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RANDOM SUMS OF INDEPENDENT RANDOM VECTORS ATTRACTED BY (SEMI)-STABLE HEMIGROUPS

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Abstract. Let (X_n) be a sequence of independent not necessarily identically distributed random vectors belonging to the domain of attraction of a stable or semistable hemigroup, i.e. for an increasing sampling sequence (k_n) such that $k_{n+1}/k_n \to c \geq 1$ and linear operators A_n, the normalized sums $A_n \sum_{k=\lfloor k_n s\rfloor+1}^{\lfloor k_n t\rfloor} X_k$ converge in distribution uniformly on compact subsets of $\{0 \leq s < t\}$ to some full probability $\mu_{s,t}$. Suppose that (T_n) is a sequence of positive integer valued random variables such that T_n/k_n converges in probability to some positive random variable, where we do not assume (X_n) and (T_n) to be independent. Then weak limit theorems of random sums, where the sampling sequence (k_n) is replaced by random sample sizes (T_n), are presented.

1. Introduction

Let (X_n) be a sequence of independent not necessarily identically distributed random vectors on $\mathbb{R}^d$ with distributions $\nu_n = X_n(P)$ and let (k_n) be an increasing sampling sequence with $k_{n+1}/k_n \to c \geq 1$. Assume the

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existence of norming operators $A_n \in \mathrm{End}(\mathbb{R}^d)$ such that for any sequences $0 \leq s_n < t_n$ with $s_n \to s$ and $t_n \to t$ we have

$$A_n\Big(\sum_{k=\lfloor k_n s_n\rfloor+1}^{\lfloor k_n t_n\rfloor} X_k\Big)(P) = A_n\left(\nu_{\lfloor k_n s_n\rfloor+1} * \cdots * \nu_{\lfloor k_n t_n\rfloor}\right) \xrightarrow{\mathrm{w}} \mu_{s,t} \tag{1.1}$$

for some probability measures $\mu_{s,t}$ on $\mathbb{R}^d$, that we assume to be full if $s < t$. Here full means not concentrated on any proper hyperplane of $\mathbb{R}^d$ and "$\xrightarrow{\mathrm{w}}$" denotes weak convergence. Then, as is described in the next section, the distributions $\mu_{s,t}$, $0 \leq s \leq t$, define a stable or semistable (continuous) hemigroup, which is a two parameter generalization of stable respectively semistable continuous convolution semigroups of probability measures. The sequence of random vectors (X_n) is then said to belong to the domain of attraction of the stable or semistable hemigroup. Stability respectively semistability is characterized by a scaling property of the limits, which shows that $\mu_{0,t}$ is operator (semi)-selfdecomposable for every $t > 0$ and there exists an operator (semi)-selfsimilar process $(Y_t)_{t\geq 0}$ with independent increments such that $\mu_{s,t}$ is the distribution of the increment $Y_t - Y_s$ for any $0 \leq s \leq t$. Conversely, full operator (semi)-selfdecomposable probability measures μ can be embedded into (semi)-stable hemigroups with $\mu = \mu_{0,1}$ and the distributions of the increments of any operator (semi)-selfsimilar independent increment process $(Y_t)_{t\geq 0}$ with $Y_0 = 0$ almost surely and with Y_1 possessing a full distribution, define a (semi)-stable hemigroup. Hence the limits in (1.1) contain well known classes of probabilities and the underlying structural properties enable us to prove limit theorems for random sums, as follows.

Let (T_n) be a sequence of positive integer valued random variables, defined on the same probability space as (X_n), such that $T_n/k_n \to D$ in probability for some positive random variable D with distribution ρ. Replace k_n by T_n in (1.1) and consider the random sums

$$A_n \sum_{k=\lfloor T_n s_n\rfloor+1}^{\lfloor T_n t_n\rfloor} X_k, \tag{1.2}$$

where we do not assume (X_n) and (T_n) to be independent, neither require information about the dependence structure of both sequences. In case of independence of (X_n) and (T_n) we can apply Gnedenko's transfer-theorem under the milder condition of T_n/k_n being convergent in distribution, to obtain weak limits of (1.2); see [8]. If (X_n) and (T_n) are arbitrarily dependent, weak convergence of T_n/k_n is in general not sufficient for a weak limit law in (1.2), as Rényi [21] illustrates by an example. Rényi assumes convergence in probability for the random times to obtain the classical random central limit theorem

$$A_n\Big(\sum_{k=1}^{T_n} X_k\Big)(P) \xrightarrow{\mathrm{w}} \int_0^\infty \mathcal{N}_{0,t}\, d\rho(t),$$

where (X_n) is a standardized sequence of real valued i.i.d. random variables with finite second moments, $k_n = n$, $A_n = 1/\sqrt{n}$ and $\mathcal{N}_{0,t}$ denotes normal distribution with mean zero and variance t. This theorem has been proved in three consecutive steps, first for random samples $T_n = \lfloor nD \rfloor$ with discrete random variable $D > 0$ and then for $T_n/n \to D$ in probability, again for discrete $D > 0$, both by Rényi [21]. The last step for arbitrary $D > 0$ was shown independently in [4] and [20]. Whereas in the first step a mixing property suffices, the last two steps require an Anscombe-condition (see [1]) to be fulfilled. There are numerous extensions of the random central limit theorem in literature, extensions concerning multivariate results, results for random stable limit theorems or even more general limit theorems for triangular arrays of independent random variables; e.g. see [7], [9], [27], [13], [10], [2], [26], [6], [15], [25] or [14] and the literature cited therein. If (X_n) is not an i.i.d. sequence, all of these results need additional assumptions; essentially these are mixing conditions, variants of Anscombe's condition and regular variation of norming sequences. Especially the validation of Anscombe's condition is shown only in those cases, where finiteness of second moments is assumed, which is due to the application of Kolmogorov's inequality.

