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ON MARCZEWSKI-BURSTIN REPRESENTATIONS OF ALGEBRAS AND IDEALS

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Abstract. We study MB-representations of algebras and ideals when they are relativized to a subset, and when one considers the operations of sum and intersection for families of algebras and ideals. We observe that the algebras Δ^0_α, $3 \leq \alpha < \omega_1$, on $\mathbb{R}$ are MB-representable under GCH. We find a class of topological spaces in which the algebra of clopen sets is MB-representable.

1. Introduction

Our notation is standard. (See [7].) Let $X \neq \emptyset$. Define two operations $S_X,\ S^0_X : \mathcal{P}(\mathcal{P}(X) \setminus \{\emptyset\}) \to \mathcal{P}(\mathcal{P}(X))$ given by

$$S_X(\mathcal{F}) = \{E \subset X : (\forall\ A \in \mathcal{F})\ (\exists B \in \mathcal{F})\ B \subset A \cap E\ \vee\ B \subset A \setminus E\},$$

$$S^0_X(\mathcal{F}) = \{E \subset X : (\forall\ A \in \mathcal{F})\ (\exists B \in \mathcal{F})\ \ B \subset A \setminus E\}.$$

In [2] (see also [19]) it was observed that $S_X(\mathcal{F})$ forms an algebra of sets and $S^0_X(\mathcal{F})$ forms an ideal of sets. Obviously $S^0_X(\mathcal{F}) \subset S_X(\mathcal{F})$ and $X \in S_X(\mathcal{F})$. Throughout the paper we assume that an algebra of subsets of X contains

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the set X. Operations S_X and S_X^0 were considered by Marczewski [22], and earlier by Burstin [6]. (See also [17].) The authors of [5], [2] and [1] proved that several algebras and ideals of subsets of X are of the form $S_X(\mathcal{F})$ and $S_X^0(\mathcal{F})$. This problem was also investigated for pairs $(\Sigma, \mathcal{I})$ where $\mathcal{I}$ is an ideal in an algebra Σ. If the respective representation exists, we say that Σ (or $(\Sigma, \mathcal{I})$) is *Marczewski-Burstin representable* or briefly, *MB-representable*. Some questions on this topic remain open, for instance it is still not known whether it can be proved in ZFC that the algebra of Borel subsets of $\mathbb{R}$ is MB-representable. (See [1].) In our paper we continue studies connected with MB-representations. If X is fixed, we write S and S^0 instead of S_X and S_X^0.

2. Relativization, sums and intersections

Let Σ be an algebra of subsets of X and let $\emptyset \neq Y \in \Sigma$. Then the family $\Sigma_Y = \{A \cap Y : A \in \Sigma\}$ forms an algebra of subsets of Y. This is a natural relativization of the algebra Σ to subsets of Y. Also, if $\mathcal{I} \subset \Sigma$ is an ideal of sets and $Y \notin \mathcal{I}$ then $\mathcal{I}_Y = \{A \cap Y : A \in \mathcal{I}\}$ forms an ideal of subsets of Y.

Theorem 1. *Assume that $\mathcal{F} \subset \mathcal{P}(X) \setminus \{\emptyset\}$, $\Sigma = S_X(\mathcal{F})$, $\mathcal{I} = S_X^0(\mathcal{F})$ and $Y \in \Sigma \setminus \mathcal{I}$. Then $\Sigma_Y = S_Y(\mathcal{F}_Y)$ and $\mathcal{I}_Y = S_Y^0(\mathcal{F}_Y)$ where $\mathcal{F}_Y = \mathcal{F} \cap \mathcal{P}(Y)$.*

Proof. Since $Y \in \Sigma$, we have $\Sigma_Y = \Sigma \cap \mathcal{P}(Y)$ and $\mathcal{I}_Y = \mathcal{I} \cap \mathcal{P}(Y)$.

We will show that $\Sigma_Y \subset S_Y(\mathcal{F}_Y)$. Let $E \in \Sigma_Y$, thus $E \in \Sigma$ and $E \subset Y$. Let $A \in \mathcal{F}_Y$, thus $A \in \mathcal{F}$ and $A \subset Y$. Since $E \in \Sigma = S_X(\mathcal{F})$, there is a $C \in \mathcal{F}$ such that either $C \subset A \cap E$ or $C \subset A \setminus E$. Obviously $C \in \mathcal{F}_Y$. Hence $E \in S_Y(\mathcal{F}_Y)$.

We will show that $S_Y(\mathcal{F}_Y) \subset \Sigma_Y$. Let $E \in S_Y(\mathcal{F}_Y)$. It is enough to prove that $E \in S_X(\mathcal{F})$. Let $A \in \mathcal{F}$. If there is a $B \in \mathcal{F}$ such that $B \subset A \cap Y$, then $B \in \mathcal{F}_Y$ and from $E \in S_Y(\mathcal{F}_Y)$ it follows that there is a $C \in \mathcal{F}_Y$ such that either $C \subset B \cap E$ or $C \subset B \setminus E$. Since $C \in \mathcal{F}_Y$, we have $C \in \mathcal{F}$. If there is no $B \in \mathcal{F}$ such that $B \subset A \cap Y$ then, since $Y \in \Sigma = S_X(\mathcal{F})$, there is a $C \in \mathcal{F}$ for which $C \subset A \setminus Y$. Hence $C \subset A \setminus E$. So $E \in S_X(\mathcal{F})$.

