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EXISTENCE OF GLOBAL WEAK SOLUTIONS FOR COUPLED THERMOELASTICITY WITH BARBER'S HEAT EXCHANGE CONDITION

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Abstract. The existence of global weak solutions for coupled thermoelasticity with the nonlinear contact boundary condition and Barber's heat exchange condition is proved via the Faedo-Galerkin, monotonicity and compactness methods. Some *a priori* bounds obtained with Gronwalls inequality in connection with the embedding and trace theorems lead to accomplishing a generalization of our previous study [5]. The heat-exchange coefficient associated with Barber's heat exchange condition is dependent only on the normal displacement. This dependence is described by a bounded Lipschitz function. Moreover, this study is some extension of works due to Andrews et al. [3] and Elliot et al. [12].

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1. Introduction and the result

1.1. Formulation of the problem.

In this paper we are concerned with the existence of global weak solutions to the problem describing the evolution of N-dimensional thermoelastic body occupying a bounded domain $\Omega \subset \mathbb{R}^N$ ($N = 2$ or 3) with boundary $\partial\Omega$. Some part of the boundary may be brought into contact with an elastic obstacle. Moreover, this portion of the boundary is subject to the heat exchange, which results in the so-called Barber's heat exchange condition. The problem under consideration consists of:

- *the equations of coupled thermoelasticity* (cf. Section 2)

$$\rho_r \partial_t^2 U - \mathrm{div}\sigma = \rho_r b \quad \text{in } Q, \tag{1.1}$$

$$\alpha\partial_t\theta + \mathrm{div} q = -\eta \mathrm{div}(\partial_t U) + \rho_r r \quad \text{in } Q; \tag{1.2}$$

- *the initial conditions*

$$U(\cdot,0) = U_0 \quad \text{and} \quad \partial_t U(\cdot,0) = U_1 \text{ on } \Omega \quad \text{and} \quad \theta(\cdot,0) = \theta_0 \text{ on } \Omega; \tag{1.3}$$

- *the boundary conditions*

$$U = g \text{ on } \Gamma_d \times [0,T], \quad \sigma\nu = F \text{ on } \Gamma_f \times [0,T], \tag{1.4}$$

$$\sigma_\nu = -p_0(\cdot)(U_\nu - g_\nu)_+^\xi \quad \text{and} \quad \sigma_\tau = 0 \text{ on } \Gamma_c \times [0,T], \tag{1.5}$$

$$\begin{aligned} &\theta = \theta_a \text{ on } \Gamma_d \times [0,T], \\ &q \cdot \nu = \beta(x, U_\nu - g_\nu)(\theta - \theta_a) \text{ on } \Gamma_d' : \times [0,T]. \end{aligned} \tag{1.6}$$

Here $Q := \Omega \times]0,T[$ with $\partial\Omega =: \Gamma =: \Gamma_d \cup \Gamma_f \cup \Gamma_c$ and $\Gamma_d' := \Gamma \setminus \Gamma_d$. The functions

$$U\colon Q \to \mathbb{R}^N \quad \text{and} \quad \theta\colon Q \to [0,\infty[$$

stand for the vector field of displacements and the scalar field of the absolute temperature, respectively. The quantities

$$b\colon Q \to \mathbb{R}^N \quad \text{and} \quad r\colon Q \to \mathbb{R}$$

are the vector fields of external body forces and the distributed heat source in the body, respectively; $\sigma\colon Q \to \mathbb{R}^{N^2}$ denotes the stress tensor, which is furnished by the Duhamel-Neumann relations

$$\sigma_{jk} = A_{jklm}\varepsilon_{lm}(U) - \gamma(\theta - \theta_r)\delta_{jk} =: \Sigma_{jk}(U, \theta - \theta_r) \quad (\gamma > 0)$$

with the strain tensor

$$\varepsilon_{lm}(U) := \frac{1}{2}(U_{l,m} + U_{m,l})$$

and with the coefficients

$$A_{jklm} := \lambda\delta_{jk}\delta_{lm} + \mu\left(\delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl}\right),$$

where θ_r denotes the *uniform* reference temperature. (The rule of the usual summation convention is applied here and in the further part of the paper.) Symbols δ_{jk} $(j, k = 1, \dots, N)$ stands for the Kronecker delta, $\lambda > 0$ and $\mu > 0$ are Lamé's constants, $q\colon Q \to \mathbb{R}^N$ means the heat flux which is postulated by a constitutive relation of the form (see Section 2 for its justification)

$$q := -\kappa \nabla\theta - k\nabla\partial_t\theta$$

with the coefficient of thermal conductivity of the body κ and some non-negative parameter k. Then $\boldsymbol{\nu}\colon \Gamma \to \mathbb{R}^N$ is the field of the outward unit vectors normal to the boundary Γ, and $U_\nu(x', t) = U_j(x', t) \cdot \nu_j(x')$ is the displacement normal to the boundary at time t with the boundary point x';

$$\sigma_\nu = \sigma_{jk} \cdot \nu_j \nu_k$$

is the normal component of σ on Γ;

$$\sigma_\tau = (\sigma_{jk} \cdot \nu_k - \sigma_\nu \nu_j)_{j=1}^N \in \operatorname{Span}\left\{\boldsymbol{\tau}_1(x'), \dots, \boldsymbol{\tau}_{N-1}(x')\right\}$$

is the tangential vector, where the vectors: $\boldsymbol{\nu}(x'), \boldsymbol{\tau}_1(x'), \dots, \boldsymbol{\tau}_{N-1}(x')$ form an orthonormal set of the normal and tangential vectors at every point $x' \in \Gamma$ with $\boldsymbol{\tau}_j\colon \Gamma \to \mathbb{R}^N$. Next

$$F\colon \Gamma_f \times [0, T] \to \mathbb{R}^N \quad \text{and} \quad p_0\colon \Gamma_c \to [0, \infty[$$

stand for the traction and the pressure, respectively, $\xi \geq 1$, θ_a denotes the ambient outside temperature, $\beta\colon \Omega \times \mathbb{R} \to \mathbb{R}^+ := [0, \infty[$ is a function defining the heat-exchange coefficient along boundary $\Gamma_f \cup \Gamma_c$ such that

$$\beta(x, s) := \begin{cases} \beta_f(x) & \text{for } x \in \Gamma_f, \\ \beta_c(s) & \text{for } x \in \Gamma_c \end{cases}$$

with $\beta_f\colon \Omega \to \mathbb{R}^+$ and $\beta_c\colon \mathbb{R} \to \mathbb{R}^+$. Function $g\colon \Gamma \to \mathbb{R}^N$ stands for the gap between the boundary of the body and the boundary of the obstacle, $g_\nu(x') = g_k(x') \cdot \nu_k(x')$ is the displacement normal to the boundary Γ with the frontier point x', ρ_r denotes the reference mass density of the body, $\alpha = \rho_r c_V$, where c_V is the specific heat of the body at constant volume; ε, γ and η are some positive constants; $\pi_+ := \max\{\pi, 0\}$ for any function π. Finally, ∂_t, ∇, div and Δ denote the t-derivative, the gradient operator, the divergence operator and the Laplacian operator, respectively. With the exception of U and θ all the quantities occurring in (1.1)–(1.5) are given.

Let $\bar{g}$ denote the extension of g (cf. [26], [16]). Then we may define new functions by

$$u := U - \bar{g} \quad \text{and} \quad \vartheta := \theta - \theta_r. \tag{1.7}$$

Substituting (1.7) into (1.1)–(1.5) leads to the following problem

$$\rho_r \partial_t^2 u - \operatorname{div}\Sigma(u,\vartheta) = h \quad \text{in } Q, \tag{1.8}$$

$$\alpha\partial_t\vartheta - \kappa\Delta\vartheta - k\Delta\partial_t\vartheta = -\eta\operatorname{div}\partial_t u + \omega \quad \text{in } Q, \tag{1.9}$$

$$u(\cdot,0) = u_0 \text{ and } \partial_t u(\cdot,0) = u_1 \text{ on } \Omega \text{ and } \vartheta(\cdot,0) = \vartheta_0 \text{ on } \Omega \tag{1.10}$$

$$u = 0 \text{ on } \Gamma_d \times [0,T],\ \Sigma(u,\vartheta)\nu = f \text{ on } \Gamma_f \times [0,T], \tag{1.11}$$

$$\Sigma_\nu(u,\vartheta) = -p_0(\cdot)(u_\nu)_+^\xi \text{ and } \Sigma_\tau(u,\vartheta) = 0 \text{ on } \Gamma_c \times [0,T], \tag{1.12}$$

$$\vartheta = 0 \text{ on } \Gamma_d \times [0,T],$$

$$\kappa\partial_\nu\vartheta + k\partial_\nu\partial_t\vartheta = -\beta(x,u_\nu)\vartheta + \zeta(x,u_\nu) \text{ on } \Gamma_d' \times [0,T], \tag{1.13}$$

where

$$h := \rho_r b + \Sigma(\bar{g},0),\ \zeta(x,u_\nu) := (\vartheta_a - \vartheta_r)\beta(x,u_\nu) \text{and } \omega = \rho_r r. \tag{1.14}$$

Systematic studies aimed at obtaining a basic understanding of what goes on in the thermoelastic body during heating or cooling has not been completed yet, but some progress in the field of coupled thermoelasticity with Barber's heat exchange condition has been made by generalizing works [3] and [12] due to Andrews et al. and Elliot et al., respectively, and our previous investigation [5]. This generalization consists in introducing the mixed non-linear boundary condition for the temperature instead of the Dirichlet one in [5]. Such a boundary condition better suits the phenomena occurring in these processes, because the Dirichlet boundary condition for the temperature is peculiar to the paradoxical phenomenon which is connected with coupled thermoelasticity under contact boundary conditions (cf. [11]) and does not appear fully acceptable from the physical point of view. Although such a modification is sufficient to show existence of global weak solutions to the problem, which may be seen from a priori estimates in Section 3, it gives rise to some difficulties to be surmounted before regularity of the solution may be accomplished. This drawback is removed by introducing the short memory term $-k\nabla\partial_t\vartheta$ into the heat flux. Such an introduction is, to a certain extent, in agreement with the modified law of Fourier (cf. Section 2).