The purpose of this paper is to generalize the random central limit theorem to domains of attraction of stable and semistable hemigroups, which includes the majority of the cited situations above. The proof of our results in the third section will follow the above mentioned steps of proof of the random central limit theorem, containing the verification of Anscombe's condition. We further take a brief look at the special case of appropriately centered i.i.d. sequences (X_n) in the last section, since our results differ slightly from known results in view of random centering.

2. Stable and semistable hemigroups

This section summarizes results of an earlier paper [3] about stable and semistable hemigroups, their domains of attraction and structural properties, which is our basis of distributional assumptions for attaining limit theorems for random sums of independent random vectors in the next section.

Assume that (1.1) holds for any sequences $0 \leq s_n < t_n$ with $s_n \to s$, $t_n \to t$ and for some increasing sampling sequence (k_n) with $k_{n+1}/k_n \to c \geq 1$ and for some linear operators $A_n \in \mathrm{End}(\mathbb{R}^d)$, where the limits $\mu_{s,t}$ are full if $s < t$. In view of the fullness we can assume the norming operators A_n to be invertible for all $n \in \mathbb{N}$. Then it is shown in Theorem 3.2 of [3], that in case $c > 1$ the set $\mathcal{H} = \{\mu_{s,t} \mid 0 \le s \le t\}$ is a *semistable hemigroup*, i.e.

$$\mu_{s,r} * \mu_{r,t} = \mu_{s,t} \text{ for all } 0 \le s \le r \le t, \tag{2.1}$$

$$\text{the mapping } (s,t) \mapsto \mu_{s,t} \text{ is weakly continuous,} \tag{2.2}$$

$$\begin{aligned}&\text{there exists } B \in \mathrm{GL}(\mathbb{R}^d) \text{ such that } B\mu_{s,t} = \mu_{cs,ct}\\ &\text{for all } 0 \le s \le t.\end{aligned} \tag{2.3}$$

Note that since $\mu_{s,s} = \mu_{s,s} * \mu_{s,s}$ is an idempotent with respect to convolution by (2.1), we have $\mu_{s,s} = \varepsilon_0$ for every $s \ge 0$, where ε_0 denotes the point measure in the origin. The sequence of random vectors (X_n), respectively their distributions (ν_n), is said to belong to the *domain of semistable attraction* of $\mathcal{H}$, shortly $(X_n) \in \mathrm{DOSA}(\mathcal{H}, c)$. Moreover the proof of Theorem 3.1 in [3] shows, that either B or B^2 belongs to the image of the exponential mapping $\exp : \mathrm{End}(\mathbb{R}^d) \to \mathrm{GL}(\mathbb{R}^d)$, $\exp(A) = \sum_{k=0}^{\infty} A^k/k!$, such that there exists an *exponent* $Q \in \mathrm{End}(\mathbb{R}^d)$ with $\tilde{c}^Q \mu_{s,t} = \mu_{\tilde{c}s,\tilde{c}t}$ for all $0 \le s \le t$ and some $\tilde{c} \in \{c, c^2\}$, where t^Q is defined as $\exp(Q \log t)$ for $t > 0$. The real part of any eigenvalue of the exponent Q is positive, hence Q is invertible and $c^{-tQ} \to 0$ as $t \to \infty$; see [3, Theorem 3.1]. Since by Theorem 3.3(b) of [3], $\mathrm{DOSA}(\mathcal{H}, c) = \mathrm{DOSA}(\mathcal{H}, c^2)$ we will always assume the existence of an exponent for parameter c, else redefine the sampling sequence as (k_{2n}). Now the semistable scaling relation (2.3) becomes

$$c^Q \mu_{s,t} = \mu_{cs,ct} \quad \text{for all } 0 \le s \le t. \tag{2.4}$$

By Theorem 3.5 of [3], we will choose the norming operators (A_n) to be embeddable into a regularly varying sequence of invertible operators (B_n) with index $-Q$, where Q is an exponent of $\mathcal{H}$, i.e. we can choose $A_n = B_{k_n}$, where

$$B_{\lfloor \lambda n \rfloor} B_n^{-1} \to \lambda^{-Q} \text{ uniformly on compact subsets of } \{\lambda > 0\}. \tag{2.5}$$

Moreover by Theorem 3.6 of [3], the sequence $B_n(\nu_{\lfloor ns_n \rfloor + 1} * \cdots * \nu_{\lfloor nt_n \rfloor})$ is weakly relatively compact for any sequences $0 \le s_n < t_n$ with $s_n \to s$ and $t_n \to t$, where the set of weak limit points is

$$\mathrm{LIM}\left\{B_n\left(\nu_{\lfloor ns_n \rfloor + 1} * \cdots * \nu_{\lfloor nt_n \rfloor}\right)\right\} = \left\{\lambda^{-Q}\mu_{\lambda s, \lambda t} \middle| \lambda \in [1, c)\right\}. \tag{2.6}$$

Note that in view of (2.2) the convergence in (1.1) can be interpreted as pointwise weak convergence ($s_n \equiv s$, $t_n \equiv t$, $0 \le s < t$), which holds uniformly on compact subsets of $\{0 \le s < t\}$. Moreover Lemma 3.7 of [3] shows that (1.1) is equivalent to pointwise weak convergence together with the uniform infinitesimality condition

$$\max_{1 \le k \le n} P\{\|B_n X_k\| > \varepsilon\} \to 0 \quad \text{for any } \varepsilon > 0, \tag{2.7}$$

hence semistable hemigroups are closely related to operator semiselfdecomposable laws as defined in [3]. In fact (2.4) shows that for any $t > 0$ and $n \in \mathbb{N}$ we have $\mu_{0,t} = c^{-nQ}\mu_{0,t} * \mu_{c^{-n}t,t}$ such that $\mu_{0,t}$ is operator semiselfdecomposable for every $t > 0$. Conversely, every full operator semiselfdecomposeable probability measure μ fulfills $\mu = c^{-Q}\mu * \nu$ for some probability distribution ν and some invertible linear operator Q, whose eigenvalues have positive real part. If ν is full then μ can be embedded into a semistable hemigroup with exponent Q and $\mu = \mu_{0,1}$; see [3], Theorem 3.11 for details.