Analogously, one can show that $\mathcal{I}_Y = S_Y^0(\mathcal{F}_Y)$. □

Now, let us consider a kind of the inverse problem. Let $T \neq \emptyset$. For each $t \in T$, let Y_t be a nonempty subset of X. Assume that for each $t \in T$, an algebra Σ^t and an ideal $\mathcal{I}^t$ of subsets of Y_t are given. It is easy to check that the families

$$\Sigma = \bigcap_{t \in T} \{E \subset X : \ E \cap Y_t \in \Sigma^t\} \text{ and } \mathcal{I} = \bigcap_{t \in T} \{E \subset X : \ E \cap Y_t \in \mathcal{I}^t\}$$

form an algebra and an ideal of subsets of X, respectively. We denote $\Sigma = \bigoplus_{t\in T} \Sigma^t$ and $\mathcal{I} = \bigoplus_{t\in T} \mathcal{I}^t$. Observe that if any two distinct members of $\mathcal{Y} = \{Y_t : t \in T\}$ are disjoint, we have $\Sigma_{Y_t} = \Sigma^t$ and $\mathcal{I}_{Y_t} = \mathcal{I}^t$ for each $t \in T$. The operation $\bigoplus$ is analogous to the sum of topological spaces [9], however, in general we do not require the disjointness of sets Y_t, $t \in T$.

Theorem 2. *Assume that $T \neq \emptyset$ and let $Y_t \in \mathcal{P}(X) \setminus \{\emptyset\}$, $\mathcal{F}_t \subset \mathcal{P}(Y_t) \setminus \{\emptyset\}$ be given for each $t \in T$. Assume additionally, for any $t_1,\ t_2 \in T$, the following condition*

$$(\forall A_1 \in \mathcal{F}_{t_1},\ \forall A_2 \in \mathcal{F}_{t_2})\ \ A_1 \subset A_2 \Rightarrow A_1 \in \mathcal{F}_{t_2}. \tag{$\star$}$$

If $\Sigma^t = S_{Y_t}(\mathcal{F}_t)$ and $\mathcal{I}^t = S^0_{Y_t}(\mathcal{F}_t)$ then, for $\Sigma = \bigoplus_{t\in T} \Sigma^t$ and $\mathcal{I} = \bigoplus_{t\in T} \mathcal{I}^t$, we have $\Sigma = S_X(\mathcal{F})$ and $\mathcal{I} = S^0_X(\mathcal{F})$ where $\mathcal{F} = \bigcup_{t\in T} \mathcal{F}_t$.

Proof. To show $\Sigma \subset S_X(\mathcal{F})$, consider an $E \in \Sigma$ and an $A \in \mathcal{F}$. Hence $A \in \mathcal{F}_t$ for some $t \in T$. Clearly $E \cap Y_t \in \Sigma^t = S_{Y_t}(\mathcal{F}_t)$. Thus there exists a $B \in \mathcal{F}_t$ such that either $B \subset (A \cap Y_t) \cap (E \cap Y_t)$ or $B \subset (A \cap Y_t) \setminus (E \cap Y_t)$. Consequently, $B \in \mathcal{F}$ and either $B \subset A \cap E$ or $B \subset A \setminus E$. Hence $E \in S_X(\mathcal{F})$. (In this part of proof we do not use $(\star)$.)

To show $S_X(\mathcal{F}) \subset \Sigma$, consider an $E \in S_X(\mathcal{F})$ and a $t \in T$. We want to prove that $E \cap Y_t \in S_{Y_t}(\mathcal{F}_t)$. Let $A \in \mathcal{F}_t$. Obviously, $A \in \mathcal{F}$. Since $E \in S_X(\mathcal{F})$, there is a $B \in \mathcal{F}$ such that either $B \subset A \cap E$ or $B \subset A \setminus E$. By the definition of $\mathcal{F}$ there is a $t_1 \in T$ such that $B \in \mathcal{F}_{t_1}$. Since $B \subset A$, we have $B \in \mathcal{F}_t$, by $(\star)$. Thus, either $B \subset A \cap (E \cap Y_t)$ or $B \subset A \setminus (E \cap Y_t)$. Hence $E \cap Y_t \in S_{Y_t}(\mathcal{F}_t)$.

The proof of $\mathcal{I} = S^0_X(\mathcal{F})$ is analogous. □

Remark 1. A. Bartoszewicz has observed that $(\star)$ in Theorem 2 can be replaced by a weaker condition

$$(\forall A_1 \in \mathcal{F}_{t_1},\ \forall A_2 \in \mathcal{F}_{t_2})(A_1 \subset A_2 \Rightarrow (\exists A \in \mathcal{F}_{t_2})A \subset A_1).$$

The proof needs only minor modification. Note that condition $(\star)$ is fulfilled, if any two distinct sets $Y_{t_1},\ Y_{t_2}$ are disjoint. This enables one to produce new examples of MB-representable algebras and ideals from the known examples.

