.5) exists. By continuity we have $h(w) > 0$ on $(0, t_0)$, t_0 small. It is easily seen that $w'(t) > 0$ and then $w(t) > r_0$ on $(0, t_0)$. Assume that $w'(t_1) = 0$ for $t_1 > 0$, then inequality (2.1) cannot be satisfied at $t = t_1$. This shows in particular that $w'(t) > 0$ on $(0, +\infty)$.

Let us prove now that $w(t)$ goes to infinity with t. To this end we suppose the contrary; that is $w(t) \to l > r_0$. Hence there exists (t_n) converging to infinity such that $w'(t_n)$ tends to 0. Now multiplying (2.1) by w and integrating the result over $(0, t_n)$ we deduce that

$$w(t_n)w'(t_n) - w(0)w'(0) + \frac{K}{2}\left(w^2(t_n) - w^2(0)\right) \ge \int_0^{t_n} h(w)w\,ds,$$

thanks to (2.4). Therefore $h(w)w \in L^1(0, +\infty)$, and then $h(l)l = 0$. A contradiction, and then $w(t)$ tends to infinity with t. Next the auxiliary function

$$v := \frac{1}{2}w^2,$$

satisfies the inequality

$$v'' \ge wh(w) - Kww' + (w')^2.$$

If $K \le 0$ we have

$$v'' \ge wh(w),$$

and if $K > 0$ we deduce from the Young inequality

$$v'' \ge wh(w) - \frac{K^2}{4}w^2.$$

Now from (2.3) and the fact that w goes to infinity, we deduce

$$v'' \geq \overline{K}w^{p+1} = \overline{K}2^{(p+1)/2}v^{(p+1)/2},$$

for t large. A simple analysis of this inequality asserts that v is not global since v and v' are nonnegative functions and $(p+1)/2 > 1$. This contradicts our assumption and finishes the proof. □

Remark 2.1. Note that (as inequality (2.1) is autonomous) if there exists $t_0 > 0$ such that $w(t_0) > r_0$ and $w'(t_0) \geq 0$, the conclusion of the preceding proposition remains true.

Corollary 2.1. *Let* $p > 1$, $p > q > 0$, $\delta, \beta \in \mathbb{R}$ *and* α *is positive real. There is no global solution* w *to*

$$w'' + \delta w' \geq \alpha w^p - \beta w^q, \tag{2.6}$$

such that

$$w(0) > \left(\frac{\beta_+}{\alpha}\right)^{1/(p-q)}, \quad w'(0) \geq 0, \tag{2.7}$$

where $\beta_+ = \max\{\beta, 0\}$.

Remark 2.2. It is possible to obtain a global solution to (2.6) such that $\beta > 0$ and

$$0 < w(0) < \left(\frac{\beta}{\alpha}\right)^{1/(p-q)}.$$

For example the function

$$w(t) = \frac{r}{(1 + e^{st})^{2/(p-1)}},$$

where

$$r = \left(\frac{\beta}{\alpha}\right)^{1/(p-q)}, \quad s^2 = \frac{\alpha(p-1)^2}{2(p+1)}\left(\frac{\beta}{\alpha}\right)^{(p-1)/(p-q)}, \quad q = \frac{p+1}{2},$$

satisfies

$$w'' + \frac{2}{p-1}sw' = \alpha w^p - \beta w^{(p+1)/2},$$

on $\mathbb{R}_+$. The following result shows that solutions to (2.1) may blow-up at a finite point in the case where $w'(0) < 0$.

Put

$$H(r) = \int_0^r h(s)ds.$$

Proposition 2.2. *Assume, in addition, that $rg(r) \geq 0$ for any $r \leq 0$. Then inequality (2.1) has no global solution w such that*

$$\frac{w_1^2}{2} - H(w_0) \leq -H(r_0), \tag{2.8}$$

where

$$w_0 = w(0) > r_0, \quad w_1 = w'(0) < 0.$$

Proof. Let w be a global solution satisfying (2.8). Since $w_1 < 0$ the function $w(t)$ is decreasing for small t. Assume that there exists $0 < t_0 < \infty$, such that

$$w(t_0) > r_0, \quad w'(t_0) = 0.$$

Using the previous result we deduce that w is not global. Now assume that $w > r_0$ on $(0, +\infty)$, and then $w' < 0$. Integrating (2.1) over $(0, +\infty)$ yields