Now we turn to the case $k_{n+1}/k_n \to c = 1$. Then Theorem 3.3(a) of [3] shows that without loss of generality we can assume $k_n = n$ in (1.1). Thus by Theorem 2.1 of [3] we have that $\mathcal{H} = \{\mu_{s,t} \mid 0 \leq s \leq t\}$ is a *stable hemigroup*, i.e. (2.1) and (2.2) hold and there exist $B_\lambda \in \mathrm{GL}(\mathbb{R}^d)$ such that

$$B_\lambda \mu_{s,t} = \mu_{\lambda s, \lambda t} \quad \text{for all } 0 \leq s \leq t \text{ and } \lambda > 0. \tag{2.8}$$

In this case the sequence of random vectors (X_n), respectively their distributions (ν_n), is said to belong to the *domain of attraction* of $\mathcal{H}$, shortly $(X_n) \in \mathrm{DOA}(\mathcal{H})$. Moreover Theorem 1.15 of [11] shows, that there exists an *exponent* $Q \in \mathrm{GL}(\mathbb{R}^d)$ such that (2.8) holds for $B_\lambda = \lambda^Q$ and the real part of any eigenvalue of Q is positive. Now the stable scaling relation (2.8) becomes

$$\lambda^Q \mu_{s,t} = \mu_{\lambda s, \lambda t} \quad \text{for all } 0 \leq s \leq t \text{ and } \lambda > 0. \tag{2.9}$$

By Theorem 2.3 of [3], we will always assume the norming operators (A_n) to vary regularly with index $-Q$, i.e. (2.5) holds. Note that $A_n = B_{k_n} = B_n$ in (2.5), since we assume $k_n = n$ in the stable case $c = 1$. Although there is no need to rename the norming sequence (A_n), we will frequently switch from A_n to B_n in the following, to make notation consistent with the semistable case. Especially (2.6) holds in the stable case, since (2.9) shows that the only limit point in (2.6) is $\mu_{s,t}$ in agreement with (1.1). Again (1.1) is equivalent to pointwise weak convergence together with the uniform infinitesimality (2.7) by Lemma 2.5 of [3], hence stable hemigroups are closely related to operator selfdecomposable laws. In fact (2.9) shows that for any $s, t > 0$ we have $\mu_{0,t} = e^{-sQ}\mu_{0,t} * \mu_{e^{-s}t,t}$ such that $\mu_{0,t}$ is operator selfdecomposable for every $t > 0$. Conversely every full operator selfdecomposeable probability measure μ can be embedded into a stable hemigroup with $\mu = \mu_{0,1}$; see [23], Theorem 3.2 for details.

The connection of stable respectively semistable hemigroups to operator selfsimilar and semiselfsimilar processes with independent increments, as pointed out in the introduction, is directly apparent from above, refering to [12], [16], [17] and [24].

3. Random sums of independent random vectors

In this section we present the main results including the proofs. As stated in the introduction, the proofs will follow corresponding results on the random central limit theorem. Therefore we give some preparatory results about a mixing property in the sense of Rényi and a verification of Anscombe's condition. Throughout this section we take over the notations and assumptions about stable and semistable hemigroups and their domains of attraction laid out in the previous section. For simplicity we will write

$$S_n^{s,t} = \sum_{k=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} X_k$$

for all $0 \le s < t$ and $n \in \mathbb{N}$ in the sequel; as usual, $S_n^{s,t} = 0$ if $\lfloor ns\rfloor = \lfloor nt\rfloor$.

Lemma 3.1. *For any sequences $0 \le s_n < t_n$ with $s_n \to s$, $t_n \to t$ the sequence of random variables*

$$Z_n = A_n S_{k_n}^{s_n,t_n} = A_n \sum_{k=\lfloor k_n s_n\rfloor+1}^{\lfloor k_n t_n\rfloor} X_k$$

is mixing in the sense of Rényi, i.e.

$$P(Z_n \in C \mid B) \longrightarrow \mu_{s,t}(C)$$

for every Borel set $C \subseteq \mathbb{R}^d$, whose boundary ∂C fulfills $\mu_{s,t}(\partial C) = 0$, and every event B with positive probability.

Proof. By Theorem 1 of [22, Chapter VII, §10], it is sufficient to show that

$$P(Z_n \in C \mid Z_m \in C) \to \mu_{s,t}(C) \qquad \text{for all } m \in \mathbb{N} \text{ with } P\{Z_m \in C\} > 0. \tag{3.1}$$

This is obviously fulfilled if $\lfloor k_n s_n\rfloor \to \infty$, since for sufficiently large n we have $\lfloor k_n s_n\rfloor \ge \lfloor k_m t_m\rfloor$ and hence Z_n and Z_m are independent. If $\lfloor k_n s_n\rfloor \equiv 0$, Proposition 2.1 in [10] together with its attached remark shows that (3.1) is valid. We now give a slight modification of Hazod's proof in [10] for the case that $\lfloor k_n s_n\rfloor$ is bounded. Let $n > m$ and write $Z_n = U_n + V_n$, where

$$U_n = \begin{cases} A_n \sum\limits_{k=\lfloor k_n s_n\rfloor+1}^{\lfloor k_m t_m\rfloor} X_k & \text{if } \lfloor k_n s_n\rfloor < \lfloor k_m t_m\rfloor, \\ -A_n \sum\limits_{k=\lfloor k_m t_m\rfloor+1}^{\lfloor k_n s_n\rfloor} X_k & \text{if } \lfloor k_n s_n\rfloor \ge \lfloor k_m t_m\rfloor, \end{cases} \qquad \text{and}$$