It is obvious that the intersection of a family of algebras (ideals) is again an algebra (ideal). Is that intersection MB-representable, provided all the factors are MB-representable? From Theorem 2 we can infer the affirmative answer in some special case.

Corollary 1. *Let $T \neq \emptyset$ and let $\mathcal{F}_t \subset \mathcal{P}(X) \setminus \{\emptyset\}$ be given for each $t \in T$. Assume, for any $t_1,\ t_2 \in T$, the following condition*

$$(\forall A_1 \in \mathcal{F}_{t_1},\ \forall A_2 \in \mathcal{F}_{t_2})\ \ A_1 \subset A_2 \Rightarrow A_1 \in \mathcal{F}_{t_2}.$$

Then $\bigcap\limits_{t\in T} S(\mathcal{F}_t) = S(\bigcup\limits_{t\in T} \mathcal{F}_t)$ *and* $\bigcap\limits_{t\in T} S^0(\mathcal{F}_t) = S^0(\bigcup\limits_{t\in T} \mathcal{F}_t)$.

Proof. Put $Y_t = X$ for each $t \in T$, observe that $\bigoplus_{t\in T} S(\mathcal{F}_t) = \bigcap_{t\in T} S(\mathcal{F}_t)$, $\bigoplus_{t\in T} S^0(\mathcal{F}_t) = \bigcap_{t\in T} S^0(\mathcal{F}_t)$ and use Theorem 2. □

Now, we are going to show some situations where Corollary 1 applies. For an ideal $\mathcal{I} \subset \mathcal{P}(X)$ we denote

$$\operatorname{add}(\mathcal{I}) = \min\{|\mathcal{G}| :\ \mathcal{G} \subset \mathcal{I}\ \&\ \bigcup \mathcal{G} \notin \mathcal{I}\}.$$

We say that two ideals $\mathcal{I},\ \mathcal{J} \subset \mathcal{P}(X)$ are *orthogonal* if there is a set $E \subset X$ such that $E \in \mathcal{I}$ and $X \setminus E \in \mathcal{J}$. We say that two families $\mathcal{F},\ \mathcal{G} \subset \mathcal{P}(X) \setminus \{\emptyset\}$ are *mutually coinitial* (in symbols $\mathcal{F} \sim \mathcal{G}$), if

$$(\forall A \in \mathcal{F})(\exists B \in \mathcal{G}) B \subset A \text{ and } (\forall A \in \mathcal{G})(\exists B \in \mathcal{F}) B \subset A.$$

(See [2].) It is easy to check that relation $\sim$ is transitive and that $\mathcal{F} \sim \mathcal{G}$ implies $S(\mathcal{F}) = S(\mathcal{G})$ and $S^0(\mathcal{F}) = S^0(\mathcal{G})$.

Corollary 2. *Let $0 < |T| < \kappa$ and, for each $t \in T$, let a family $\mathcal{F}_t \subset \mathcal{P}(X) \setminus \{\emptyset\}$ be given. Assume that:*

1^0 **:** *$\mathcal{F}_t \subset S(\mathcal{F}_t)$ for each $t \in T$;*
2^0 **:** *the ideals $S^0(\mathcal{F}_t)$, $t \in T$, are pairwise orthogonal;*
3^0 **:** $\operatorname{add}(S^0(\mathcal{F}_t)) \geq \kappa$ *for each $t \in T$.*

Then $\bigcap\limits_{t\in T} S(\mathcal{F}_t) = S(\bigcup\limits_{t\in T} \mathcal{F}_t)$ *and* $\bigcap\limits_{t\in T} S^0(\mathcal{F}_t) = S^0(\bigcup\limits_{t\in T} \mathcal{F}_t)$.

Proof. Denote $\Sigma^t = S(\mathcal{F}_t)$ and $\mathcal{I}^t = S^0(\mathcal{F}_t)$ for $t \in T$. By 2^0, for any $t_1,\ t_2 \in T,\ t_1 \neq t_2$, we can pick a set $A(t_1, t_2) \in \mathcal{I}^{t_1}$ with $X \setminus A(t_1, t_2) \in \mathcal{I}^{t_2}$. We may assume that $A(t_2, t_1) = X \setminus A(t_1, t_2)$. For each $t_0 \in T$ define

$$A(t_0) = \bigcup_{t\in T\setminus\{t_0\}} A(t_0, t).$$

Then $A(t_0) \in \mathcal{I}^{t_0}$, by 3^0 and $|T| < \kappa$. It can easily be checked that $\mathcal{F}_t \subset \Sigma^t$ (see 1^0) implies $\Sigma^t \setminus \mathcal{I}^t \sim \mathcal{F}_t$ for each $t \in T$. (See [2, Proposition 1.1].) Put

$$\mathcal{F}_t^* = \{A \setminus A(t) :\ A \in \Sigma^t \setminus \mathcal{I}^t\},\ \ t \in T.$$