The important point to note here is the form of solutions to that problem for the one space dimensional case of linear thermoelasticity (cf. [12], [3]). First, in the work [12] due to Elliot and Tang, the method of compensated compactness is used to prove that their problem admits at least one pair of solutions to a dynamic contact problem in thermoelasticity with Barber's heat exchange condition. Second, in the paper [3] due to Andrews et al., the thermoelastic contact with Barber's heat exchange condition is investigated, where the acceleration of the rod is assumed to be equal to zero. The novelty in all these considerations is the appearance of the non-linear function β occurring in Barber's heat exchange condition. The function β defines the heat-exchange coefficient that, in its full generality, depends on the normal displacement u_ν; the reader is referred to [3] and references therein for the characterization of this coefficient. The function β may be treated as a multivalued one [3] or as a single-valued one [11]. Although there has been much recent interest in a question of finding the solution to a dynamic contact problem [12], [3] for these two cases, all of these studies deal with the one space dimensional problem of linear thermoelasticity. Our study concerning that kind of problems is carried over to the space dimension greater than one, i.e., $N = 2$ or 3. We succeeded in finding a global weak solution to that problem in the case where the data satisfy some relevant assumptions. Moreover, these solutions are proved to be regular.

1.2. Notations.

We employ the usual notation for the standard functional spaces that will be used throughout the paper (see [1], [2] or/and [25]). The symbol $L_q(\Omega)$, $1 \le q \le \infty$, denotes the usual Lebesgue space of real valued functions with norm $\| \cdot \|_{q,\Omega}$, in particular $\| \cdot \| := \| \cdot \|_{2,\Omega}$. Moreover, $W^{m,q}(\Omega)$ stand for the usual Sobolev spaces with norm $\| \cdot \|_{W^{m,q}(\Omega)}$. In particular, we recall $H^m(\Omega) := W^{m,2}(\Omega)$. The inner products on $L_2(\Omega)$ and $L_2(\Gamma)$ are denoted by $(\cdot,\cdot)_\Omega \equiv (\cdot,\cdot)$ and $\langle \cdot,\cdot \rangle_\Gamma$, respectively. We recall the Hilbert space $H(\Delta,\Omega) := \{u \in H^1(\Omega) \colon \Delta u \in L_2(\Omega)\}$ with the norm [20]

$$\| u \|_{H(\Delta,\Omega)} := \left(\| u \|^2_{H^1(\Omega)} + \| \Delta u \|^2 \right)^{1/2}.$$

The mappings $\gamma_0 \colon H^1(\Omega) \to H^{1/2}(\Gamma)$ and $\gamma_1 \colon H(\Delta,\Omega) \to H^{-1/2}(\Gamma)$ are the trace maps such that

$$\gamma_0(u) = u_{|\Gamma}, \quad \gamma_1(u) = \left(\frac{\partial u}{\partial \nu} \right)_{|\Gamma} \text{ for all } u \in D(\bar{\Omega}).$$

Moreover, the generalized Green's formula

$$\int_\Omega (-\Delta u) w dx = \int_\Omega \nabla u \cdot \nabla w dx - \langle \gamma_1(u), \gamma_0(w) \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}$$

holds for all $u \in H(\Delta,\Omega)$ and $w \in H^1(\Omega)$.

For further consideration we need to introduce the Hilbert space

$$V := \left\{ w \in H^1(\Omega) \colon \gamma_0(w) = 0 \text{ on } \Gamma_d \right\}$$

as the closure in $H^1(\Omega)$ of $\left\{ w \in C^1(\bar{\Omega}) \colon w = 0 \text{ on } \Gamma_d \right\}$ which is valid for a bounded domain $\Omega \subset \mathbb{R}^N$ whose boundary Γ is Lipschitzian (cf. [16]). We may endow the space V with the norm defined by

$$\| w \|_V = \left(\int_\Omega \left(| \nabla w |^2 + | w |^2 \right) dx \right)^{1/2}.$$

The space V is a closed linear subspace of $H^1(\Omega)$. The topological dual of V is denoted by V^* . The symbol $\langle \cdot, \cdot \rangle_V$ stands for the duality pairing on $V^* \times V$ and $\| \cdot \|_{V^*}$ denotes the V^* norm.

1.3. Assumptions on the given data.

We make the following assumptions on the given data to prove the desired result:

- (A1) the boundary $\Gamma =: \Gamma_d \cup \Gamma_f \cup \Gamma_c$ is Lipschitzian and piecewise C^1, where each Γ_j $(j = d, f, c)$ is a measurable subset of Γ such that $\text{meas}_{N-1}\{\Gamma_d\} \neq 0$ and $\bar{\Gamma}_d \cap \Gamma_f \cap \Gamma_c = \emptyset$;
- (A2) the functions $\beta_c \in C^1(R, R^+)$ and $\beta_f \in C(\Gamma_f, R^+)$ satisfy the following conditions:

$$0 < m_\beta \leq \beta_c(s);\ \beta_f(x) \leq M_\beta < \infty \text{ for } s \in R,\ x \in \Gamma_f \quad \text{and for some } m_\beta \leq M_\beta \tag{1.15}$$

$$| \beta_c(s_1) - \beta_c(s_2) | \leq l_\beta \mid s_1 - s_2 \mid \ \text{ for } s_1, s_2 \in R \quad \text{and for some } 0 < l_\beta < \infty; \tag{1.16}$$

- (A3) $u_0 \in V^N$, $u_1 \in L_2(\Omega)^N$ and $\vartheta_0 \in L_2(\Omega)$;
- (A4) the functions g and ϑ_a are assumed to be constant on Γ_c and Γ, respectively;
- (A5) $h \in L_\infty(0, T; V^{*N})$, $\partial_t h \in L_2(0, T; V^{*N})$ and $\omega \in L_2(0, T; V^*)$;
- (A6) the exponent ξ in the contact boundary condition (1.5) satisfies the following relations:

$$\xi \geq 1 \text{ and } \xi = q_1(q-1)/q$$

where

$$q, q_1 \in \begin{cases} [1, \infty[& \text{for } N = 2 \\ [1, 4] & \text{for } N = 3; \end{cases}$$

- (A7) $f \in L_\infty\left(0, T; L_p(\Gamma_f)^N\right)$, $\partial_t f \in L_2\left(0, T; L_p(\Gamma_f)^N\right)$ with $p = 1 + 1/\xi$;
- (A8) the function $p_0(\cdot) \in L_\infty(\Gamma_c)$ is merely required to take on non-negative values.

1.4. Weak formulation of the problem.

In order to state the problem in the variational form we need to introduce the bilinear form $a\colon V \times V \to \mathbb{R}$ by

$$a(u, \varphi) := \int_\Omega A_{ijkl}\varepsilon_{kl}(u)\varepsilon_{ij}(\varphi)dx = \int_\Omega A_{ijkl}u_{k,l}\varphi_{i,j}dx \tag{1.17}$$

where the symbol $_{,k}$ denotes differentiation with respect to x_k and the symmetry of the fourth order elasticity tensor A has been used; this tensor satisfies the condition:

$$\exists(\alpha_a > 0)\forall\left((\xi_{ij})_{N\times N}\colon \xi_{ij}=\xi_{ji}\right) A_{ijkl}\xi_{ij}\xi_{kl} \geq \alpha_a \mid \xi \mid^2 . \tag{1.18}$$

Moreover, we will need to consider the following surface integral:

$$\begin{aligned}\int_\Gamma \sigma_{jk}\nu_k\varphi_j \, d\Gamma &= \int_{\Gamma_f} f_j\varphi_j \, d\Gamma + \int_{\Gamma_c} (\sigma_\nu\varphi_\nu + \sigma_\tau\varphi_\tau) \, d\Gamma \\ &= \int_{\Gamma_f} f_j\varphi_j \, d\Gamma - \int_{\Gamma_c} [p_0(u_\nu)_+^\xi]\varphi_\nu \, d\Gamma \\ &= \langle f,\varphi\rangle_{\Gamma_f} - \langle P\left(u(t)\right),\varphi\rangle_{\Gamma_c}\end{aligned}$$

for any $\varphi \in V$, where $P\colon V \to V^*$ is a non-linear map.