$$V_n = A_n \sum_{k=\lfloor k_m t_m \rfloor+1}^{\lfloor k_n t_n \rfloor} X_k.$$

Then $U_n \to 0$ in probability by uniform infinitesimality (2.7). Hence the distributions of (Z_n) and (V_n) have the same weak limit behaviour. This remains true, if we switch to the underlying probability measure $Q = P(\cdot \mid Z_m \in C)$. Since V_n and Z_m are independent we get

$$\begin{aligned}\lim_{n\to\infty} P(Z_n \in C \mid Z_m \in C) &= \lim_{n\to\infty} P(V_n \in C \mid Z_m \in C) \\ &= \lim_{n\to\infty} P\{V_n \in C\} = \lim_{n\to\infty} P\{Z_n \in C\} = \mu_{s,t}(C),\end{aligned}$$

which proves the lemma. □

Theorem 3.2. *The following* Anscombe-condition *is fulfilled.*
For any $\varepsilon > 0$ and any sequences $0 \le s_n < t_n$ with $s_n \to s$, $t_n \to t$ and $s < t$ we have

$$\lim_{a\downarrow 0} \limsup_{n\to\infty} P\left\{ \max_{|m-n|\le an} \left\| B_n\left(S_m^{s_n,t_n} - S_n^{s_n,t_n}\right)\right\| > \varepsilon \right\} = 0.$$

Proof. Choose $a \in (0,1)$ so small such that $(2s+t)/(s+2t) \le 1-a$. Then for sufficiently large n we have

$$\frac{s_n}{t_n} \le \frac{s+\frac{t-s}{3}}{t-\frac{t-s}{3}} = \frac{2s+t}{s+2t} \le 1-a \le \frac{1}{1+a}.$$

In case $(1-a)n \le m \le n$ we get $\lfloor ms_n \rfloor \le \lfloor ns_n \rfloor \le \lfloor n(1-a)t_n \rfloor \le \lfloor mt_n \rfloor \le \lfloor nt_n \rfloor$ and hence we have

$$\left\|B_n\left(S_m^{s_n,t_n} - S_n^{s_n,t_n}\right)\right\| = \left\|B_n\Big(\sum_{k=\lfloor ms_n\rfloor+1}^{\lfloor ns_n\rfloor} X_k - \sum_{k=\lfloor mt_n\rfloor+1}^{\lfloor nt_n\rfloor} X_k\Big)\right\|.$$

In case $n \le m \le (1+a)n$ we get $\lfloor ns_n \rfloor \le \lfloor ms_n \rfloor \le \lfloor n(1+a)s_n \rfloor \le \lfloor nt_n \rfloor \le \lfloor mt_n \rfloor$ and hence we have

$$\left\|B_n\left(S_m^{s_n,t_n} - S_n^{s_n,t_n}\right)\right\| = \left\|B_n\Big(-\sum_{k=\lfloor ns_n\rfloor+1}^{\lfloor ms_n\rfloor} X_k + \sum_{k=\lfloor nt_n\rfloor+1}^{\lfloor mt_n\rfloor} X_k\Big)\right\|.$$

If n is sufficiently large and $0 < a \le (t-s)/(2t+s)$, we obtain

$$\begin{aligned}&\max_{|m-n|\le an} \left\|B_n\left(S_m^{s_n,t_n} - S_n^{s_n,t_n}\right)\right\| \\ &\le \max_{(1-a)n\le m\le n} \left\|B_n\left(S_m^{s_n,t_n} - S_n^{s_n,t_n}\right)\right\| + \max_{n\le m\le(1+a)n} \left\|B_n\left(S_m^{s_n,t_n} - S_n^{s_n,t_n}\right)\right\|\end{aligned}$$

$$\begin{aligned}
&\leq \max_{(1-a)n\leq m\leq n}\Big\|B_n\sum_{k=\lfloor ms_n\rfloor+1}^{\lfloor ns_n\rfloor}X_k\Big\| + \max_{(1-a)n\leq m\leq n}\Big\|B_n\sum_{k=\lfloor mt_n\rfloor+1}^{\lfloor nt_n\rfloor}X_k\Big\|\\
&\quad+\max_{n\leq m\leq (1+a)n}\Big\|B_n\sum_{k=\lfloor ns_n\rfloor+1}^{\lfloor ms_n\rfloor}X_k\Big\| + \max_{n\leq m\leq (1+a)n}\Big\|B_n\sum_{k=\lfloor nt_n\rfloor+1}^{\lfloor mt_n\rfloor}X_k\Big\|\\
&\leq \max_{0\leq l\leq an}\Big\|B_n\sum_{k=1}^{\lfloor ns_n\rfloor-\lfloor(l+\lfloor(1-a)n\rfloor)s_n\rfloor}X_{\lfloor ns_n\rfloor-k+1}\Big\|\\
&\quad+\max_{0\leq l\leq an}\Big\|B_n\sum_{k=1}^{\lfloor nt_n\rfloor-\lfloor(l+\lfloor(1-a)n\rfloor)t_n\rfloor}X_{\lfloor nt_n\rfloor-k+1}\Big\|\\
&\quad+\max_{0\leq l\leq an}\Big\|B_n\sum_{k=1}^{\lfloor(l+n)s_n\rfloor-\lfloor ns_n\rfloor}X_{\lfloor ns_n\rfloor+k}\Big\|\\
&\quad+\max_{0\leq l\leq an}\Big\|B_n\sum_{k=1}^{\lfloor(l+n)t_n\rfloor-\lfloor nt_n\rfloor}X_{\lfloor nt_n\rfloor+k}\Big\|.
\end{aligned}$$