$$\int_0^\infty h(w(s))ds < +\infty.$$

Therefore $h(w(\infty)) = 0$, and then $w(\infty) \leq r_0$. Consequently there exists $0 < t_0 \leq +\infty$ such that

$$w(t) > r_0, \quad w'(t) < 0,$$

on $(0, t_0)$ and

$$w(t_0) = r_0.$$

Next by (2.1) the function

$$E = \frac{1}{2}\left(w'\right)^2 - H(w)$$

is strictly decreasing. Hence $E(t) > E(t_0)$, for any $t < t_0$. This implies in particular that

$$\frac{w_1^2}{2} - H(w_0) > E(t_0) > -H(r_0),$$

which contradicts (2.8). This finishes the proof. $\square$

Corollary 2.2. *Let $p > 1$, $p > q > 0$, $\delta \geq 0$, $1 \geq s > 0$ and α, β are positive reals. There is no global solution w to*

$$\begin{cases} w'' + \delta|w'|^{s-1}w' \geq \alpha w^p - \beta w^q, \\ w(0) = w_0, \quad w'(0) = w_1, \end{cases} \tag{2.9}$$

such that

$$\frac{w_1^2}{2} - \frac{\alpha}{p+1}w_0^{p+1} + \frac{\beta}{q+1}w_0^{q+1}$$

$$\leq \frac{p-q}{(p+1)(q+1)}\alpha^{-(q+1)/(p-q)}\beta^{(p+1)/(p-q)}, \tag{2.10}$$

where

$$w_0 > r_0 = \left(\frac{\beta}{\alpha}\right)^{1/(p-q)}, \quad w_1 < 0. \tag{2.11}$$

We finish this section by simple results concerning the problem

$$w'' + \delta w' = \alpha e^{\tau t}|w|^{p-1}w + \beta w, \tag{2.12}$$

where $\alpha > 0$, $\beta \geq 0$ and $\tau \geq 0$.

When $\delta = 1$ a change of variable $y = e^t$ transforms this equation into

$$\frac{d}{dy}\left(y^2\frac{dz}{dy}\right) = \alpha y^\tau |z|^{p-1}z + \beta z, \quad y \geq 1,$$

which is of the type of the Emden-Fowler equations [4]. First we consider the case $\tau = 0$.

Theorem 2.1. *Let $\tau = 0$, $p > 1$, $\delta \in \mathbb{R}$ and α, β be positive reals. For any $\Gamma \neq 0$ there exists a unique global non trivial solution, w_Γ, to (2.12) such that $w_\Gamma(0) = \Gamma$. The function w_Γ does not change sign and satisfies*

$$\lim_{t\to+\infty}\frac{w'_\Gamma(t)}{w_\Gamma(t)} = -\frac{\delta + \sqrt{\delta^2 + 4\beta}}{2}. \tag{2.13}$$

Moreover, if in addition $\delta = 0$, we have

$$w_\Gamma(t) = \Gamma\left(\frac{2\sqrt{\beta(p+1)}}{Ae^{(p-1)\sqrt{\beta}t} + B}\right)^{2/(p-1)} e^{\sqrt{\beta}t}, \tag{2.14}$$

where

$$A := \sqrt{\beta(p+1) + 2\alpha|\Gamma|^{p-1}} + \sqrt{\beta(p+1)},$$
$$B := -\sqrt{\beta(p+1) + 2\alpha|\Gamma|^{p-1}} + \sqrt{\beta(p+1)}.$$

Proof. Since the function $-w$ is also a solution of (2.12) we restrict our considerations to the case $w(0) = \Gamma > 0$. Let Γ_0 be any real number and consider

$$\begin{cases} w'' + \delta w' = \alpha|w|^{p-1}w + \beta w, \\ w(0) = \Gamma, \quad w'(0) = \Gamma_0. \end{cases} \tag{2.15}$$