We are now in a position to state the problem in the variational form. It reads

$$\langle \rho\partial_t^2 u - \operatorname{div}\sigma - h, \varphi\rangle_V = 0 \text{ for any } \varphi \in V,$$

which is equivalent to

$$\langle \rho\partial_t^2 u, \varphi\rangle_V + a(u,\varphi) - \gamma(\theta, \operatorname{div}\varphi)_\Omega - \langle f,\varphi\rangle_{\Gamma_f} + \langle P\left(u(t)\right),\varphi\rangle_{\Gamma_c} = \langle h,\varphi\rangle_V$$

for any $\varphi \in V$.

By Korn's inequality in the form

$$\int_\Omega \mid \nabla u \mid^2 dx \leq K \int_\Omega \mid \varepsilon(u) \mid^2 dx \text{ for any } u \in H^1(\Omega)^N, \tag{1.19}$$

which holds for the domains with piecewise C^1 boundaries, and a result in Morrey's monograph (cf. [21, p. 82])

$$\parallel w \parallel^2 \leq C_m(\parallel \nabla w \parallel^2 + \mid \int_{\Gamma_d} w dx \mid) \text{ for any } w \in H^1(\Omega), \tag{1.20}$$

which is valid for the Γ_d with positive measure $\text{meas}_{N-1}(\Gamma_d) > 0$ and some constant C_m which is independent of w, we may deduce the coercivity of the form a, i.e.

$$a(w,w) \geq m_a \parallel w \parallel_V^2, \ w \in V \text{ with } m_a = \frac{\alpha_a \min(1, C_m)}{2KC_m}. \tag{1.21}$$

Moreover, the continuity of form a is expressed by

$$a(u,w) \leq M_a \parallel u \parallel_V \parallel w \parallel_V, \text{ for } u, w \in V. \tag{1.22}$$

It should be noticed that for the Hilbert triplet

$$V \subset H \subset V^*$$

the embeddings are compact. The first inclusion is a consequence of (1.20), whereas its compactness is ensured by the compact embedding theorem [14], [16]. The second inclusion may be proved in a standard way.

1.5. Main result.

Definition 1.1. A pair (u, θ) is a global weak solution to problem (1.8)–(1.13) if and only if the functions

$$u \in L_\infty(0,T;V^N) \text{ for which } \partial_t u \in L_\infty\left(0,T;L^2(\Omega)^N\right) \text{ and}$$
$$\partial_t^2 u \in L_\infty\left(0,T;V^{*N}\right)$$

and

$$\vartheta \in L_\infty(0,T;V)$$

satisfy the following integral identities

$$\int_0^T \{\rho_r\langle\partial_t^2 u,\varphi\rangle_V + a(u,\varphi) - \gamma(\vartheta,\mathrm{div}\varphi)_\Omega - \langle f,\varphi\rangle_{\Gamma_f} + \langle P\,(u(t))\,,\varphi\rangle_{\Gamma_c} - \langle h,\varphi\rangle_V\}\,dt = 0, \tag{1.23}$$

$$\begin{aligned}&\int_0^T [-(\alpha\vartheta + \eta\mathrm{div}u, \partial_t\psi) + \kappa(\nabla\vartheta,\nabla\psi) - k(\nabla\vartheta,\nabla\partial_t\psi) - \langle\omega,\psi\rangle_V]\,dt\\ &+ \int_0^T\int_{\Gamma_d'}[\beta(\cdot,u_\nu)\vartheta - \zeta(\cdot,u_\nu)]\,\psi\,dA\,dt\\ &= \int_\Omega [\alpha\vartheta_0(x) + \eta\mathrm{div}\,(u_0(x))]\,\psi(x,0)\,dx\\ &+ k\int_\Omega \nabla\vartheta_0(\cdot)\cdot\nabla\partial_t\psi(\cdot,0)\,dx\end{aligned} \tag{1.24}$$

for any $\varphi \in L_2(0,T;V^N)$ and any $\psi \in H^1(0,T;V)$ with $\varphi(\cdot,T) = 0$ and $\psi(\cdot,T) = 0$ for all $0 < T < \infty$, respectively.

Our main results are existence and regularity theorems for global in time weak solutions to problem (1.8)–(1.13).

Theorem 1.2. *Assume* (A1)–(A8). *Then problem* (1.8)–(1.13) *admits a global weak solution in the sense of Definition* 1.1.

Remark 1.1. It is worth mentioning that the case where the number k vanishes is considered in [4], therefore we will avoid discussing it here.

Theorem 1.3. *Let the assumptions of Theorem* 1.2 *with* $\xi = 1$ *are satisfied. Moreover, we additionally assume that there are positive constants* β_1 *and* ζ_1 *such that*

$$\mid\beta_s(x,s)\mid\le\beta_1 \ \textit{and}\ \mid\zeta_s(x,s)\mid\le\zeta_1 \ \textit{for all}\ (x,s)\in\Omega\times\mathbb{R}$$

and that

$$\dot{\omega}(\cdot,\cdot) \in L_2(0,T;V^*),\ \dot{h}(\cdot,\cdot) \in L_\infty(0,T;V^*),\ \dot{f}(\cdot,\cdot) \in L_\infty\left(0,T;L_p(\Gamma_f)\right),$$
$$\ddot{h}(\cdot,\cdot) \in L_2(0,T;V^*),\ \ddot{f}(\cdot,\cdot) \in L_2\left(0,T;L_p(\Gamma_f)\right),$$
$$\ddot{u}(\cdot,0) \in H \text{ and } \dot{\vartheta}(\cdot,0) \in L_2(\Omega),$$

where p is defined in (A7). *Then problem* (1.8)–(1.13) *admits a global regular solution.*

1.6. Plan of the remaining sections.

Section 2 provides some justification for the postulated constitutive relations occurring in the posed problem. In Section 3 a priori estimates of solutions to the problem are derived. Section 4 contains the proof of Theorem 1.2.

2. Basic equations

2.1. Motion of the body.

Let $\chi\colon \Omega \times [0,T[\to \mathbb{R}^N$ be a function defining the motion of the thermoelastic body. We recall that $U(X,t) := \chi(X,t) - X$ stands for the field of displacements and the deformation is given by $F(X,t) = \nabla\chi(X,t) = \nabla U(X,t) - I$, where I denotes the unit tensor. Then the equations of thermoelasticity reads as follows (cf. [6], [9] and [23]):

$$\rho_r \ddot{U} - \mathrm{div}\sigma = \rho_r b \text{ in } Q, \tag{2.1}$$

$$\rho_r \dot{e} + \mathrm{div} q = \sigma \cdot \dot{F} + \rho_r r \text{ in } Q, \tag{2.2}$$

where σ denotes the first Piola-Kirchhoff stress tensor, e is the internal energy per unit mass and the heat flux q is calculated in the reference configuration, r is the internal heat supply per unit mass. The divergence operator div $\sum_{k=1}^N \partial/\partial X_k$ acts in the reference configuration, the superposed dot stands for the material time derivative.

In order for the thermodynamic process $(U(\cdot,\cdot),\theta(\cdot,\cdot))$ to be compatible with the second law of thermodynamics, it is necessary and sufficient that the local dissipation inequality

$$\rho_r \dot{\psi} + \rho_r s \dot{\theta} - \sigma \cdot \dot{F} + \frac{q \cdot g}{\theta} \leq 0 \tag{2.3}$$

holds on its domain, where $g := \nabla\theta$ is the temperature gradient in the reference configuration. This inequality is consequence of the Clausius-Duhem inequality for the entropy production

$$\rho_r \dot{s} \geq \mathrm{div}\left(\frac{-q}{\theta}\right) + \frac{\rho_r r}{\theta}, \tag{2.4}$$

where s stands for the specific entropy per unit mass, the specific free energy function

$$\psi = e - s\theta, \tag{2.5}$$

and the first law of thermodynamics (2.2).

2.2. Restrictions imposed on thermoelastic materials by the second law of thermodynamics.

We now postulate the following constitutive relations:

$$\begin{aligned} &\psi = \Psi(F,\theta,g), \quad s = S(F,\theta,g), \quad \sigma = \Sigma(F,\theta,g),\\ &q = Q(F,\theta,g,\dot{g}) = Q^I(F,\theta,g) - \kappa_1(\theta)\dot{g}, \end{aligned} \tag{2.6}$$

where the heat flux q has been split up into two terms: the classical one and the following $-\kappa_1(\theta)\dot{g}$. The latter heat flux has been derived from investigations within the framework of Statistical Mechanics in [7] and then it has been revised in [8].

Making use of (2.3) and (2.6) we can derive the following inequality:

$$\begin{aligned} &(\rho_r \partial_F \Psi(F,\theta,g) - \Sigma(F,\theta,g)) \cdot \dot{F} + (\rho_r \partial_\theta \Psi(F,\theta,g) + \rho_r S(F,\theta,g))\,\dot{\theta}\\ &+ \left(\rho_r \partial_g \Psi(F,\theta,g) - \frac{\kappa_1(\theta) g}{\theta}\right) \cdot \dot{g} + \frac{Q^I(F,\theta,g)\cdot g}{\theta} \le 0. \end{aligned} \tag{2.7}$$

Lemma 2.1. *The local dissipation inequality* (2.3) *is satisfied for all admissible thermodynamic processes if and only if the following three statements:*

1) $\Psi(F,\theta,g) = \dfrac{1}{\rho_r}\left(\Psi^I(F,\theta) + \kappa_1(\theta)\dfrac{g\cdot g}{2\theta}\right)$;
2) Σ *is determined through the* **stress relation**
   $\Sigma(F,\theta) = \partial_F \Psi^I(F,\theta)$
   and S *through the* **entropy relation**
   $S(F,\theta,g) = -\partial_\theta \Psi(F,\theta,g)$;
3) Q^I *obeys the* **heat conduction inequality**
   $Q^I(F,\theta,g)\cdot g \le 0$

hold on its domain.