So we have

$$\begin{aligned}
&\limsup_{n\to\infty}P\Big\{\max_{|m-n|\leq an}\big\|B_n\big(S_m^{s_n,t_n}-S_n^{s_n,t_n}\big)\big\|>\varepsilon\Big\}\\
&\leq \limsup_{n\to\infty}P\Big\{\max_{0\leq l\leq an}\Big\|B_n\sum_{k=1}^{\lfloor ns_n\rfloor-\lfloor(l+\lfloor(1-a)n\rfloor)s_n\rfloor}X_{\lfloor ns_n\rfloor-k+1}\Big\|>\frac{\varepsilon}{4}\Big\}\\
&\quad+\limsup_{n\to\infty}P\Big\{\max_{0\leq l\leq an}\Big\|B_n\sum_{k=1}^{\lfloor nt_n\rfloor-\lfloor(l+\lfloor(1-a)n\rfloor)t_n\rfloor}X_{\lfloor nt_n\rfloor-k+1}\Big\|>\frac{\varepsilon}{4}\Big\}\\
&\quad+\limsup_{n\to\infty}P\Big\{\max_{0\leq l\leq an}\Big\|B_n\sum_{k=1}^{\lfloor(l+n)s_n\rfloor-\lfloor ns_n\rfloor}X_{\lfloor ns_n\rfloor+k}\Big\|>\frac{\varepsilon}{4}\Big\} \qquad (3.2)\\
&\quad+\limsup_{n\to\infty}P\Big\{\max_{0\leq l\leq an}\Big\|B_n\sum_{k=1}^{\lfloor(l+n)t_n\rfloor-\lfloor nt_n\rfloor}X_{\lfloor nt_n\rfloor+k}\Big\|>\frac{\varepsilon}{4}\Big\}.
\end{aligned}$$

We now apply Ottaviani's inequality (see Lemma 3.21 in [5]) to each of the four parts on the right-hand side of (3.2). For the first two parts observe that for any positive sequence $r_n\to r$ Ottaviani's inequality yields

$$P\Big\{\max_{0\leq l\leq an}\Big\|B_n\sum_{k=1}^{\lfloor nr_n\rfloor-\lfloor(l+\lfloor(1-a)n\rfloor)r_n\rfloor}X_{\lfloor nr_n\rfloor-k+1}\Big\|>\frac{\varepsilon}{4}\Big\}$$

$$\leq \frac{1}{1-m_n(r_n,a)}\, P\Big\{\Big\|B_n \sum_{k=1}^{\lfloor nr_n\rfloor-\lfloor\lfloor(1-a)n\rfloor r_n\rfloor} X_{\lfloor nr_n\rfloor-k+1}\Big\| > \frac{\varepsilon}{8}\Big\}$$

with

$$\begin{aligned} m_n(r_n,a) &= \max_{0\leq l\leq an} P\Big\{\Big\|B_n \sum_{k=\lfloor nr_n\rfloor-\lfloor(l+\lfloor(1-a)n\rfloor)r_n\rfloor+1}^{\lfloor nr_n\rfloor-\lfloor\lfloor(1-a)n\rfloor r_n\rfloor} X_{\lfloor nr_n\rfloor-k+1}\Big\| > \frac{\varepsilon}{8}\Big\} \\ &= \max_{0\leq l\leq an} P\Big\{\Big\|B_n \sum_{k=\lfloor\lfloor(1-a)n\rfloor r_n\rfloor+1}^{\lfloor(l+\lfloor(1-a)n\rfloor)r_n\rfloor} X_k\Big\| > \frac{\varepsilon}{8}\Big\} \\ &= \max_{j\in I_n} P\Big\{\big\|B_n S_{\lfloor(1-a)n\rfloor}^{r_n,(j+1)r_n}\big\| > \frac{\varepsilon}{8}\Big\} \end{aligned}$$

where $I_n = \big\{l/\lfloor(1-a)n\rfloor \mid 0\leq l\leq an,\, l\in\mathbb{N}_0\big\}$. In particular we get for $m_n(r_n,a)<1$

$$\begin{aligned} &P\Big\{\max_{0\leq l\leq an}\Big\|B_n \sum_{k=1}^{\lfloor nr_n\rfloor-\lfloor(l+\lfloor(1-a)n\rfloor)r_n\rfloor} X_{\lfloor nr_n\rfloor-k+1}\Big\| > \frac{\varepsilon}{4}\Big\} \\ &\leq \frac{m_n(r_n,a)}{1-m_n(r_n,a)}. \end{aligned} \tag{3.3}$$

Now choose $j_a(n)\in I_n$ such that

$$m_n(r_n,a) = P\Big\{\big\|\big(B_nB_{\lfloor(1-a)n\rfloor}^{-1}\big)B_{\lfloor(1-a)n\rfloor}S_{\lfloor(1-a)n\rfloor}^{r_n,(j_a(n)+1)r_n}\big\| > \frac{\varepsilon}{8}\Big\}.$$

By (2.5) and (2.6) we have

$$\begin{aligned} &\mathrm{LIM}\Big\{\big(B_nB_{\lfloor(1-a)n\rfloor}^{-1}\big)B_{\lfloor(1-a)n\rfloor}S_{\lfloor(1-a)n\rfloor}^{r_n,(j_a(n)+1)r_n}(P)\Big\} \\ &= \big\{(1-a)^Q\lambda^{-Q}\mu_{\lambda r,\lambda(j_a+1)r}\big|\lambda\in[1,c),\ j_a\in\mathrm{LIM}\{j_a(n)\}\big\}, \end{aligned}$$

where $0\leq j_a\leq a/(1-a)$ is an arbitrary limit point of $(j_a(n))$. In view of the portmanteau theorem, for some $\lambda_a\in[1,c)$ and some $0\leq l_a\leq a/(1-a)$ we get

$$\limsup_{n\to\infty} m_n(r_n,a) \leq (1-a)^Q\lambda_a^{-Q}\mu_{\lambda_a r,\lambda_a(l_a+1)r}\{\|x\|\geq\varepsilon/8\}$$

and since $l_a\to 0$ if $a\downarrow 0$ we further have by (2.2)

$$\limsup_{a\downarrow 0}(1-a)^Q\lambda_a^{-Q}\mu_{\lambda_a r,\lambda_a(l_a+1)r}\{\|x\|\geq\varepsilon/8\} \leq \varepsilon_0\{\|x\|\geq\varepsilon/8\} = 0.$$