Obviously $\mathcal{F}_t^* \sim \Sigma^t \setminus \mathcal{I}^t$ and thus $\mathcal{F}_t^* \sim \mathcal{F}_t$ for each $t \in T$. Consider any $t_1,\ t_2 \in T,\ t_1 \neq t_2$, and $A_1 \in \mathcal{F}_{t_1}^*,\ A_2 \in \mathcal{F}_{t_2}^*$. Then $A_1 \subset X \setminus A(t_1) \subset X \setminus A(t_1, t_2) = A(t_2, t_1)$ and $A_2 \subset X \setminus A(t_2) \subset X \setminus A(t_2, t_1)$, so $A_1 \subset A_2$ is impossible. Hence by Corollary 1 we obtain $\bigcap_{t\in T} S(\mathcal{F}_t^*) = S(\bigcup_{t\in T}\mathcal{F}_t^*)$ and $\bigcap_{t\in T} S^0(\mathcal{F}_t^*) = S^0(\bigcup_{t\in T}\mathcal{F}_t^*)$. But $\mathcal{F}_t^* \sim \mathcal{F}_t$ for each $t \in T$, and thus also $\bigcup_{t\in T}\mathcal{F}_t^* \sim \bigcup_{t\in T}\mathcal{F}_t$. Hence the assertion follows. □

Example 1. In [18] Mycielski introduced a class of σ-ideals on the Cantor space 2^ω; we shall call them Mycielski ideals. Further results on that topic were obtained in [3] and [20], [21]. Lemma 1.1 in [3] states that one can find a Mycielski ideal othogonal to each ideal of a given countable family of Mycielski ideals. This easily leads to the family $\{\mathcal{M}_\alpha : \alpha < \omega_1\}$ of pairwise orthogonal Mycielski ideals. To apply Corollary 2 we put $\mathcal{F}_\alpha = \{2^\omega \backslash E : E \in \mathcal{M}_\alpha\}$. Then $\mathcal{F}_\alpha \subset \mathcal{M}_\alpha \cup \mathcal{F}_\alpha = S(\mathcal{F}_\alpha)$ and $\mathcal{M}_\alpha = S^0(\mathcal{F}_\alpha)$ for each $\alpha < \omega_1$. (See [2, Proposition 1.5].) Since $\mathcal{M}_\alpha$ are σ-ideals, we have $\mathrm{add}(\mathcal{M}_\alpha) \geq \omega_1$ for $\alpha < \omega_1$ (in fact $\mathrm{add}(\mathcal{M}_\alpha) = \omega_1$, cf. [20]). Hence Corollary 2 applies.

Example 2. Let $X = \mathbb{R}$. Let Σ_1 and Σ_2 stand for the algebra of all Lebesgue measurable sets, and for the algebra of all sets with the Baire property (in $\mathbb{R}$). Let $\mathcal{I}_1$ and $\mathcal{I}_2$ denote the ideal of null sets, and the ideal of meager sets (in $\mathbb{R}$). In [6] it is proved that $\Sigma_1 = S(\mathcal{F}_1)$ and $\mathcal{I}_1 = S^0(\mathcal{F}_1)$ where $\mathcal{F}_1$ consists of perfect sets of positive measure. In [5] it is proved that $\Sigma_2 = S(\mathcal{F}_2)$ and $\mathcal{I}_2 = S^0(\mathcal{F}_2)$ where $\mathcal{F}_2$ consists of sets of the form $U \setminus A$ where U is nonempty open and $A \subset U$ is meager of type F_σ. By Corollary 2 we infer that $\Sigma_1 \cap \Sigma_2 = S(\mathcal{F}_1 \cup \mathcal{F}_2)$ and $\mathcal{I}_1 \cap \mathcal{I}_2 = S^0(\mathcal{F}_1 \cup \mathcal{F}_2)$, so the pair $(\Sigma_1 \cap \Sigma_2, \mathcal{I}_1 \cap \mathcal{I}_2)$ is MB-representable. This result can be also derived from the general theorem by Baldwin [4] who proved that if the pair $(\Sigma, \mathcal{I})$, consisting of an algebra Σ and an ideal $\mathcal{I} \subset \Sigma$, possesses the so-called hull property, then $\Sigma = S(\Sigma \setminus \mathcal{I})$ and $\mathcal{I} = S^0(\Sigma \setminus \mathcal{I})$. We say that $(\Sigma, \mathcal{I})$ has the *hull property* if whenever $U \subset X$ there is a $V \in \Sigma$ such that $U \subset V$, and if $W \in \Sigma$ is such that $U \subset W$, then $V \setminus W \in \mathcal{I}$. It is known that each of the pairs $(\Sigma_1, \mathcal{I}_1)$, $(\Sigma_2, \mathcal{I}_2)$ has the hull property, and the same follows for $(\Sigma_1 \cap \Sigma_2,\ \mathcal{I}_1 \cap \mathcal{I}_2)$. Thus $\Sigma_1 \cap \Sigma_2 = S(\Sigma_1 \cap \Sigma_2 \setminus (\mathcal{I}_1 \cap \mathcal{I}_2))$ and $\mathcal{I}_1 \cap \mathcal{I}_2 = S^0(\Sigma_1 \cap \Sigma_2 \setminus (\mathcal{I}_1 \cap \mathcal{I}_2))$. Finally, observe that $\mathcal{F}_1 \sim \Sigma_1 \setminus \mathcal{I}_1$, $\mathcal{F}_2 \sim \Sigma_2 \setminus \mathcal{I}_2$ and $\mathcal{F}_1 \cup \mathcal{F}_2 \sim \Sigma_1 \cap \Sigma_2 \setminus (\mathcal{I}_1 \cap \mathcal{I}_2)$.