This problem has a unique local solution, $w_{\Gamma,\Gamma_0} \in C^2([0, T_{\Gamma,\Gamma_0}))$. We shall investigate whether w_{Γ,Γ_0} is global and its properties. It follows from above that if $\Gamma_0 \geq 0$ then w_{Γ,Γ_0} is not global. Therefore we assume that $\Gamma_0 < 0$. Let us note that if there exists $t_0 \in [0, T_{\Gamma,\Gamma_0})$ such that $w_{\Gamma,\Gamma_0}(t_0) > 0$ and $w'_{\Gamma,\Gamma_0}(t_0) \geq 0$ then w_{Γ,Γ_0} is not global. Hence if w_{Γ,Γ_0} is global necessarily

$$w_{\Gamma,\Gamma_0}(t) > 0, \qquad w'_{\Gamma,\Gamma_0}(t) < 0,$$

for all $t \geq 0$. Using this we can deduce that w_{Γ,Γ_0} and w'_{Γ,Γ_0} tend to 0 as t approaches infinity. Next a linearized equation of (2.15), at the origin, is

$$w'' + \delta w' = \beta w,$$

and the characteristic equation has the roots $(-\delta + \sqrt{\delta^2 + 4\beta})/2$ and $-(\delta + \sqrt{\delta^2 + 4\beta})/2$. From the above consideration we deduce that $(2.15)_1$ has exactly one global non trivial solution w_Γ such that $w_\Gamma(0) = \Gamma$. The function w_Γ is positive and satisfies (2.13). Finally, for $\delta = 0$, equation $(2.15)_1$ reads

$$w'' = \alpha |w|^{p-1} w + \beta w.$$

Using this and the fact that w and w' tend to 0 as $t \to \infty$, we find

$$w'^2 = \frac{2\alpha}{p+1} |w|^{p+1} + \beta w^2.$$

The rest of the proof is done by a simple integration (see [5, p. 360]).   □

Remark 2.3. For the case where $\beta = 0$, we can deduce that if $\delta \geq 0$, any global positive solution satisfies the following estimate

$$w(t) \leq \frac{\Gamma}{\left(1 + \Gamma^{(p-1)/2} \dfrac{p-1}{2} \sqrt{\dfrac{2\alpha}{p+1}} t\right)^{2/(p-1)}} = w_\Gamma(t),$$

where $\Gamma = w(0)$. Note that if $\delta = 0$, w_Γ is the unique global solution to (2.15) such that $w_\Gamma(0) = \Gamma$.

Let us now turn to the case $\tau > 0$ and $\beta = 0$.

Theorem 2.2. *Let* $\tau, \alpha > 0$ *and* $\delta > -\tau/2$. *Assume that* $p > 1$, *then any global solutions* w *to* (2.12) *is monotone and satisfies*

$$0 < |w| \tag{2.16}$$

$$\leq \frac{|w(t_0)|}{\left(1 + \dfrac{p-1}{2} \sqrt{\dfrac{2\alpha}{p+1}} |w(t_0)|^{(p-1)/2} \left(e^{(\tau/2)t} - e^{(\tau/2)t_0}\right)\right)^{2/(p-1)}},$$

for all $t \geq t_0$, *where* t_0 *is large.*

Proof. It is not difficult to observe that any global non trivial solution does not change sign and it is strictly monotone and bounded. So we may consider only positive global solutions. Let w be a global positive solution. Put $L = \lim_{t \to \infty} w(t)$. Integrating (2.12) over $(0, +\infty)$ yields $L = 0$. Now the function

$$E(t) = \frac{1}{2} e^{-\tau t} w'^2(t) - \frac{\alpha}{p+1} w^{p+1}(t)$$

satisfies
$$E'(t) = -(\frac{\tau}{2} + \delta)e^{-\tau t}w'^2(t) < 0.$$

Next, because $L = 0$ and $\liminf w' = 0$, it follows that $E(t) \geq 0$ and then
$$-w' \geq \sqrt{\frac{2\alpha}{p+1}}e^{(\tau/2)t}w^{(p+1)/2}.$$

Integrating the above inequality we conclude that (2.16) holds. This completes the proof.   □

For the case $\delta = -\tau/2$ we have $E = \text{const}$. Hence

Corollary 2.3. *Assume that $\delta = -\tau/2$ and $\beta = 0$. Then Problem (2.12) has a unique family of global solutions*
$$w(t) = \frac{w(0)}{\left(1 + \frac{p-1}{2}\sqrt{\frac{2\alpha}{p+1}}|w(0)|^{(p-1)/2}\left(e^{(\tau/2)t} - 1\right)\right)^{2/(p-1)}}.$$