Then in view of (3.3) we obtain

$$\lim_{a\downarrow 0}\limsup_{n\to\infty} P\Big\{\max_{0\leq l\leq an}\Big\|B_n \sum_{k=1}^{\lfloor nr_n\rfloor-\lfloor(l+\lfloor(1-a)n\rfloor)r_n\rfloor} X_{\lfloor nr_n\rfloor-k+1}\Big\| > \frac{\varepsilon}{4}\Big\} = 0.$$

Analogously we observe for the last two parts of the right-hand side of (3.2), that for any positive sequence $r_n \to r$ we have

$$\lim_{a\downarrow 0} \limsup_{n\to\infty} P\Big\{ \max_{0\le l\le an} \Big\| B_n \sum_{k=1}^{\lfloor (l+n)r_n\rfloor - \lfloor nr_n\rfloor} X_{\lfloor nr_n\rfloor + k} \Big\| > \frac{\varepsilon}{4} \Big\} = 0.$$

This concludes the proof. □

The next Lemma is crucial for the proof of random limit theorems. Recall the three steps of proof for the random central limit theorem given in the introduction. Whereas in the first step $T_n = \lfloor nD \rfloor$ a mixing property suffices, in the last two steps approximations of T_n by $\lfloor nD\rfloor$ with discrete D, and of arbitrary $D > 0$ by discrete random variables, respectively, apply. Anscombe's condition tells us about the quality of these approximations for random sums, as follows.

Lemma 3.3. *Let (U_n) be a sequence of positive integer valued random variables with $U_n/n \to U$ in probability for some positive random variable U. If U is discrete we have for any sequences $0 \le s_n < t_n$ with $s_n \to s$, $t_n \to t$ and $s < t$*

$$B_n \left(S^{s_n,t_n}_{U_n} - S^{s_n,t_n}_{\lfloor nU\rfloor} \right) \to 0. \tag{3.4}$$

in probability. Further define positive discrete random variables V_m by

$$V_m = k \cdot 2^{-m} \quad \text{if} \quad (k-1)2^{-m} < U \le k \cdot 2^{-m}$$

and let positive integer valued random variables be given by $U_{m,n} = U_n + \lfloor n(V_m - U)\rfloor$, then for every $\varepsilon > 0$ we have

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\left\{ \big\| B_n \big(S^{s_n,t_n}_{U_n} - S^{s_n,t_n}_{\lfloor nV_m\rfloor} \big) \big\| > \varepsilon \right\} = 0 \tag{3.5}$$

and

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\left\{ \big\| B_n \big(S^{s_n,t_n}_{U_{m,n}} - S^{s_n,t_n}_{\lfloor nV_m\rfloor} \big) \big\| > \varepsilon \right\} = 0. \tag{3.6}$$

Proof. Let (u_k) be an at most countable sequence with $p_k = P\{U = u_k\} > 0$ and $\sum_k p_k = 1$. For $\delta > 0$ choose $N \in \mathbb{N}$ such that $\sum_{k>N} p_k \le \delta/2$. For any $a > 0$ we have

$$\limsup_{n\to\infty} P\big\{ |U_n - \lfloor nU\rfloor| > a\lfloor nU\rfloor \big\} = \limsup_{n\to\infty} P\big\{ \big|(U_n/\lfloor nU\rfloor) - 1\big| > a \big\} = 0.$$

Thus, with the events $E_{n,k} = \{|U_n - \lfloor nu_k\rfloor| \le a\lfloor nu_k\rfloor\}$, we obtain

$$\limsup_{n\to\infty} P\Big\{ \big\| B_n \big(S^{s_n,t_n}_{U_n} - S^{s_n,t_n}_{\lfloor nU\rfloor} \big) \big\| > \varepsilon \Big\}$$

$$
\begin{aligned}
&\le \limsup_{n\to\infty} P\Big\{\big\|B_n\big(S_{U_n}^{s_n,t_n}-S_{\lfloor nU\rfloor}^{s_n,t_n}\big)\big\|>\varepsilon,\ |U_n-\lfloor nU\rfloor|\le a\lfloor nU\rfloor\Big\}\\
&\quad+\limsup_{n\to\infty} P\big\{|U_n-\lfloor nU\rfloor|>a\lfloor nU\rfloor\big\}\\
&\le \frac{\delta}{2}+\sum_{k\le N} p_k \limsup_{n\to\infty} P\Big(\big\{\big\|B_n\big(S_{U_n}^{s_n,t_n}-S_{\lfloor nu_k\rfloor}^{s_n,t_n}\big)\big\|>\varepsilon\big\}\cap E_{n,k}\Big| U=u_k\Big)\\
&\le \frac{\delta}{2}+\sum_{k\le N} \limsup_{n\to\infty} P\Big\{\max_{|m-\lfloor nu_k\rfloor|\le a\lfloor nu_k\rfloor}\big\|B_n\big(S_m^{s_n,t_n}-S_{\lfloor nu_k\rfloor}^{s_n,t_n}\big)\big\|>\varepsilon\Big\}.
\end{aligned}
$$

Let $M=2\max_{k\le N}\|u_k^Q\|$ then for sufficiently large n we have $\|B_nB_{\lfloor nu_k\rfloor}^{-1}\|\le M$ by (2.5). Choose $a>0$ sufficiently small such that by Theorem 3.2 we have for all $k\le N$

$$
\limsup_{n\to\infty} P\Big\{\max_{|m-\lfloor nu_k\rfloor|\le a\lfloor nu_k\rfloor}\big\|B_{\lfloor nu_k\rfloor}\big(S_m^{s_n,t_n}-S_{\lfloor nu_k\rfloor}^{s_n,t_n}\big)\big\|>\frac{\varepsilon}{M}\Big\}\le\frac{\delta}{2N}.
$$