Condition $(\star)$ in Theorem 2 is rather restrictive. Let us mention two important algebras of the form $\bigoplus_{Y\in\mathcal{Y}} \Sigma^Y$. Because of condition $(\star)$, our Theorem 2 seems useless in these cases. Namely, consider the algebra Σ of all sets in $\mathbb{R}$ that are of types F_σ and G_δ simultaneously. Following [14, §34,VI] we have $\Sigma = \bigoplus_{Y\in\mathcal{Y}} \Sigma^Y$ where $\mathcal{Y}$ is the family of all nonempty closed sets in $\mathbb{R}$, and Σ^Y stands for the algebra of subsets of Y with nowhere dense boundary, relatively to Y. It is known that $\Sigma^Y = S_Y(\mathcal{F}_Y)$ where $\mathcal{F}_Y$ consists of all nonempty sets open in the topology of Y. A trouble with condition $(\star)$ appears in the following situation. Let Y_1, Y_2 be nonempty perfect sets where $Y_1 \subset Y_2$ and Y_1 is nowhere dense in Y_2. Then for each $A_1 \in \mathcal{F}_{Y_1}$ we have $A_1 \notin \mathcal{F}_{Y_2}$.

Let us consider another example. Let $\mathcal{Y}$ be as above and now let Σ^Y stand for the algebra of subsets of $Y \in \mathcal{Y}$, with the Baire property, relatively to Y. Thus the sets of the algebra $\Sigma = \bigoplus_{Y \in \mathcal{Y}} \Sigma^Y$ are called sets with the *Baire property in the restricted sense*. (See [14, §11, VI].) Theorem 2 again seems unapplicable.

We leave open the question whether these two algebras are MB-representable. The problem concerning sets with the Baire property in the restricted sense has been suggested to the first author by P. Reardon.

3. MB-representations of algebras Δ^0_α

It was shown in [1] that, under GCH (more precisely, under $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$), the algebra of all Borel sets in $\mathbb{R}$ is MB-representable. We will observe that the same method leads to the analogous result for the algebras of ambiguous Borel sets of classes $\alpha \geq 3$. However, this does not work in the case $\alpha = 2$.

Recall necessary definitions from [1]. Let $\mathcal{A} \subset \mathcal{P}(X)$ and $\mathcal{I} \subset \mathcal{A}$ be an algebra and an ideal. We say that $\mathcal{A}$ is *inner* (*outer*) *MB-representable* if there is an $\mathcal{F} \subset \mathcal{P}(X)$ such that $\mathcal{A} = S(\mathcal{F})$ and $\mathcal{F} \subset \mathcal{A}$ ($\mathcal{F} \cap \mathcal{A} = \emptyset$). We say that $\mathcal{A}$ is *strongly outer MB-representable* if for each family $\mathcal{C} \subset \mathcal{P}(X)$ with $\mathcal{A} \subset \mathcal{C}$ and $|\mathcal{C}| = |\mathcal{A}|$ there is an $\mathcal{F} \subset \mathcal{P}(X) \setminus \mathcal{C}$ such that $\mathcal{A} = S(\mathcal{F})$. If moreover, $\mathcal{I} = S^0(\mathcal{F})$, we say that the pair $(\mathcal{A}, \mathcal{I})$ is (respectively) inner, or outer, or strongly outer MB-representable. We shall use, in the role of $\mathcal{I}$, the ideal

$$H(\mathcal{A}) = \{A \subset X \colon (\forall B \subset A) B \in \mathcal{A}\}$$

of sets which hereditarily belong to $\mathcal{A}$.

Let us quote two theorems from [1].

Theorem 3. *Let $|X| = \kappa \geq \omega$ and let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra such that $H(\mathcal{A}) \subset [X]^{<\kappa}$, $\mathcal{A} \cap [X]^{<\kappa} \subset H(\mathcal{A})$ and $S(\mathcal{A} \setminus [X]^{<\kappa}) \setminus \mathcal{A} \neq \emptyset$. Then $\mathcal{A}$ is not inner MB-representable.*

Theorem 4. *Let $|X| = \kappa \geq \omega$ and let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra such that $[X]^{<\kappa} \subset \mathcal{A}$. If $2^\kappa = \kappa^+$ and $|\mathcal{A}| < 2^\kappa$ then the pair $(\mathcal{A}, H(\mathcal{A}))$ is strongly outer MB-representable.*

Fix an uncountable Polish space X. Thus $|X| = \mathfrak{c}$ where $\mathfrak{c} = 2^\omega$ is the cardinality of continuum. Consider Δ^0_α, $\alpha < \omega_1$, the algebra of ambiguous Borel sets of class α in X, i.e. $\Delta^0_2 = F_\sigma \cap G_\delta$, $\Delta^0_3 = F_{\sigma\delta} \cap G_{\delta\sigma}$, etc. (See

[11].) The algebra $\Delta^0_{\omega_1} = \bigcup_{\alpha<\omega_1} \Delta^0_\alpha$ consists of all Borel sets in X. The family of G_δ sets in X is written as Π^0_2.