Proof. The sufficiency of 1)–3) follows from (2.7), whereas the necessity of the conditions can be showed in the same way as in [6], which completes the proof. □

We now want to eliminate the energy e in (2.2). To this end, we differentiate (2.5) with respect to time to get

$$\dot{e} = \dot{\psi} + \theta\dot{s} + s\dot{\theta}$$

while 1) and 2) in the Lemma 2.1 imply that

$$\dot{\psi} = \frac{1}{\rho_r}\Sigma \cdot \dot{F} - S\dot{\theta} + \frac{\kappa_1(\theta)}{\rho_r \theta} g \cdot \dot{g}.$$

Hence, the last two equalities lead to

$$\rho_r \dot{e} = \rho_r \theta \dot{s} + \Sigma \cdot \dot{F} + \frac{\kappa_1(\theta)}{\theta} g \cdot \dot{g}. \tag{2.8}$$

By (2.2) and (2.8),

$$\rho_r \theta \dot{s} + \frac{\kappa_1(\theta)}{\theta} g \cdot \dot{g} + \mathrm{div} q = \rho_r r. \tag{2.9}$$

But

$$\dot{s} = \partial_F S(F,\theta,g) \cdot \dot{F} + \partial_\theta S(F,\theta,g)\dot{\theta} - \frac{1}{\rho_r}\frac{\partial}{\partial \theta}\left[\frac{\kappa_1(\theta)}{\theta}\right] g \cdot \dot{g}. \tag{2.10}$$

By Lemma 2.1 and Ψ being of class C^2 in its domain, one may infer what follows

$$\frac{1}{\rho_r}\partial_\theta \Sigma(F,\theta) = -\partial_F S(F,\theta,g). \tag{2.11}$$

Taking account of (2.11) and both statements 1) and 2) from Lemma 2.1 into (2.10) we get

$$\begin{aligned} \rho_r \theta \dot{s} = &- \theta \left\{ \partial_\theta \Sigma(F,\theta) \cdot \dot{F} + \left[\partial_\theta^2 \Psi^I(F,\theta) + \frac{d^2}{d\theta^2}\left(\frac{\kappa_1(\theta)}{\theta}\right) \frac{g \cdot g}{2} \right] \dot{\theta} \right. \\ &\left. + \frac{d}{d\theta}\left[\frac{\kappa_1(\theta)}{\theta}\right] g \cdot \dot{g} \right\}. \end{aligned} \tag{2.12}$$

Inserting (2.12) into (2.9) leads to the required equation

$$\begin{aligned} &- \theta \left\{ \partial_\theta \Sigma(F,\theta) \cdot \dot{F} + \left[\partial_\theta^2 \Psi^I(F,\theta) + \frac{d^2}{d\theta^2}\left(\frac{\kappa_1(\theta)}{\theta}\right) \frac{g \cdot g}{2} \right] \dot{\theta} \right\} - 2\kappa_1'(\theta) g \cdot \dot{g} \\ &+ \frac{2\kappa_1(\theta)}{\theta} g \cdot \dot{g} - \mathrm{div}\left(\kappa_0(\theta)\nabla\theta\right) - \kappa_1(\theta)\mathrm{div}\dot{g} = \rho_r r, \end{aligned} \tag{2.13}$$

where the superposed prime in κ_1' stands for the derivative of κ_1.

2.3. Derivation of the linear theory.

In order to derive the linear theory of generalized thermoelasticity, it is convenient to put restrictions on κ_1. We form them into the following Cauchy problem:

$$\kappa_1'(\theta) = \frac{\kappa_1(\theta)}{\theta}, \quad \kappa_1(\theta_r) = \kappa_{1r}, \tag{2.14}$$

where κ_{1r} is a given nonnegative number. Its solution is of the form

$$\kappa_1(\theta) = \kappa_r \theta \quad \text{with} \quad \kappa_r = \kappa_{1r}/\theta_r. \tag{2.15}$$

On substituting (2.15) into (2.13) we arrive at the reduced equation

$$\begin{aligned} &-\theta\lfloor\partial_\theta\Sigma(F,\theta)\cdot\dot{F}+\partial_\theta^2\Psi^I(F,\theta)\dot{\theta}\rfloor-\operatorname{div}\left(\kappa_0(\theta)\nabla\theta\right)-\kappa_r\theta\operatorname{div}\dot{g}\\ &=\rho_r r. \end{aligned} \tag{2.16}$$

We are now in a position to determine the linear approximation to this system under the following assumptions: the material in question is *isotropic*; the displacement gradient and its rate of change are small; the temperature field nearly equals a given *uniform* temperature θ_r called the *reference temperature*; the temperature rate and temperature gradient are small; the function $\kappa_0\colon [0,\infty[\to]0,\infty[$ representing the *heat conductivity* is equal to a positive constant κ; i.e.,

$$\begin{aligned} &|\ \nabla U\ |,\ |\ \nabla\dot{U}\ |,\ |\ \theta-\theta_r\ |,\ |\ \dot{\theta}\ |,\ |\ g\ |\le\delta_r\\ &\qquad\qquad\qquad\qquad\theta_r\equiv\text{const and }\kappa_0(\theta)=\kappa>0. \end{aligned} \tag{2.17}$$

We can now proceed analogously to the procedure carried over in [6] to derive the **basic equations of linearized thermoelasticity theory**:

$$\begin{aligned} &\rho_r\ddot{U}-\operatorname{div}\Sigma=\rho_r b\\ &\alpha\dot{\theta}+\operatorname{div}q=-\eta\cdot\operatorname{tr}\dot{E}+\rho_r r \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} &\Sigma=2\mu E+\lambda(\operatorname{tr}E)I-\gamma(\theta-\theta_r)I,\\ &E=\frac{1}{2}\left(\nabla U+\nabla U^T\right),\ q=-\kappa\nabla\theta-k\nabla\dot{\theta},\\ &\gamma=\alpha_e(3\lambda+2\mu),\ \eta=\theta_r\gamma,\ k=\theta_r\kappa_r, \end{aligned} \tag{2.19}$$

with α_e being the *coefficient of thermal expansion*, the superscript T denoting the transposition and the symbol **tr** standing for the trace of a matrix. According to (2.18) and (2.19) we were able to justify that the problem posed in Section 1 is consistent with the second law of thermodynamics. As is seen from the considerations carried out above, the classical Fourier law has been modified by a term describing heat conduction with short thermal memory (see [8], [22]). Such a modification turns out to be helpful to get a more regular solution to the problem posed in the previous section, which may be seen from a priori estimates in the next section. It is worth while mentioning here that only the case where k vanishes is investigated in the thesis [4]. Another approach dealing with generalized thermal conductivity is researched by J. Ignaczak in [15]. In his paper the heat flux is defined by the ordinary differential equation of first order of the form

$$\tau_0\dot{q}+q=-\kappa\nabla\vartheta$$

where $\tau_0 > 0$ is a *thermal relaxation time*. That assumption changes the type of the energy equation into the hyperbolic one instead of the parabolic one.

3. A priori bounds

In this section two a priori estimates for the solutions to problem (1.8)–(1.13) are derived. The first estimate is a main tool for proving existence of global weak solutions, whereas the second one is convenient to show their regularity. To this end let us define

$$P_0\left(u(\cdot,t)\right) := \frac{2}{\xi+1}\int_{\Gamma_c} p_0(\cdot)(u_\nu)_+^{\xi+1} d\Gamma. \tag{3.1}$$

Lemma 3.1. *Let the hypotheses of Theorem* 1.2 *be fulfilled. If* (u,ϑ) *is a sufficiently smooth solution to problem* (1.8)–(1.13)*, then for all* $0 < T < \infty$ *the first a priori estimate*

$$\begin{aligned}&\frac{\eta}{2}\left[\frac{\gamma}{\eta}\min(\alpha,k) \parallel \vartheta(\cdot,t) \parallel_V^2 + \rho_r \parallel \partial_t u(\cdot,t) \parallel^2 + \frac{m_a}{2} \parallel u(\cdot,t) \parallel_V^2 + P_0\left(u(\cdot,t)\right)\right] \\ &+ \kappa\gamma\int_0^t \parallel \nabla\vartheta(\cdot,s) \parallel^2 ds + \gamma m_\beta \int_0^t\int_{\Gamma'_d} \mid \vartheta(\cdot,s) \mid^2 d\Gamma\, ds \le K_1\end{aligned} \tag{3.2}$$

holds, where positive constant K_1 *depends only on the given data. Moreover, the second a priori estimate*

$$\begin{aligned}&\frac{\eta}{2}\left[\frac{\gamma}{\eta}\min(\alpha,k) \parallel \dot\vartheta(\cdot,t) \parallel_V^2 + \rho_r \parallel \partial_t \dot u(\cdot,t) \parallel^2 + \frac{m_a}{2} \parallel \dot u(\cdot,t) \parallel_V^2 + P_0\left(\dot u(\cdot,t)\right)\right] \\ &+ \kappa\gamma\int_0^t \parallel \nabla\dot\vartheta(\cdot,s) \parallel^2 ds + \gamma m_\beta \int_0^t\int_{\Gamma'_d} \mid \dot\vartheta(\cdot,s) \mid^2 d\Gamma\, ds \le K_2\end{aligned} \tag{3.3}$$

is valid with positive constant K_2 *depending on the given data provided that the assumptions of Theorem* 1.3 *are fulfilled and* (u,ϑ) *is a sufficiently smooth solution to* (1.8)–(1.13)*.*