Remark 2.4. Estimate (2.16) remains true if we have greater than in (2.12) instead of equality. Note that $|w| \leq Ce^{-[\tau/(p-1)]t}$ for large t and the function
$$Ke^{-[\tau/(p-1)]t},$$
satisfies (2.12) for any $\delta \in \mathbb{R}$ and any $\beta \geq 0$, where τ and K are subject to the following conditions
$$\tau > \frac{p-1}{2}\left(\delta + \sqrt{\delta^2 + 4\beta}\right),$$
and
$$\tau^2 - (p-1)\delta\tau - (p-1)^2\beta = (p-1)^2\alpha|K|^{p-1}.$$

Using the function
$$E_0(t) = \frac{1}{2}e^{-\tau t}\left[w'^2(t) - \beta w^2(t)\right] - \frac{\alpha}{p+1}w^{p+1}(t)$$
and Proposition 2.2, the preceding theorem is completed by

Proposition 2.3. *Let $\tau, \alpha > 0$, $\beta \leq 0$ and $\delta > -\tau/2$. Assume that $p > 1$, then there is no global solution to (2.15) such that*
$$(w'(0))^2 < \beta(w(0))^2 + \frac{2\alpha}{p+1}|w(0)|^{p+1}.$$

3. Nonglobal results for the quasilinear hyperbolic-elliptic inequalities

The results of Section 2 will allow us to prove the absence of global nonnegative solutions to the following

$$
\begin{cases}
u_{tt} + \varepsilon a \Delta u + \varphi \left(\int_\Omega |\nabla u|^2 dx \right) \Delta u + \delta u_t \geq f(x,t), \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \Omega \times (0,T), \\
u = 0, \qquad\qquad\qquad\qquad \text{in } \partial\Omega \times (0,T),
\end{cases}
\tag{3.1}
$$

where Ω is a regular open subset of $\mathbb{R}^N$, $N \geq 1$, $a > 0$, $\varepsilon = \pm 1$ and $f \in L^\infty(0, \infty; L^2(\Omega))$. The function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is locally Lipschitz satisfying

$$
\varphi(s) \geq bs^\gamma,
\tag{3.2}
$$

where $b > 0$ and $\gamma > 0$.

3.1. The case $\varepsilon = -1$.

Let Φ_1 be a positive eigenfunction of

$$
-\Delta \Phi_1 = \lambda_1 \Phi_1, \quad \Phi_{1|\partial\Omega} = 0,
$$

where λ_1 is the first eigenvalue. We assume that

$$
\|\Phi_1\|_2 = 1.
\tag{3.3}
$$

Theorem 3.1. *Let $\delta \in \mathbb{R}$ and $\gamma > 0$. Assume*

$$
\int_\Omega f(x,t)\Phi_1 dx \geq 0,
\tag{3.4}
$$

for all $t \geq 0$. Then there exists no global solution to (3.1) such that

$$
\int_\Omega u(x,0)\Phi_1(x)dx > \lambda_1^{-1/2} \left(\frac{a}{b}\right)^{1/2\gamma}, \quad \int_\Omega u_t(x,0)\Phi_1(x)dx \geq 0.
$$

Proof. Let u be a global solution. Multiplying equation $(3.1)_1$ by Φ_1, we get

$$
w'' + \delta w' \geq \lambda_1 \varphi \left(\int_\Omega |\nabla u|^2 dx \right) w - a\lambda_1 w,
\tag{3.5}
$$

whe re

$$
w(t) = \int_\Omega u(x,t)\Phi_1(x)dx \quad \text{and} \quad w' := \frac{dw}{dt}.
$$

As $w(0) > 0$ one sees $w(t) > 0$ on $(0, t_0)$, t_0 small. On the other hand we have

$$
\varphi \left(\int_\Omega |\nabla u|^2 dx \right) \geq b \left(\int_\Omega |\nabla u|^2 dx \right)^\gamma,
$$

therefore

$$\varphi\left(\int_\Omega |\nabla u|^2 dx\right) \geq b\lambda_1^\gamma \left(\int_\Omega |u|^2 dx\right)^\gamma.$$

Next, using Hölder inequality and (3.3) to deduce that

$$w^2(t) \leq \int_\Omega u^2 dx,$$

and then

$$\varphi\left(\int_\Omega |\nabla u|^2 dx\right) \geq b\lambda_1^\gamma w^{2\gamma}.$$

Substituting this into (3.5) yields

$$w'' + \delta w' \geq b\lambda_1^{\gamma+1} w^{2\gamma+1} - a\lambda_1 w, \tag{3.6}$$

on $[0, t_0[$.