Lemma 1. $H(\Delta^0_2) = [X]^{\leq\omega} \cap \Pi^0_2$.

Proof. To show "$\subset$" suppose that $A \in H(\Delta^0_2)$ is uncountable. Pick a perfect subset of A and its subset which is not in Δ^0_2. (See [11, 13.6, 22.4].) Contradiction.

To show "$\supset$" consider an $A \in [X]^{\leq\omega} \cap \Pi^0_2$. Let $B \subset A$. Thus $X \setminus B = (X \setminus A) \cup (A \setminus B)$ and so, B is of type G_δ. Hence $B \in [X]^{\leq\omega} \cap \Pi^0_2 \subset \Delta^0_2$. Consequently $A \in H(\Delta^0_2)$. $\square$

Lemma 2. $H(\Delta^0_\alpha) = [X]^{\leq\omega}$ *for each* α, $3 \leq \alpha \leq \omega_1$.

Proof. The argument for "$\subset$" is similar to that in the proof of Lemma 1. Inclusion "$\supset$" follows from $[X]^{\leq\omega} \subset \Delta^0_3 \subset \Delta^0_\alpha$. $\square$

As an application of Theorems 3 and 4 we obtain the following

Theorem 5. *In an uncountable Polish space X, we have:*

(I) *algebras Δ^0_α, $2 \leq \alpha \leq \omega_1$, are not inner MB-representable;*

(II) *if $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$, then the pairs $(\Delta^0_\alpha, [X]^\omega)$, $3 \leq \alpha \leq \omega_1$, are strongly outer MB-representable.*

Proof. (I) (Cf. [1, Corollary 14].) Fix α, $2 \leq \alpha \leq \omega_1$, and put $\mathcal{A} = \Delta^0_\alpha$. From Lemmas 1 and 2 we infer that $H(\mathcal{A}) \subset [X]^{<\mathfrak{c}}$ and $\mathcal{A} \cap [X]^{<\mathfrak{c}} \subset H(\mathcal{A})$. Observe that $\mathcal{A} \setminus [X]^{<\mathfrak{c}}$ and the family of all perfect sets in X are mutually coinitial. Hence $S(\mathcal{A} \setminus [X]^{<\mathfrak{c}})$ is equal to the algebra of Marczewski measurable sets. Since there is a non-Borel Marczewski measurable set [15], we may use Theorem 3.

(II) (Cf. [1, Corollary 5].) We have $|X| = \mathfrak{c} = \omega_1$, and $[X]^{<\omega_1} = [X]^{\leq\omega} = H(\Delta^0_\alpha) \subset \Delta^0_\alpha$ by Lemma 2. Also $|\Delta^0_\alpha| = \mathfrak{c} = \omega_1 < \omega_2 = 2^{\omega_1}$. Then apply Theorem 4. $\square$

Remark 2. In the case $\alpha = 2$ we cannot repeat the argument for Theorem 5 (II) since $[X]^\omega \subset \Delta^0_2$ is false. Indeed, each set from $[X]^\omega \setminus \Pi^0_2$ is not in Δ^0_2.

Finally, let us show that Theorem 4 can be applied to an algebra on a set of cardinality 2^λ where a cardinal $\lambda \geq \omega$ is arbitrarily large.

Example 3. Let $\lambda \geq \omega$ be a cardinal and consider the Cantor cube $X = \{0,1\}^\lambda$. The basis for the product topology τ on X is of size $|[\lambda]^{<\omega}| = \lambda$ which easily implies that $|\tau| = 2^\lambda$. Hence $|X| = |\tau| = 2^\lambda$ and we denote $2^\lambda = \kappa$. An algebra $\mathcal{A} \subset \mathcal{P}(X)$ will be called λ^+-additive if $\bigcup_{\nu<\lambda} A_\nu \in \mathcal{A}$ for any function $\nu \mapsto A_\nu \in \mathcal{A}$, $\nu < \lambda$. Now, let $\mathcal{A}$ stand for the smallest λ^+-additive algebra containing τ. We can classify sets in $\mathcal{A}$ analogously as Borel sets in a Polish space, considering the classes analogous to Σ^0_α, Π^0_α (cf. [11], [7]) but now $\alpha < \lambda^+$. Each of this class is of size κ since $\kappa^\lambda = (2^\lambda)^\lambda = \kappa$. Hence we conclude that $|\mathcal{A}| \leq \lambda^+\kappa = \kappa$. Assume $2^\lambda = \lambda^+$ (that is $\kappa = \lambda^+$) and $2^\kappa = \kappa^+$, which is a part of GCH. Thus $[X]^{<\kappa} = [X]^{\leq\lambda} \subset \mathcal{A}$ and $|\mathcal{A}| = \kappa < 2^\kappa$. Consequently, Theorem 4 applies.