Proof. Multiplying (1.8) and (1.9) by $\eta\partial_t u$ and $\gamma\vartheta$, respectively, then integrating the resulting identities over Ω, and, in turn, adding up the recently obtained equalities we find

$$\begin{aligned}&\frac{\eta}{2}\frac{d}{dt}\left[\frac{\gamma}{\eta}\left(\alpha \parallel \vartheta(\cdot,t) \parallel^2 + k \parallel \nabla\vartheta(\cdot,t) \parallel^2\right) + \rho_r \parallel \partial_t u(\cdot,t) \parallel^2 + a\left(u(\cdot,t), u(\cdot,t)\right)\right. \\ &\quad \left. + P_0\left(u(\cdot,t)\right)\right] + \kappa\gamma \parallel \nabla\vartheta(\cdot,t) \parallel^2 + \gamma\int_{\Gamma'_d} \beta(x,u_\nu) \mid \vartheta \mid^2 dA\end{aligned}$$

$$= \eta \frac{d}{dt} \int_{\Gamma_f} f \cdot u \, dA - \eta \int_{\Gamma_f} \partial_t f \cdot u \, dA + \eta \frac{d}{dt} \int_{\Omega} h \cdot u \, dA - \eta \int_{\Omega} \partial_t h \cdot u \, dx$$
$$+ \gamma \int_{\Omega} \omega \vartheta \, dx + \gamma \int_{\Gamma'_d} \vartheta \zeta(x, u_\nu) \, dA. \tag{3.4}$$

Integrating (3.4) with respect to time over $]0, t[$, using the coercivity and continuity properties of $a(\cdot, \cdot)$, applying the Hölder and Young inequalities and taking into account the inequality coming from the trace theorem leads to what follows:

$$\frac{\eta}{2} \left[\frac{\gamma}{\eta} \min(\alpha, k) \parallel \vartheta(\cdot, t) \parallel_V^2 + \rho_r \parallel \partial_t u(\cdot, t) \parallel^2 + m_a \parallel u(\cdot, t) \parallel_V^2 + P_0 \left(u(\cdot, t) \right) \right]$$
$$+ \kappa \gamma \int_0^t \parallel \nabla \vartheta(\cdot, s) \parallel^2 ds + \gamma m_\beta \int_0^t \int_{\Gamma'_d} \mid \vartheta(\cdot, s) \mid^2 d\Gamma \, ds \tag{3.5}$$
$$\leq c + \frac{\eta}{2\delta_2} \parallel h(\cdot, t) \parallel_{V^*}^2 + \frac{\eta}{2\delta_1} \parallel f(\cdot, t) \parallel_{L_p(\Gamma_f)}^2 + \frac{\eta \left[\delta_1 C_e^2 + \delta_2 \right]}{2} \parallel u(\cdot, t) \parallel_V^2$$
$$+ \frac{\eta}{2} \int_0^t \left[\parallel \partial_s h(\cdot, s) \parallel_{V^*}^2 + \parallel \partial_s f(\cdot, s) \parallel_{L_p(\Gamma_f)}^2 + \left(1 + C_e^2 \right) \parallel u(\cdot, s) \parallel_V^2 \right] ds$$
$$+ \frac{\gamma}{2} \int_0^t \left[\parallel \omega(\cdot, s) \parallel_{V^*}^2 + \parallel \zeta(x, u_\nu) \parallel_{L_p(\Gamma_f)}^2 + \left(1 + C_e^2 \right) \parallel \vartheta(\cdot, s) \parallel_V^2 \right] ds$$

where δ_j $(j = 1, 2)$ are positive parameters to be chosen, c is a constant dependent on the data and C_e is a constant occurring in

$$\parallel u(\cdot, t) \parallel_{L_q(\Gamma)}^2 \leq C_e^2 \parallel u(\cdot, t) \parallel_V^2$$

with $q \in [1, \infty[$ for $N = 2$ or $q \in [1, 4]$ for $N = 3$ (cf. [16, Chapter 6, Theorems 6.4.1 and 6.4.2]), $p = q/(q-1)$. Let us choose δ_j $(j = 1, 2)$ in such a way that

$$\delta_1 = \frac{m_a}{4C_e^2} \quad \text{and} \quad \delta_2 = \frac{m_a}{4}.$$

Then application of the Gronwall inequality to (3.5) furnishes the estimate (3.2).

Means of achieving the inequality (3.3) are similar to those of obtaining (3.2), so that we give only a sketch of its derivation. To this end, differentiate (1.8) and (1.9) with respect to time, take the scalar product in $L^N(\Omega)$ and $L(\Omega)$ of the resulting equations with $\eta \partial_t \dot{u}$ and $\gamma \dot{\vartheta}$, respectively, and carry out the procedure as that at the beginning of this proof to get

$$\begin{aligned}
&\frac{\eta}{2}\frac{d}{dt}\left[\frac{\gamma}{\eta}\left(\alpha \parallel \dot{\vartheta}(\cdot,t) \parallel^2 + k \parallel \nabla\dot{\vartheta}(\cdot,t) \parallel^2\right) + \rho_r \parallel \partial_t \dot{u}(\cdot,t) \parallel^2 + a\left(\dot{u}(\cdot,t), \dot{u}(\cdot,t)\right)\right.\\
&\quad \left. + P_0\left(\dot{u}(\cdot,t)\right)\right] + \kappa\gamma \parallel \nabla\dot{\vartheta}(\cdot,t) \parallel^2 + \gamma \int_{\Gamma'_d} \beta(\cdot,u_\nu) \mid \dot{\vartheta} \mid^2 dA\\
&= \eta\frac{d}{dt}\int_{\Gamma_f} \dot{f}\cdot\dot{u}\,dA - \eta\int_{\Gamma_f} \partial_t\dot{f}\cdot\dot{u}\,dA + \eta\frac{d}{dt}\int_\Omega \dot{h}\cdot\dot{u}\,dA - \eta\int_\Omega \partial_t\dot{h}\cdot\dot{u}\,dx\\
&\quad + \gamma\int_\Omega \dot{\omega}\dot{\vartheta}\,dx + \gamma\int_{\Gamma'_d}\left[-\beta_{u_\nu}(x,u_\nu)\vartheta\dot{\vartheta}\dot{u}_\nu + \zeta_{u_\nu}(x,u_\nu)\dot{\vartheta}\dot{u}_\nu\right] dA. \qquad (3.6)
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{\eta}{2}\left[\frac{\gamma}{\eta}\min(\alpha,k) \parallel \dot{\vartheta}(\cdot,t) \parallel_V^2 + \rho_r \parallel \partial_t\dot{u}(\cdot,t) \parallel^2 + m_a \parallel \dot{u}(\cdot,t) \parallel_V^2 + P_0\left(\dot{u}(\cdot,t)\right)\right]\\
&+ \kappa\gamma\int_0^t \parallel \nabla\dot{\vartheta}(\cdot,s) \parallel^2 ds + \gamma m_\beta \int_0^t \int_{\Gamma'_d} \mid \dot{\vartheta}(\cdot,s) \mid^2 d\Gamma\, ds\\
&\le c_1 + \frac{\eta}{2\delta_4} \parallel \dot{h}(\cdot,t) \parallel_{V^*}^2 + \frac{\eta}{2\delta_3} \parallel \dot{f}(\cdot,t) \parallel_{L_p(\Gamma_f)}^2 + \frac{\eta\left[\delta_3 C_e^2 + \delta_4\right]}{2} \parallel \dot{u}(\cdot,t) \parallel_V^2\\
&+ \frac{\eta}{2}\int_0^t \left[\parallel \partial_s^2 h(\cdot,s) \parallel_{V^*}^2 + \parallel \partial_s^2 f(\cdot,s) \parallel_{L_p(\Gamma_f)}^2 + \left(1 + C_e^2\right) \parallel \dot{u}(\cdot,s) \parallel_V^2\right] ds\\
&+ \frac{\gamma}{2}\int_0^t \left[\parallel \dot{\omega}(\cdot,s) \parallel_{V^*}^2 + C_e^2 \parallel \dot{\vartheta}(\cdot,s) \parallel_V^2\right] ds \qquad (3.7)\\
&+ \frac{\gamma}{2}\int_0^t \left[\zeta_1 + \beta_1 \parallel \vartheta(\cdot,s) \parallel_{L_{r_1}(\Gamma'_d)}\right]\left[\parallel \dot{\vartheta}(\cdot,s) \parallel_{L_{p_1}(\Gamma'_d)}^2 + \parallel \dot{u}_\nu(\cdot,s) \parallel_{L_{q_1}(\Gamma'_d)}^2\right] ds,
\end{aligned}$$

where $p_1^{-1} + q_1^{-1} + r_1^{-1} = 1$ $(p_1, q_1, r_1, \ge 1)$. We will use the following inequality concerning the property of the trace operator

$$\parallel \vartheta(\cdot,s) \parallel_{L_{r_1}(\Gamma'_d)} \le C_e \parallel \vartheta(\cdot,s) \parallel_V \le C_e\sqrt{\frac{2K_1}{\gamma\min(\alpha,K)}} \quad \text{for all } s \ge 0 \qquad (3.8)$$

with $1 \le r_1 < \infty$ for $N = 2$ or $r_1 \in [1,4]$ for $N = 3$, where the estimate (3.2) was used. By (3.8) and by choosing δ_j $(j = 3,4)$ in such a way that

$$\delta_3 = \frac{m_a}{4C_e^2} \quad \text{and} \quad \delta_4 = \frac{m_a}{4},$$

and applying the Gronwall lemma we may arrive at (3.3), which is possible under appropriate selections of p_1, q_1 and r_1 in (3.7). This concludes the proof of the lemma. □

4. Construction of a solution to the problem

In this section the existence of a solution to the problem posed in Section 1 will be established by means of the Faedo-Galerkin method with some *a priori* estimates and the compactness method.