Finally, since $\gamma > 0$, $w'(0) \geq 0$ and $w(0)$ is larger than $\left(a/(b\lambda_1^\gamma)\right)^{1/2\gamma}$, we deduce from Proposition 2.1 that w is not global. A contradiction. This ends the proof. □

The following result is an immediate consequence of Corollary 2.2.

Proposition 3.1. *Let $\gamma > 0$ and $\delta \geq 0$. Let u be a solution to (3.1). Assume*

$$B^2 + a\lambda_1 A^2 - \frac{\lambda_1^\gamma}{\gamma+1} bA^{2(\gamma+1)} \leq \frac{\gamma}{\gamma+1}\left(\frac{a^{\gamma+1}}{b}\right)^{1/\gamma}, \tag{3.7}$$

where

$$A = \int_\Omega u(x,0)\Phi_1(x)dx > \lambda_1^{-1/2}\left(\frac{a}{b}\right)^{1/2\gamma}, \tag{3.8}$$

and

$$B = \int_\Omega u_t(x,0)\Phi_1(x)dx < 0.$$

Then the function

$$w(t) = \int_\Omega u(x,t)\Phi_1(x)dx$$

is not global.

Remark 3.1. A similar result can be obtained for the following problems

$$u_{tt} + \delta u_t - a\Delta u - \varphi\left(\int_\Omega u^2 dx\right) u \geq f,$$

and

$$u_{tt} - a\Delta u + \varphi\left(\int_\Omega u^2 dx\right)\Delta u - \Delta u_t \geq f.$$

3.2. The case $\varepsilon = 1$.

Here we shall investigate the elliptic inequality

$$u_{tt} + \delta u_t + a\Delta u + \varphi \left(\int_\Omega |\nabla u|^2 dx \right) \Delta u \geq f(x,t), \qquad (3.9)$$

posed in the half-infinite cylinder

$$\Sigma := \Omega \times (0, +\infty),$$

subject to zero condition on the lateral boundary. Here $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $a > 0$ and $\delta \in \mathbb{R}$. We are concerned with the nonglobal results of solutions to (3.9) and, in the case where the solutions exist, we investigate the asymptotic behaviour as t tends to infinity. A number of papers have appeared in the study of the asymptotic behavior of global solutions. In [20] Véron studied the following

$$u_{tt} + \Delta u = f(u), \quad \text{in } \Omega \times (0, +\infty), \qquad (3.10)$$

with Dirichlet condition, where f is a continuous and monotonic function such that $f(0) = 0$. The author proved that for any $u_0 \in L^p(\Omega)$, $p \geq 1$, the solution u is bounded, in particular if $p = 2$, u satisfies

$$\|u(t, .)\|_{L^\infty(\Omega)} \leq Ct^{-N/2} \|u_0\|_{L^2(\Omega)}, \quad C = \text{const} > 0.$$

Later Brada [5] showed that for $F(s)/s^2 \in L^2(-1, 1)$, where $F' = f$, then

$$\lim_{t \to \infty} e^{\sqrt{\lambda_1} t} u(t, .) = \alpha \Phi_1.$$

Here we shall give a sufficient condition which asserts the nonexistence of global solutions. As in Section 3.1, λ_1 denotes the first eigenvalue and Φ_1 is the positive eigenfunction normalized by $\|\Phi_1\|_2 = 1$. We have the following:

Theorem 3.2. *Let $\gamma > 0$, $a > 0$ and assume that φ and f satisfy (3.2), (3.4). Then Problem (3.9) has no global solution such that*

$$\int_\Omega u(x,0)\Phi_1 > 0,$$

and

$$\int_\Omega u_t(x,0)\Phi_1 \geq 0.$$

Proof. Let u be a global solution. The function

$$w(t) = \int_\Omega u(x,t)\Phi_1 dx$$

is positive and satisfies

$$w'' + \delta w' \geq \lambda_1 a w + b\lambda_1^{\gamma+1} w^{2\gamma+1},$$

in $(0, t_0)$, t_0 small enough. Arguing as in the proof of Proposition 2.1 one sees that if w exists, it remains positive and satisfies the inequality for all $t > 0$. Now as $\gamma > 0$ we deduce from Section 2 that w is not global. $\square$