Then (3.4) follows, since

$$
\begin{aligned}
&P\Big\{\max_{|m-\lfloor nu_k\rfloor|\le a\lfloor nu_k\rfloor}\big\|B_n\big(S_m^{s_n,t_n}-S_{\lfloor nu_k\rfloor}^{s_n,t_n}\big)\big\|>\varepsilon\Big\}\\
&\le P\Big\{\max_{|m-\lfloor nu_k\rfloor|\le a\lfloor nu_k\rfloor}\big\|B_nB_{\lfloor nu_k\rfloor}^{-1}\big\|\,\big\|B_{\lfloor nu_k\rfloor}\big(S_m^{s_n,t_n}-S_{\lfloor nu_k\rfloor}^{s_n,t_n}\big)\big\|>\varepsilon\Big\}\\
&\le P\Big\{\max_{|m-\lfloor nu_k\rfloor|\le a\lfloor nu_k\rfloor}\big\|B_{\lfloor nu_k\rfloor}\big(S_m^{s_n,t_n}-S_{\lfloor nu_k\rfloor}^{s_n,t_n}\big)\big\|>\frac{\varepsilon}{M}\Big\}\le\frac{\delta}{2N}
\end{aligned}
$$

and thus

$$
\limsup_{n\to\infty} P\Big\{\big\|B_n\big(S_{U_n}^{s_n,t_n}-S_{\lfloor nU\rfloor}^{s_n,t_n}\big)\big\|>\varepsilon\Big\}\le\frac{\delta}{2}+\sum_{k\le N}\frac{\delta}{2N}=\delta.
$$

We further have $0\le V_m-U<2^{-m}$, thus $V_m\to U$ in probability and for any fixed $m\in\mathbb{N}$ we have

$$
\frac{U_{m,n}}{n}=\frac{U_n}{n}+\frac{\lfloor n(V_m-U)\rfloor}{n}\to V_m
$$

in probability. Choose $m_0\in\mathbb{N}$ with $P\{U>m_0\}\le\delta/3$ and $m_1\ge m_0$ such that for all $m\ge m_1$ we have $P\{U\le m2^{-m}\}\le\delta/3$. Let $0<M=2\sup_{0\le r\le m_0}\|r^Q\|<\infty$ then in view of (2.5) we have for sufficiently large n

$$
\max_{k=m+1,\dots,m_02^m}\|B_nB_{\lfloor nk2^{-m}\rfloor}^{-1}\|\le\sup_{0\le r\le m_0}\|B_nB_{\lfloor nr\rfloor}^{-1}\|\le M.
$$

In view of Theorem 3.2 further choose $m_2\ge m_1$ such that for all $m\ge m_2$ we have

$$
\limsup_{n\to\infty} P\Big\{\max_{|l-n|\le m^{-1}n}\big\|B_n\big(S_l^{s_n,t_n}-S_n^{s_n,t_n}\big)\big\|>\frac{\varepsilon}{M}\Big\}\le\frac{\delta}{3}. \tag{3.7}
$$

Let $p_k(n) = \lfloor nk \cdot 2^{-m} \rfloor$ and define the events

$$\begin{aligned} E_k &= \big\{(k-1)2^{-m} < U \le k \cdot 2^{-m}\big\} = \big\{V_m = k \cdot 2^{-m}\big\}, \\ G_{k,n} &= \big\{|U_n - p_k(n)| \le n \cdot 2^{-m}\big\}, \\ \widetilde{G}_{k,n} &= \big\{|U_n - p_k(n)| \le m^{-1} p_k(n)\big\}, \\ A_{k,n} &= \Big\{ \max_{|l - p_k(n)| \le m^{-1} p_k(n)} \big\| B_{p_k(n)} \big(S_l^{s_n,t_n} - S_{p_k(n)}^{s_n,t_n} \big) \big\| > \frac{\varepsilon}{M} \Big\}. \end{aligned}$$

Then in view of (3.7) and Lemma 3 in [4] we have for all $k \in \mathbb{N}$

$$\limsup_{n\to\infty} P(A_{k,n} \mid E_k) \le \delta/3. \tag{3.8}$$

Further observe $n \cdot 2^{-m} \le p_k(n)/(k-1) \le m^{-1} p_k(n)$ for sufficiently large n and $k \ge m+1$, hence $G_{k,n}$ is contained in $\widetilde{G}_{k,n}$. In view of

$$\limsup_{n\to\infty} P\big\{|U_n - \lfloor nV_m \rfloor| > n \cdot 2^{-m}\big\} \le P\big\{|U - V_m| \ge 2^{-m}\big\} = 0$$

(3.5) follows from (3.8), since for $m \ge m_2$ we have

$$\begin{aligned} &\limsup_{n\to\infty} P\Big\{ \big\| B_n \big(S_{U_n}^{s_n,t_n} - S_{\lfloor nV_m \rfloor}^{s_n,t_n} \big) \big\| > \varepsilon \Big\} \\ &\le \limsup_{n\to\infty} P\Big\{ \big\| B_n \big(S_{U_n}^{s_n,t_n} - S_{\lfloor nV_m \rfloor}^{s_n,t_n} \big) \big\| > \varepsilon,\ |U_n - \lfloor nV_m \rfloor| \le n \cdot 2^{-m} \Big\} \\ &\quad + \limsup_{n\to\infty} P\big\{|U_n - \lfloor nV_m \rfloor| > n \cdot 2^{-m}\big\} \\ &\le \limsup_{n\to\infty} \sum_{k=m+1}^{m_0 2^m} P\Big(\Big\{ \big\| B_n \big(S_{U_n}^{s_n,t_n} - S_{p_k(n)}^{s_n,t_n} \big) \big\| > \varepsilon \Big\} \cap G_{k,n} \cap E_k \Big) \\ &\quad + P\{U > m_0\} + P\{U \le m\, 2^{-m}\} \\ &\le \frac{2\delta}{3} + \limsup_{n\to\infty} \sum_{m+1}^{m_0 2^m} P\Big(\Big\{ \big\| B_n \big(S_{U_n}^{s_n,t_n} - S_{p_k(n)}^{s_n,t_n} \big) \big\| > \varepsilon \Big\} \cap \widetilde{G}_{k,n} \cap E_k \Big) \\ &\le \frac{2\delta}{3} + \sum_{k=m+1}^{m_0 2^m} P(E_k) \limsup_{n\to\infty} P(A_{k,n} \mid E_k) \\ &\le \frac{2\delta}{3} + \frac{\delta}{3} \sum_{k=m+1}^{m_0 2^m} P(E_k) \le \delta. \end{aligned}$$