Let $\{w^k\}_{k=1}^{\infty}$ and $\{e^k\}_{k=1}^{\infty}$ be a basis for V and for V^N, respectively, such that

$$(w^j, w^k)_\Omega + (\nabla w^j, \nabla w^k)_\Omega = \delta_{jk},$$
$$(e^j, e^k)_\Omega = \delta_{jk}.$$

We look for an approximate solution to problem (1.8)–(1.13) as a sequence of pairs $\{u^n, \vartheta^n\}_{n=1}^{\infty}$ such that

$$u^n(x,t) := \sum_{k=1}^{n} c_k^n(t) e^k(x) \tag{4.1}$$

$$\vartheta^n(x,t) := \sum_{k=1}^{n} d_k^n(t) w^k(x) \tag{4.2}$$

$$\begin{aligned}&\rho_r \frac{d^2 c_j^n(t)}{dt^2} + a(u^n, e^j) - \gamma(\vartheta^n, \mathrm{div} e^j)_\Omega \\ &- \langle f, e^j\rangle_{\Gamma_f} + \langle P\left(u^n(t)\right), e^j\rangle_{\Gamma_c} = \langle h, e^j\rangle_V\end{aligned} \tag{4.3}$$

$$\begin{aligned}&\frac{d d_i^n(t)}{dt}\left[\alpha(w^i, w^j)_\Omega + k(\nabla w^i, \nabla w^j)_\Omega\right] + \kappa\left(\nabla\vartheta^n, \nabla w^j\right)_\Omega \\ &+ \langle\beta(u_\nu^n)\vartheta^n - \zeta(\cdot, u_\nu^n), w^j\rangle_{\Gamma_d'} + \left(\eta \mathrm{div}(\partial_t u^n) - \omega, w^j\right)_\Omega = 0\end{aligned} \tag{4.4}$$

$$c_j^n(0) = c_j^0,\ \frac{d}{dt} c_j^n(0) = \bar{c}_j^0,\ d_j^n(0) = d_j^0 \quad (j = 1, 2, \ldots, n), \tag{4.5}$$

where

$$\begin{aligned}u_0(x) &:= \sum_{k=1}^{\infty} (u_0, e^k) e^k(x) =: \sum_{k=1}^{\infty} c_k^0 e^k(x),\\ u_1(x) &:= \sum_{k=1}^{\infty} (u_1, e^k) e^k(x) =: \sum_{k=1}^{\infty} \bar{c}_k^0 e^k(x),\\ \vartheta_0(x) &:= \sum_{k=1}^{\infty} (\vartheta_0, w^k) w^k(x) =: \sum_{k=1}^{\infty} d_k^0 w^k(x).\end{aligned} \tag{4.6}$$

The existence of $\{u^n, \vartheta^n\}_{n=1}^{\infty}$ may be established by the standard argument of ordinary differential equations. Indeed, the implicit part of the

Galerkin equations (4.4) may be expressed by

$$\mathbf{A}^n \begin{bmatrix} \dot{d}_1^n(t) \\ \vdots \\ \dot{d}_n^n(t) \end{bmatrix} = \begin{bmatrix} F_1^n\left(t, c^n(t), \dot{c}^n(t), d^n(t)\right) \\ \vdots \\ F_n^n\left(t, c^n(t), \dot{c}^n(t), d^n(t)\right) \end{bmatrix} \tag{4.7}$$

where $F^n \colon [0,T] \times \mathbb{R}^{3n} \to \mathbb{R}$ is globally Lipschitz continuous, and the entries of the matrix $\mathbf{A}^n$ are defined by

$$a_{ij}^n := \left[\alpha(w^i, w^j)_\Omega + k(\nabla w^i, \nabla w^j)_\Omega\right]_{i,j=1}^n . \tag{4.8}$$

This matrix may be inverted because of

$$\det(\mathbf{A}^n) = \det\left(\left[\alpha(w^i, w^j)_\Omega + k(\nabla w^i, \nabla w^j)_\Omega\right]_{i,j=1}^n\right) \geq \min(\alpha, k) \tag{4.9}$$

by the orthogonality of w^i and w^j in $H = L_2(\Omega)$, and ∇w^i and ∇w^j in H^N. By the Picard theorem, there exists $t_0 > 0$ such that the initial problem (4.3)–(4.5) has a unique solution on $[0, t_0[$. We intend to show that in fact the solution exists on $[0,T]$. To this end, we carry out the same procedure as that in Section 3 to derive the estimates for $\{u^n, \vartheta^n\}_{n=1}^\infty$ instead of (u, ϑ) occurring in Lemma 3.1. First, we multiply (4.3) by $dc_j^n(t)/dt$ and (4.4) by $d_j^n(t)$, respectively, and then sum up the resulting equalities with respect to j ranging from 1 to n. Next we perform integration by parts to obtain the identities

$$\begin{aligned} &\frac{\rho_r}{2}\frac{d}{dt} \parallel \partial_t u^n \parallel^2 + a(u^n, \partial_t u^n) - \gamma(\vartheta^n, \mathrm{div}\partial_t u^n)_\Omega \\ &- \langle f, \partial_t u^n\rangle_{\Gamma_f} + \langle P\left(u^n(t)\right), \partial_t u^n\rangle_{\Gamma_c} = \langle h, \partial_t u^n\rangle_V \end{aligned} \tag{4.10}$$

$$\begin{aligned} &\frac{\alpha}{2}\frac{d}{dt} \parallel \vartheta^n \parallel^2 + \kappa \parallel \nabla\vartheta^n \parallel^2 + \frac{k}{2}\frac{d}{dt} \parallel \nabla\vartheta^n \parallel^2 \\ &+ \langle \beta(\cdot, u_\nu^n)\vartheta^n - \zeta(\cdot, u_\nu^n), \vartheta^n\rangle_{L_2(\Gamma_d')} + (\eta \mathrm{div}(\partial_t u^n) - \omega, \vartheta^n)_\Omega = 0. \end{aligned} \tag{4.11}$$

We are now in a position to proceed as in Lemma 3.1 from Section 3 to derive the desired estimates:

$$\begin{aligned} &\frac{\eta}{2}\left[\frac{\gamma}{\eta}\min(\alpha, k) \parallel \vartheta^n(\cdot, t) \parallel_V^2 + \rho_r \parallel \partial_t u^n(\cdot, t) \parallel^2 + \frac{m_a}{2} \parallel u^n(\cdot, t) \parallel_V^2 \right. \\ &\quad \left. + P_0\left(u^n(\cdot, t)\right)\right] + \kappa\gamma \int_0^t \parallel \nabla\vartheta^n(\cdot, s) \parallel^2 ds + \gamma m_\beta \int_0^t \parallel \vartheta^n(\cdot, s) \parallel_{2,\Gamma_d'}^2 ds \\ &\leq K_1, \end{aligned} \tag{4.12}$$

where positive constant K_1 depends only on the given data. By the Banach-Alaoglu theorem (cf. [24], [27]), from these estimates it follows that there exists a subsequence of $\{u^n, \vartheta^n\}_{n=1}^\infty$, also denoted by $\{u^n, \vartheta^n\}_{n=1}^\infty$, and a pair of functions (u, θ) such that

$$u^n \xrightarrow[n\to\infty]{} u \text{ weak-star in } L_\infty(0,T;V^N), \tag{4.13}$$

$$\partial_t u^n \xrightarrow[n\to\infty]{} \partial_t u \text{ weak-star in } L_\infty(0,T;L_2(\Omega)^N), \tag{4.14}$$

$$\vartheta^n \xrightarrow[n\to\infty]{} \vartheta \text{ weak-star in } L_\infty(0,T;V). \tag{4.15}$$

Lemma 4.1.

$$(\partial_t^2 u^n)_{n=1}^1 \textit{ is uniformly bounded in } L_\infty(0,T;V^{*N}). \tag{4.16}$$