The following result will show that a solution may blow-up in the case where

$$\int_\Omega u_t(x, 0)\Phi_1 < 0.$$

Theorem 3.3. *Let $\gamma > 0$, $a \geq 0$ and $\delta > \max(0, 1 - \lambda_1 a)$. Assume that φ and f satisfy (3.2), (3.4). Then there exists no global positive solution to (3.9) such that*

$$\int_\Omega (\sqrt{\delta + \lambda_1 a}\, u(x, 0) + u_t(x, 0))\Phi_1 dx \geq 0. \qquad (3.11)$$

Proof. Arguing as in the proof of Theorem 3.2, the mean function satisfies

$$w'' + \delta w' \geq \lambda_1 aw + b\lambda_1^{\gamma+1}|w|^p, \qquad (3.12)$$

where $p := 2\gamma + 1$, subject to the condition

$$\sqrt{\delta + \lambda_1 a}\, w(0) + w'(0) \geq 0.$$

The case $w'(0) \geq 0$ was treated in the preceding theorem. Suppose now that $w'(0) < 0$. Put

$$v(t) = w - ce^{-\sqrt{\delta + \lambda_1 a}\, t},$$

where the constant c satisfies

$$-\frac{w'(0)}{\sqrt{\delta + \lambda_1 a}} < c < w(0).$$

Using inequality (3.12) we infer

$$v'' + \delta v' \geq \lambda_1 av + b\lambda_1^{\gamma+1}\left|v + ce^{-\sqrt{\delta + \lambda_1 a}\, t}\right|^p.$$

As $v(0), v'(0) > 0$ we deduce from the last inequality that v and v' are positive as soon as v exists. Therefore v satisfies

$$v'' + \delta v' \geq \lambda_1 av + b\lambda_1^{\gamma+1}v^p.$$

Hence v is not global. This ends the proof. $\square$

The following result gives the large time behavior of any possible positive global solution.

Theorem 3.4. *Let $\gamma > 0$, $a \geq 0$ and $\delta > \max(0, 1 - \lambda_1 a)$. Assume that φ and f satisfy (3.2), (3.4). Let u be a global positive solution to (3.9), then there exists*

$$0 \leq \alpha_0 \leq \int_\Omega u(x, 0)\Phi_1 dx$$

such that

$$\lim_{t \to +\infty} e^{\sqrt{\delta + a\lambda_1}t} \int_\Omega u(x, t)\Phi_1 dx = \alpha_0.$$

Proof. According to the previous result we can see that

$$\int_\Omega (\sqrt{\delta + \lambda_1 a}u(x, t) + u_t(x, t))\Phi_1 dx < 0,$$

for all $t \geq 0$. Therefore the function

$$W(t) = e^{\sqrt{\delta + a\lambda_1}t} \int_\Omega u(x, t)\Phi_1 dx$$

is global, positive and decreasing and then W goes to a finite limit as t tends to infinity.   □

Corollary 3.1. *Assume, in addition, that $a > 0$, $\delta > \max(0, 1 - \lambda_1 a)$ then*

$$\lim_{t \to \infty} e^{\sqrt{\lambda_1 a}t} \int_\Omega u(x, t)\Phi_1 dx = 0.$$

To complete Theorem 3.3 we have

Theorem 3.5. *Let $\gamma > 0$, $a \geq 0$ and $\delta > 0$. Assume that φ and f satisfy (3.2), (3.4) and*

$$\liminf_{t \to \infty} \int_\Omega f(x, t)\Phi_1(x)dx > 0.$$

Then there is no global positive solution to (3.9).