4. MB-representations of clopen sets

A basic question concerning MB-representations was whether every algebra of sets is MB-representable. Now, the answer is known. One of the theorems in [1] gives the negative answer under GCH: if $2^\kappa = \kappa^+$ and $|X| = \kappa \geq \omega$ then there is a non-MB-representable algebra on X. In December 2002, P. Koszmider [13] found a non-MB-representable algebra $\mathcal{A} \subset \mathcal{P}(\omega)$ in ZFC.

A related question is whether every algebra of sets is isomorphic to an MB-representable algebra where an isomorphism is meant in the Boolean theoretical sense. Suprisingly, Koszmider [13] answered it in the affirmative. A natural idea to solve this problem is to use the classical Stone representation theorem which states that every Boolean algebra (in particular, every algebra of sets) is isomorphic to the algebra Clop(X) of clopen subsets of some zero-dimensional compact Hausdorff space X. (See [12].) Koszmider [13] proved that this last algebra is isomorphic to an MB-representable algebra of sets. Independently of this result one can pose the following topological problem: describe all zero-dimensional compact Hausdorff spaces X for which Clop(X) is MB-representable. We do not solve it in this paper. We only give some conditions on a topological space X under which the algebra Clop(X) is MB-representable.

Let $\lambda \geq 2$ be a cardinal. A topological space is called λ-*resolvable* if there is a disjoint family of cardinality λ, of dense subsets of X. (See [8].) Clearly, each λ-resolvable space is dense-in-itself. Now, let λ be infinite. For $\Gamma \subset \lambda$ put $(+1)\Gamma = \Gamma$ and $(-1)\Gamma = \Gamma^c$ $(= \lambda \setminus \Gamma)$. A family $\mathcal{F} \subset \mathcal{P}(\lambda)$ is called *independent* if, whenever $\Gamma_0, ..., \Gamma_n$ is a finite sequence of distinct elements from $\mathcal{F}$ and $\varepsilon_0, ..., \varepsilon_n$ is a sequence of numbers $-1, +1$, then $\bigcap_{k=0}^n \varepsilon_k\Gamma_k \neq \emptyset$. The theorem of Fichtenholz, Kantorovitch and Hausdorff states that for each cardinal $\lambda \geq \omega$ there is an independent family $\mathcal{F} \subset \mathcal{P}(\lambda)$ of cardinality 2^λ. (See [16].) We call it Theorem FKH.

Theorem 6. *Let X, with $|X| = \kappa \geq \omega$, be a dense-in-itself a λ-resolvable topological space where $2^\lambda \geq \kappa$. Then the algebra* $\mathrm{Clop}(X)$ *is MB-representable.*

Proof. We will mimic some ideas contained in [2, Theorem 2.1] which are due to S. Wroński. The trick with an independent family, suggested us by P. Koszmider, has strenghtened the former version of our theorem.

By Theorem FKH there is an independent family $\mathcal{T} \subset \mathcal{P}(\lambda)$ of size 2^λ. Since $\kappa \leq 2^\lambda$, any subfamily of size κ is also independent. So, assume that $|\mathcal{T}| = \kappa$. Because $|X| = \kappa$, we may put $\mathcal{T} = \{T_x \colon x \in X\}$. Since X is λ-resolvable, there is a disjoint family $\{D_\alpha \colon \alpha < \lambda\}$ of dense subsets of X. We may assume that $\bigcup_{\alpha<\lambda} D_\alpha = X$. Define

$$F(x) = \bigcup_{\alpha \in T_x} D_\alpha \setminus \{x\} \text{ for } x \in X.$$

Obviously, the sets $F(x)$, $x \in X$, are dense.

Claim 1. *For each finite set $\{x_0, ..., x_n\} \subset X$, the set $X \setminus \bigcup_{k=0}^n F(x_k)$ is dense.*

Indeed, by the definition of $F(x_k)$ and the disjointness of sets D_α we have

$$X \setminus \bigcup_{k=0}^n F(x_k) \supset \bigcap_{k=0}^n \bigcup_{\alpha \in T^c_{x_k}} D_\alpha \supset \bigcup_{\alpha \in T^c_{x_0} \cap \cdots \cap T^c_{x_n}} D_\alpha.$$

Since $\mathcal{T}$ is independent, $T^c_{x_0} \cap \cdots \cap T^c_{x_n} \neq \emptyset$. Thus $X \setminus \bigcup_{k=0}^n F(x_k)$ contains at least one dense set D_α which ends the proof of Claim 1.

Now, define $\mathcal{F} = \{U \setminus F(x) :\ x \in U \text{ and } U \text{ is open}\}$.

At first we shall prove that $\mathrm{Clop}(X) \subset S(\mathcal{F})$. Let $V \in \mathrm{Clop}(X)$ and consider a $U \setminus F(x) \in \mathcal{F}$. There are two cases:

1^0**:** $x \in V$. Then $x \in U \cap V$. Put $W = U \cap V$. Hence $W \setminus F(x) \subset (U \setminus F(x)) \cap V$ and $W \setminus F(x) \in \mathcal{F}$.