Proof. We start with the identity

$$\begin{aligned} &\rho_r\langle\partial_t^2 u^n,\varphi\rangle_V + a(u^n,\varphi) - \gamma\,(\vartheta^n,\mathrm{div}\varphi)_\Omega - \langle f,\varphi\rangle_{\Gamma_f} \\ &+ \langle P\,(u^n(t))\,,\varphi\rangle_{\Gamma_c} = \langle h,\varphi\rangle_V, \end{aligned} \tag{4.17}$$

where $\varphi(x) := \sum_1^n d_k e^k(x)$ with any $d_k \in \mathbb{R}^N$ $(k = 1, 2, \dots)$. Hence, we can deduce the estimate

$$\begin{aligned} &\rho_r\langle\partial_t^2 u^n,\varphi\rangle_V \le a(u^n,\varphi) + \gamma \mid (\vartheta^n,\mathrm{div}\varphi)_\Omega \mid \\ &\quad + \mid \langle f,\varphi\rangle_{\Gamma_f} \mid + \mid \langle P\,(u^n(t))\,,\varphi\rangle_{\Gamma_c} \mid + \mid \langle h,\varphi\rangle_V \mid \\ &\le M_a \parallel u^n(\cdot,t) \parallel_V \parallel \varphi \parallel_V + C_e^{1+\xi} \parallel p_0 \parallel_\infty \parallel u^n(\cdot,t) \parallel_V^\xi \parallel \varphi \parallel_V \\ &\quad + \gamma \parallel \vartheta^n(\cdot,t) \parallel \parallel \varphi \parallel_V + C_e \parallel f(\cdot,t) \parallel_{L_p(\Gamma_f)} \parallel \varphi \parallel_V \\ &\quad + \parallel h(\cdot,t) \parallel_{V^*} \parallel \varphi \parallel_V \le K_3 \parallel \varphi \parallel_V \end{aligned} \tag{4.18}$$

where the Hölder and embedding inequalities and the continuity of the bilinear form $a(\cdot,\cdot)$ were used; here constant K_3 is independent of n. From (4.18) and the definition of V^*-norm it follows that

$$\parallel \partial_t^2 u^n(\cdot,t) \parallel_{V^*} := \sup_{\|\varphi\|_V=1} \frac{\langle\partial_t^2 u^n,\varphi\rangle_V}{\parallel \varphi \parallel_V} \le \frac{K_3}{\rho_r} \text{ for a.a. } t \text{ in } [0,T], \tag{4.19}$$

which proves the lemma. □

Lemma 4.2. *There is a subsequence of $(u^n)_1^\infty$, still denoted by $(u^n)_1^\infty$, such that*

$$p_0(\cdot)(u_\nu^n)_+^\xi \xrightarrow[n\to\infty]{} p_0(\cdot)(u_\nu)_+^\xi \textit{ weak-star in } L_\infty\left(0,T;L_p(\Gamma_c)\right). \tag{4.20}$$

Proof. Consider any $\varphi \in L_1(0,T;V)$. This function may be represented as $\varphi(x,t) = \sum_1^\infty d_l(t)e^l(x)$. By the Hölder inequality one may deduce the uniform boundedness on $[0,T]$ of what follows

$$\varphi(\cdot,t) \to \int_{\Gamma_c} p_0(\cdot)(u_\nu^n)_+^\xi \varphi_\nu \, dA$$
$$\leq \| p_0(\cdot) \|_{L_\infty(\Omega)} \| (u_\nu^n)_+ \|_{L^\xi_{p\xi}(\Gamma_c)} \| \varphi_\nu \|_{L_q(\Gamma_c)} \leq c \, \| \varphi(\cdot,t) \|_V \tag{4.21}$$

for $n = 1, 2, \dots$, where $p = q/(q-1)$ with

$$q \in \begin{cases} [1,\infty[& \text{for } N = 2, \\ [1,4] & \text{for } N = 3 \end{cases}$$

and c is a constant independent of n and t. Indeed, by the property of the trace theorem, one may conclude the uniform boundedness of $\| (u_\nu^n(\cdot,t))_+ \|_{L^\xi_{p\xi}(\Gamma_c)}$ with respect to both $n \in \mathbb{N}$ and $t \in [0,T]$:

$$\left(\int_{\Gamma_c} (u_\nu^n(\cdot,t))_+^{q_1} \, d\Gamma \right)^{1/q_1} \leq \left(\int_{\Gamma_c} | \, u^n(\cdot,t) \, |^{q_1} \, d\Gamma \right)^{1/q_1}$$
$$\leq C \, \| u^n(\cdot,t) \|_V \leq C_e \sqrt{\frac{4K_1}{m_a \eta}},$$

where

$$q_1 \in \begin{cases} [1,\infty[& \text{for } N = 2, \\ [1,4] & \text{for } N = 3. \end{cases}$$

Therefore putting $p\xi = q_1$ results in $\xi = q_1(q-1)/q$. Hence the weak-star convergence of some subsequence of $(u^n)_1^\infty$, still denoted by $(u^n)_1^\infty$, is furnished

$$p_0(\cdot)(u_\nu^n)_+^\xi \xrightarrow[n\to\infty]{} \chi \text{ weak-star in } L_\infty\left(0,T; L_p(\Gamma_c)\right) \tag{4.22}$$

with some χ in $L_\infty\left(0,T; L_p(\Gamma_c)\right)$.

It remains to show the validity of the following equality

$$\chi = p_0(\cdot)(u_\nu)_+^\xi \text{ a.e. on } \Gamma_c \times [0,T]. \tag{4.23}$$

The proof of (4.23) follows the same way as that in [5], therefore we give only its sketch. By the trace theorem ([16, Chapter 6]) the trace mapping

$$\gamma \colon H^1(Q) \xrightarrow{\text{onto}} H^{1/2}(\partial Q) \subset L_\rho(\partial Q)$$

is compact provided that

$$\rho \in [1,4[\text{ for } N = 2 \quad \text{or} \quad \rho \in [1,3[\text{ for } N = 3.$$

From the estimates

$$\| \partial_t u^n(t) \|_{L_2(\Omega)} + \| u^n(t) \|_V \leq c, \; t \in [0,T], \tag{4.24}$$

where constant c is independent of n, it follows that there exists a subsequence of $(u^n)_1^\infty$, also denoted by $(u^n)_1^\infty$, such that

$$\| u^n - u \|_{L_\rho(\partial Q)} \xrightarrow[n\to\infty]{} 0 \text{ and } u^n \xrightarrow[n\to\infty]{} u \text{ a.e. on } \partial Q. \tag{4.25}$$

Hence it follows that

$$p_0(\cdot)(u_\nu^n)_+^\xi \xrightarrow[n\to\infty]{} p_0(\cdot)(u_\nu)_+^\xi \text{ a.e. on } \Gamma_c \times [0,T].$$

Moreover, the boundedness of $(u^n)_1^\infty$ in $L_\infty(0,T;L_p(\Gamma_c))$ implies its boundedness in $L_p(\Gamma_c\times]0,T[)$. Application of Lemma 1.6 in [19] conduces to the convergence

$$p_0(\cdot)(u_\nu^n)_+^\xi \xrightarrow[n\to\infty]{} p_0(\cdot)(u_\nu)_+^\xi \text{ weakly in } L_p(\Gamma_c\times]0,T]). \tag{4.26}$$

On the other hand, from (4.22) the convergence

$$p_0(\cdot)(u_\nu^n)_+^\xi \xrightarrow[n\to\infty]{} \chi \text{ weakly in } L_p(\Gamma_c\times]0,T[) \tag{4.27}$$

is ensured. But the uniqueness of weak limits in (4.26) and (4.27) leads to the assertion of the lemma, which completes its proof. □

Lemma 4.3. *There exists a subsequence of* $\{u^n,\vartheta^n\}_{n=1}^\infty$*, also denoted by* $\{u^n,\vartheta^n\}_{n=1}^\infty$*, such that the following identities*

$$\begin{aligned} &\int_0^T \int_{\Gamma_d'} [\beta(\cdot,u_\nu^n)\vartheta^n - \beta(\cdot,u_\nu)\vartheta]\,\psi\,dA\,dt \xrightarrow[n\to\infty]{} 0 \\ &\int_0^T \int_{\Gamma_d'} [\zeta(\cdot,u_\nu^n) - \zeta(\cdot,u_\nu)]\,\psi\,dA\,dt \xrightarrow[n\to\infty]{} 0 \end{aligned} \tag{4.28}$$

hold for any $\psi \in H^1(0,T;V)$.

Proof. We start with the following identities

$$\begin{aligned} I^n &:= \int_0^T \int_{\Gamma_d'} [\beta(\cdot,u_\nu^n)\vartheta^n - \beta(\cdot,u_\nu)\vartheta]\,\psi\,dA\,dt \\ &= \int_0^T \int_{\Gamma_d'} [\beta(\cdot,u_\nu)(\vartheta^n-\vartheta)\psi + (\beta(\cdot,u_\nu^n)-\beta(\cdot,u_n))\,\vartheta^n\psi]\,dA\,dt \\ &=: I_1^n + I_2^n. \end{aligned}$$

The convergence of $(I_1^{n_k})_{k=1}^\infty$ to zero may be deduced from

$$\vartheta^{n_k} \xrightarrow[k\to\infty]{} \vartheta \text{ strongly in } L_2(\Gamma_d'\times]0,T[)$$

(see Lemma 3 in [13]) for some subsequence $(n_k)_{k=1}^\infty$

$$| I_1^{n_k} | \le \| \beta(\cdot,u_\nu) \|_{L_\infty(\Gamma_d'\times]0,T[)} \| \vartheta^{n_k} - \vartheta \|_{L_2(\Gamma_d'\times]0,T[)} \| \psi \|_{L_2(\Gamma_d'\times]0,T[)} \xrightarrow[k\to\infty]{} 0.$$

Therefore, it remains to prove the convergence to zero of $(I_2^{n_k})_{k=1}^{\infty}$.