Proof. Assume that the problem has a global positive solution u. Repeated argument of the proof of Theorem 2.1 enables us to see that $w' < 0$, $w'' > 0$ and w and w' tend to 0 as t approaches infinity. Therefore there exists a sequence $(t_n)_n$ converging to infinity such that $w''(t_n) \to 0$ as $n \to \infty$. Recall that w satisfies

$$w'' + \delta w' \geq \lambda_1 a w + b\lambda_1^{\gamma+1}w^{2\gamma+1} + \int_\Omega f(x, t)\Phi_1(x)dx.$$

A passage to the limit implies that

$$0 \geq \lim_{n \to \infty} \int_\Omega f(x, t_n)\Phi_1(x)dx,$$

which is absurd.   □

The next result is devoted to the general case where $\delta \in \mathbb{R}$.

Theorem 3.6. *Let $\gamma > 0$, $a \geq 0$ and $\delta \in \mathbb{R}$. Assume that φ and f satisfy (3.2), (3.4). Then there exists no global solution to (3.9) such that*

$$\int_\Omega (\delta + \sqrt{\delta^2 + 4\lambda_1 a}u(x,0) + 2u_t(x,0))\Phi_1 dx > 0. \tag{3.13}$$

Proof. Let u be a solution to (3.9). Arguing as in the proof of Theorem 3.1, the function $w(t) = \int_\Omega u(x,t)\Phi_1(x)dx$ satisfies

$$w'' + \delta w' \geq \lambda_1 a w + b\lambda_1^{1+\gamma} |w|^p.$$

Let $k = (-\delta + \sqrt{\delta^2 + 4\lambda_1 a})/2$ and $\eta = \sqrt{\delta^2 + 4\lambda_1 a}$, while using the change of variable $w(t) = \exp(kt)v(t)$ one obtain the following

$$v'' + \eta v' \geq b\lambda_1^{1+\gamma} \exp((p-1)kt) |v|^p. \tag{3.14}$$

Now we consider $\theta \in C_0^\infty([0, +\infty[)$, $\theta \geq 0$ such that $\theta(0) = 1$, $\theta'(0) = 0$ and

$$\int_0^\infty |\theta''|^q \theta^{1-q} dt + \int_0^\infty |\theta'|^q \theta^{1-q} dt < \infty, \tag{3.15}$$

where $q = p/(p-1)$. If one multiplies (3.14) by θ and integrates by parts, one gets

$$\eta v(0) + v'(0) + b\lambda_1^{1+\gamma} \int_0^{+\infty} \exp((p-1)kt) |v|^p \theta(t)dt$$

$$\leq M \int_0^\infty |v| (|\theta''| + |\theta'|)dt, \tag{3.16}$$

where $M = \max\{1, \eta\}$. Making use of the Young inequality, one finds, for $\varepsilon > 0$,

$$\int_0^\infty |v| (|\theta''| + |\theta'|)dt \leq \varepsilon \int_0^\infty \exp((p-1)kt) |v|^p \theta dt$$

$$+ C_\varepsilon \int_0^\infty \exp(-kt)(|\theta'|^q + |\theta''|^q)\theta^{1-q}dt.$$

Next we take $\varepsilon = b\lambda_1^{1+\gamma}/M$, to deduce, for some positive constant C

$$\eta v(0) + v'(0) \leq C \int_0^{+\infty} \exp(-kt)(|\theta'|^q + |\theta''|^q)\theta^{1-q}(t)dt, \tag{3.17}$$

thanks to (3.16). Now for $T > 0$ fixed, choosing $\theta(t) = \theta_0^r(t/T)$ where $\theta_0 \in C_0^\infty([0, +\infty[)$, $0 \le \theta_0 \le 1$, $\theta_0(t) = 1$ for $0 \le t \le 1/2$ and $\theta_0(t) = 0$ for $t \ge 1$, and where $r \ge 2q$ such that condition (3.15) holds. Using estimate (3.17), we easily obtain

$$\eta v(0) + v'(0) \le CT^{1-2q} \exp(-\frac{kT}{2}) \int_{1/2}^{1} |\theta''|^q \, \theta^{1-q} dt$$

$$+ CT^{1-q} \exp(-\frac{kT}{2}) \int_{1/2}^{1} |\theta'|^q \, \theta^{1-q} dt,$$

and then

$$\eta v(0) + v'(0) \le \overline{C} \left\{ T^{1-2q} \exp(-\frac{kT}{2}) + T^{1-q} \exp(-\frac{kT}{2}) \right\}, \quad \overline{C} > 0.$$