2^0**:** $x \notin V$. Then $x \in U \setminus V$ and from $V \in \mathrm{Clop}(X)$ it follows that the set $W = U \setminus V$ is open. Hence $W \setminus F(x) \subset (U \setminus F(x)) \setminus V$ and $W \setminus F(x) \in \mathcal{F}$.

To prove $S(\mathcal{F}) \subset \mathrm{Clop}(X)$ we need the following:

Claim 2. *If $V \setminus F(y)$ and $U \setminus F(x)$ are in $\mathcal{F}$ and $V \setminus F(y) \subset U \setminus F(x)$, then $x = y$.*

Indeed, suppose that $x \neq y$. Then

$$\begin{aligned} V &= (V \setminus F(y)) \cup (V \cap F(y)) \subset (X \setminus F(x)) \cup F(y) \\ &\subset \{x\} \cup \bigcup_{\alpha \in T^c_x} D_\alpha \cup \bigcup_{\alpha \in T_y} D_\alpha \subset \{x\} \cup (X \setminus \bigcup_{\alpha \in T_x \cap T^c_y} D_\alpha). \end{aligned}$$

By the independence of $\mathcal{T}$, the set $T_x \cap T_y^c$ is nonempty, so there is a dense set D_α, for some $\alpha \in T_x \cap T_y^c$, disjoint from the nonempty open set $V \setminus \{x\}$. Contradiction.

Now, let $A \in S(\mathcal{F})$. We want to show that A is open. Since we may replace A by $X \setminus A$, this will end the proof. Suppose that A is not open. Thus there exists an $x \in A \setminus \operatorname{int} A$. Clearly $X \setminus F(x) \in \mathcal{F}$. Since $A \in S(\mathcal{F})$ there is $U \setminus F(y) \in \mathcal{F}$ such that either

$$U \setminus F(y) \subset (X \setminus F(x)) \cap A \tag{1}$$

or $U \setminus F(y) \subset (X \setminus F(x)) \setminus A$. By Claim 2 we have $x = y$. Since $x = y \in (U \setminus F(y)) \cap A$, condition (1) holds. From $x \in U$ and $x \notin \operatorname{int} A$ it follows that $U \setminus A \neq \emptyset$. Pick a $z \in U \setminus A$. Thus $U \setminus F(z) \in \mathcal{F}$ and from $A \in S(\mathcal{F})$ it follows that there is $V \setminus F(t) \in \mathcal{F}$ such that either $V \setminus F(t) \subset (U \setminus F(z)) \cap A$ or

$$V \setminus F(t) \subset (U \setminus F(z)) \setminus A. \tag{2}$$

Using Claim 2 again we infer that $z = t$. Since $z = t \in (V \setminus F(t)) \setminus A$, condition (2) holds. By Claim 1 we have $V \setminus (F(x) \cup F(z)) \neq \emptyset$. On the other hand,

$$V \setminus (F(x) \cup F(z)) \subset U \setminus F(x) = U \setminus F(y) \subset A, \text{ by (1)}$$

and

$$V \setminus (F(x) \cup F(z)) \subset V \setminus F(z) = V \setminus F(t) \subset X \setminus A, \text{ by (2)}.$$

Contradiction. $\square$

Example 4. Let $\eta \geq \omega$ be a cardinal and put $\lambda = 2^\eta$, $\kappa = 2^\lambda$. Consider Cantor cubes $X_1 = \{0,1\}^\eta$, $X_2 = \{0,1\}^\lambda$. Then $\Delta(X_1) = \lambda$, $\Delta(X_2) = \kappa$. Let X be a topological sum of X_1 and X_2. (See [9].) Thus $|X| = \lambda + \kappa = \kappa$ and $\Delta(X) = \lambda$. The space X is compact and dense-in-itself. So by [8, Theorem 3.7] the space X is λ-resolvable. Thus we may apply Theorem 6 with $2^\lambda = \kappa$. Hence $\operatorname{Clop}(X)$ is MB-representable. Note that $\operatorname{Clop}(X)$ is nontrivial since X is zero-dimensional as a sum of zero-dimensional spaces X_1 and X_2. (See [9].) If we consider space X_2 instead of X, we get a simple example where Theorem 6 applies (thus $|X_2| = \kappa$ and X_2 is κ-resolvable) but in this case we do not use the whole power of our result.

It is an easy observation that, for a discrete topological space X, we have $\operatorname{Clop}(X) = \mathcal{P}(X) = S(\mathcal{P}(X) \setminus \{\emptyset\})$. This mixed with Theorem 6 produces the following

Corollary 3. *Let X be a topological space which is a sum of (pairwise disjoint, clopen) subspaces X_t, $t \in T$, such that each X_t is either discrete, or $|X_t| = \kappa_t \geq \omega$ and X_t is λ_t-resolvable with $2^{\lambda_t} \geq \kappa_t$. Then* $\mathrm{Clop}(X)$ *is MB-representable.*

Proof. Let $\Sigma = \mathrm{Clop}(X)$ and $\Sigma^t = \mathrm{Clop}(X_t)$ for $t \in T$. Then Σ^t are MB-representable by Theorem 6 and the above observation. The rest follows from Theorem 2. □

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