From (4.25) one may easily deduce that

$$\begin{aligned} &\| \beta(\cdot, u_\nu^{n_k}) - \beta(\cdot, u_\nu) \|_{L_\rho(\Gamma_d')\times]0,T[)} \\ &\le l_\beta \| u^{n_k} - u \|_{L_\rho(\Gamma_d'\times]0,T[)} \xrightarrow[k\to\infty]{} 0. \end{aligned} \tag{4.29}$$

By the Hölder inequality and (4.29), we get what follows:

$$\begin{aligned} | I_2^{n_k} | &\le \int_0^T \| \beta(\cdot, u_\nu^{n_k}) - \beta(\cdot, u_\nu) \|_{L_\rho(\Gamma_d')} \| \vartheta^{n_k} \|_{L_l(\Gamma_d')} \| \psi \|_{L_m(\Gamma_d')} \, dt \\ &\le C \operatorname*{ess\,sup}_{t\le T} \| \vartheta^{n_k} \|_V \int_0^T \| \beta(\cdot, u_\nu^{n_k}) - \beta(\cdot, u_\nu) \|_{L_\rho(\Gamma_d')} \| \psi \|_{L_m(\Gamma_d')} \, dt \\ &\le C_1 \| u_\nu^{n_k} - u_\nu \|_{L_\rho(\Gamma_d'\times]0,T[)} \| \psi \|_{L_{\rho/(\rho-1)}\left(0,T;L_m(\Gamma_d')\right)} \xrightarrow[k\to\infty]{} 0 \end{aligned}$$

where $1/\rho + 1/l + 1/m = 1$ $(\rho, l, m \ge 1)$ and the inequality (cf. [18, Chapter 3, Section 3])

$$\| w \|_{L_l(\Gamma_d')} \le C \| w \|_V \quad \text{for any } w \in V$$

with

$$l \in [1, 2(N-1)/(N-2)] \text{ for } N > 2 \text{ or } l \in [1, \infty[\text{ for } N = 2$$

was used. But there are some numbers $\rho, l, m \ge 1$ with $1/\rho + 1/l + 1/m = 1$ aforementioned such that the convergence holds for $\psi \in H^1(0, T; V)$ with $\| \psi \|_{L_{\rho/(\rho-1)}\left(0,T;L_m(\Gamma_d')\right)} \le c \| \psi \|_{H^1(0,T;V)}$, which concludes the proof. □

Proof of Theorem 1.2. We have to show that (u, θ) is a weak solution to the problem (1.8)–(1.13). This is done by a standard procedure. Firstly we shall prove the first identity in Definition 1.1. Multiplying (4.3) by $d_j(t) \in \mathbb{R}$ $(j = 1, 2, \dots, t \in [0, T])$ with $d_j(T) = 0$, summing up over j from 1 to n and then integrating by parts, we obtain

$$\begin{aligned} \int_0^T &\big[\rho_r \langle \partial_t^2 u^n, \varphi \rangle_V + a(u^n, \varphi) + \langle P(u^n), \varphi \rangle_{\Gamma_c} \\ &-\gamma(\vartheta^n, \operatorname{div}\varphi)_\Omega - \langle f, \varphi \rangle_{\Gamma_f} - \langle h, \varphi \rangle_V \big] \, dt = 0 \end{aligned} \tag{4.30}$$

where $\varphi(x, t) := \sum_1^n d_k(t) e^k(x)$. By Lemmas 4.1 and 4.2 there exists a subsequence of (u^n, ϑ^n), still denoted by (u^n, ϑ^n), such that (4.30) converges to the first identity from Definition 1.1 as n tends to infinity.

We now pass on to the proof of the second identity in Definition 1.1. To accomplish this, we multiply (4.4) by $a_j(t) \in \mathbb{R}$ $(j = 1, 2, \dots)$, sum up over

j from 1 to n and then integrate by parts to obtain

$$\begin{aligned}&\int_0^T [-(\alpha\vartheta^n + \eta \mathrm{div} u^n, \partial_t\psi) + \kappa(\nabla\vartheta^n, \nabla\psi) - k(\nabla\vartheta^n, \nabla\partial_t\psi) - \langle\omega, \psi\rangle_V]\, dt \\ &+ \int_0^T \int_{\Gamma'_d} [\beta(\cdot, u^n_\nu)\theta^n - \zeta(\cdot, u^n_\nu)]\, \psi\, dA\, dt \qquad (4.31) \\ &= \int_\Omega (\alpha\vartheta_0(x) + \eta\nabla\cdot u_0(x))\, \psi(x,0) dx + k\int_\Omega \nabla\vartheta_0(\cdot)\cdot\nabla\partial_t\psi(\cdot,0)dx\end{aligned}$$

for any $\psi \in H^1(0,T;V)$ with $\psi(\cdot,T) = 0$ for all $0 < T < \infty$, respectively. Thus, by Lemma 4.3 and (4.15), passing to the limit in (4.31) on a subsequence leads to (1.24), which completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. From what has already been proved and derived it may be concluded that similar considerations as that in the proof of Theorem 1.2 in connection with the estimate (3.3) from Lemma 3.1 apply to the case of regularity (see [17] for the well-known idea of the method applied in such a case), hence the proof is omitted. □

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References

[1] Adams, R.A., *Sobolev Spaces*, Academic Press, New York, 1975.

[2] Amann, H., *Parabolic evolution equations and nonlinear boundary conditions*, J. Differential Equations **72** (1988), 201–269.

[3] Andrews, K. T., Shi, P., Shillor, M., Wright, S., *Thermoelastic contact with Barber's heat exchange condition*, Appl. Math. Optim. **28** (1993), 11–48.

[4] Bień, M., *Global Weak Solutions for a Class of Problems in Mathematical Physics*, PhD thesis, Institute of Fundamental Technological Research of the Polish Academy of Sciences, Warsaw, 1998.

[5] Bień, M., *Existence of global weak solutions for coupled thermoelasticity under nonlinear boundary conditions*, Math. Methods Appl. Sci. **19** (1996), 1265–1277.

[6] Carlson, D. E., *Linear thermoelasticity*, in "Encyclopedia of Physics, Mechanics of Solids II", 6a/2, Springer, Berlin, 1972, 297–345.

[7] Cattaneo, C., *Sulla conduzione del calore*, Atti. Sem. Mat. Fis. Univ. Modena **3** (1948), 83–101.

[8] Cimmelli, V. A., Kosiński, W., *Nonequilibrium semi-empirical temperature in materials with thermal relaxation*, Arch. Mech. **43** (1991), 753–767.

[9] Day, W. A., *Heat Conduction within Linear Thermoelasicity*, Springer-Verlag, New York–Berlin, 1985.

[10] DiBenedetto, E., *Degenerate Parabolic Equations*, Springer-Verlag, New York, 1993.

[11] Duvaut, G., *Nonlinear boundary value problem in thermoelasticity*, Proceedings of the IUTAM Symposium on Finite Elasticity (Bethlehem, Pa., 1980) Nijhofft, The Hauge, 1982, 151–165.
[12] Elliott, C. M., Tang, Q. I., *A dynamic contact problem in thermoelasticity*, Nonlinear Anal. **23** (1994), 883–898.
[13] Filo, J., Kačur, J., *Local solutions of general nonlinear parabolic systems*, Nonlinear Anal. **24** (1995), 1597–1618.
[14] Friedman, A., *Partial Differential Equations*, Holt, Rinrhart and Winston, New York, 1969.
[15] Ignaczak, J., *Generalized thermoelasticity and its applications*, in "Thermal Stresses III", (ed. R. Hetnarski), Elsevier Science Publishers, Amsterdam, 1989, 279–354.
[16] Kufner, A., et al., *Function Spaces*, Noordhoff, Groningen, 1977.
[17] Ladyzhenskaya, O. A., *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 1969.
[18] Ladyzhenskaya, O. A., Solonnikov, V. A., Ural'ceva, N. N., *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Monographs **23** (1968), Amer. Math. Soc. Providence, RI.
[19] Lions, J. L., *Quelques Méthodes de Résolution des Problémes aux Limites Non-linéaires*, Dunrod Gauthier-Villars, Paris, 1969.
[20] Lions, J. L., Magenes, E., *Non-homogeneous Boundary Value Problems and Applications*, Springer-Verlag, New York-Berlin, 1972.
[21] Morrey, Ch. B, *Multiple Integrals in the Calculus of Variations*, Springer-Verlag, New York–Berlin., 1966.
[22] Niezgódka, M., Sprekels, J., *Existence of solutions for a mathematical model of structural phase transitions in shape memory alloys*, Math. Methods Appl. Sci. **10** (1988), 197–223.
[23] Nowacki, W., *Thermoelasticity*, 2nd edition, PWN-Polish Scientific Publishers, Warsaw, 1986.
[24] Rudin, W., *Functional Analysis*, McGraw-Hill, New York, 1971.
[25] Tanabe, H., *Equations of Evolutions*, Pitman, London, 1979.
[26] Wloka, J., *Partial Differential Equations*, Cambridge University Press, Cambridge, 1987.
[27] Yosida, K., *Functional Analysis*, 3rd edition, Springer-Verlag, New York–Berlin, 1971.

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