We may let $T \to \infty$ in the above inequality to deduce that

$$\eta v(0) + v'(0) \le 0,$$

which contradicts hypothesis (3.13). This ends the proof. $\qquad\square$

Remark 3.2. The results of Section 2 can also be used to obtain a family of global and nonglobal "explicit" solutions to

$$u_{tt} + \delta u_t + \left(\int_\Omega |\nabla u|^2 dx \right)^\gamma \Delta u - \beta u \ge \lambda |u|^{p-2} u, \quad \text{in } \Sigma, \qquad (3.18)$$

with Dirichlet boundary condition, where $\lambda, \beta > 0$ and $p = 2(\gamma + 1) > 1$. Here we look for solutions having the form

$$u(x, t) = w(t)v(x). \qquad (3.19)$$

In fact if we substitute (3.19) into (3.18), we obtain the following inequality

$$v(w'' + \delta w' - \beta w) \ge |w|^{2\gamma} w \left[-\left(\int_\Omega |\nabla v|^2 dx \right)^\gamma \Delta v + \lambda |v|^{2\gamma} v \right].$$

Therefore it is sufficient to demand to the required functions w and v to satisfy

$$w'' + \delta w' \ge \kappa w^{2\gamma+1} + \beta w, \qquad (3.20)$$

and

$$-\left(\int_\Omega |\nabla v|^2 dx \right)^\gamma \Delta v + \lambda |v|^{2\gamma} v = \kappa v, \ v \ge 0, \qquad (3.21)$$

where $\kappa > 0$.

We have

Theorem 3.7. *Let* $\lambda, \kappa > 0$ *and* $\gamma > 0$ *such that* $(N - 2)\gamma < 2$. *Let* $v_0 \in H_0^1(\Omega)$ *be a non trivial solution to* (3.21). *Then for any* $a > 0$, $b < 0$ *satisfying*

$$b^2 + \delta ab - \beta a^2 \geq \kappa a^{2(\gamma+1)}, \tag{3.22}$$

the function

$$U(x, t) = ae^{(b/a)t}v_0(x),$$

is a non trivial solution to (3.18).

Proof. First we show that (3.21) has a non trivial solution. To this direction we consider the problem

$$\inf \left\{ \left(\int_\Omega |\nabla v|^2 \right)^{\gamma+1} dx + \lambda \int_\Omega |v|^{2(\gamma+1)} dx; \ v \in H_0^1(\Omega), \ \int_\Omega |v|^2 = 1 \right\}.$$

As $(N - 2)(2\gamma + 1) < N + 2$, there exists $v_0 \in H_0^1(\Omega)$ which realizes the infimum; it is nonnegative and satisfies the Euler-Lagrange equation (3.21). Next we look for solution to (3.20) in the form $w(t) = ae^{(b/a)t}$, $a > 0$, $b < 0$. Insertion into (3.20) yields

$$\left(\frac{b^2}{a} + \delta b - \beta a \right) e^{(b/a)t} \geq \kappa a^{2\gamma+1} e^{(2\gamma+1)(b/a)t}.$$

This leads to the sufficient condition (3.22). $\qquad\qquad\square$

Remark 3.3. If we have equality in (3.18) instead of inequality, it is clear that w would satisfy

$$w'' + \delta w' = \kappa w^{2\gamma+1} + \beta w. \tag{3.23}$$

Then, for $\delta = 0$, the function

$$U(x, t) = \begin{cases} \dfrac{\Gamma v_0(x)}{\left(1 + \gamma \Gamma^\gamma \sqrt{\frac{\kappa}{\gamma}} t \right)^{1/\gamma}}, & \text{if } \beta = 0, \\[3ex] \Gamma \left(\dfrac{2\sqrt{\beta(\gamma+1)}}{Ae^{2\gamma\sqrt{\beta}t} + B} \right)^{1/\gamma} e^{\sqrt{\beta}t}v_0(x), & \text{for } \beta > 0, \end{cases}$$

where

$$A := \sqrt{\beta(\gamma+1) + \alpha|\Gamma|^{2\gamma}} + \sqrt{\beta(\gamma+1)},$$
$$B := -\sqrt{\beta(\gamma+1) + \alpha|\Gamma|^{2\gamma}} + \sqrt{\beta(\gamma+1)},$$

satisfies (3.18).

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