This proves (3.5) and analogously we get (3.6), since for all $m \in \mathbb{N}$

$$\lim_{n\to\infty} P\big\{|U_{m,n} - \lfloor nV_m \rfloor| > m^{-1} \lfloor nV_m \rfloor\big\} = 0$$

and with the events $H_{k,n} = \{|U_{m,n} - p_k(n)| \le m^{-1} p_k(n)\}$ we obtain for $m \ge m_2$, as before

$$\begin{aligned}
&\limsup_{n\to\infty} P\Big\{ \big\| B_n\big(S^{s_n,t_n}_{U_{m,n}} - S^{s_n,t_n}_{\lfloor nV_m \rfloor}\big) \big\| > \varepsilon \Big\} \\
&\le \frac{2\delta}{3} + \limsup_{n\to\infty} \sum_{k=m+1}^{m_0 2^m} P\Big(\Big\{ \big\| B_n\big(S^{s_n,t_n}_{U_{m,n}} - S^{s_n,t_n}_{p_k(n)}\big) \big\| > \varepsilon \Big\} \cap H_{k,n} \cap E_k \Big) \\
&\le \frac{2\delta}{3} + \sum_{k=m+1}^{m_0 2^m} P(E_k) \limsup_{n\to\infty} P(A_{k,n} \mid E_k) \le \delta.
\end{aligned}$$

This concludes the proof. □

Now we are well prepared to prove the main result of this section, a limit theorem for random sums of independent random vectors in the domain of attraction of a stable or semistable hemigroup. The proof is divided into three consecutive steps, following the proof of the random central limit theorem described in the introduction.

Theorem 3.4. *Let (T_n) be a sequence of positive integer valued random variables such that $T_n/k_n \to D$ in probability for some positive random variable D with distribution ρ. Then we have for any sequences $0 \le s_n < t_n$ with $s_n \to s$, $t_n \to t$ and $s < t$*

$$A_n S^{s_n,t_n}_{T_n}(P) = A_n\Big(\sum_{k=\lfloor T_n s_n \rfloor + 1}^{\lfloor T_n t_n \rfloor} X_k \Big)(P) \xrightarrow{\mathrm{w}} \int_0^\infty \mu_{rs,rt}\, d\rho(r).$$

Proof. *Step 1*: $T_n = \lfloor k_n D \rfloor$, where $D > 0$ is a discrete random variable. Let (d_k) be an at most countable sequence with $p_k = P\{D = d_k\} > 0$ and $\sum_k p_k = 1$. For $\delta > 0$ choose $N \in \mathbb{N}$ such that $\sum_{k>N} p_k \le \delta$. For every $k \in \mathbb{N}$ the sequence

$$A_n S^{s_n,t_n}_{\lfloor k_n d_k \rfloor} = A_n S^{s_n \lfloor k_n d_k \rfloor / k_n,\, t_n \lfloor k_n d_k \rfloor / k_n}_{k_n}$$

is a mixing sequence of random vectors in the sense of Rényi by Lemma 3.1. Thus for any closed set C we get

$$\begin{aligned}
\limsup_{n\to\infty} P\big\{ A_n S^{s_n,t_n}_{T_n} \in C \big\} &\le \delta + \sum_{k\le N} p_k \limsup_{n\to\infty} P\big(A_n S^{s_n,t_n}_{\lfloor k_n d_k \rfloor} \in C \big| D = d_k \big) \\
&\le \delta + \sum_{k\le N} p_k\, \mu_{d_k s, d_k t}(C) \\
&\le \delta + \int_0^\infty \mu_{rs,rt}(C)\, d\rho(r).
\end{aligned}$$

By the portmanteau theorem, this proves the theorem in case $T_n = \lfloor k_n D \rfloor$, since $\delta > 0$ is arbitrary.

Step 2: $T_n/k_n \to D$ in probability, where $D > 0$ is a discrete random variable.
Let us write

$$A_n S^{s_n,t_n}_{T_n} = A_n S^{s_n,t_n}_{\lfloor k_n D \rfloor} + A_n \big(S^{s_n,t_n}_{T_n} - S^{s_n,t_n}_{\lfloor k_n D \rfloor} \big),$$

then by the first step it is sufficient to show

$$A_n \big(S^{s_n,t_n}_{T_n} - S^{s_n,t_n}_{\lfloor k_n D \rfloor} \big) \to 0 \tag{3.9}$$

in probability. Let us write $n = \alpha_n k_{p_n}$ with $p_n \in \mathbb{N}$, $k_{p_n} \le n < k_{p_n+1}$ and define $U_n = \lfloor \alpha_n T_{p_n} \rfloor$. Then we have $U_{k_n} = T_n$ and $U_n/n \to D$ in probability. It follows from (3.4), that

$$B_n \big(S^{s_n,t_n}_{U_n} - S^{s_n t_n}_{\lfloor nD \rfloor} \big) \to 0$$

in probability. In particular, we get (3.9) along the subsequence (k_n), since (3.4) only depends on the parameter limits s and t.

Step 3: $T_n/k_n \to D$ in probability where $D > 0$ is arbitrary.
For $m \in \mathbb{N}$ let random variables D_m with distributions ρ_m be given by

$$D_m = k \cdot 2^{-m} \quad \text{ if } \quad (k-1)2^{-m} < D \le k \cdot 2^{-m}.$$

D_m is positive and discrete and we have $0 \le D_m - D < 2^{-m}$, thus $D_m \to D$ in probability. We further define $D_{m,n} = U_n + \lfloor n(D_m - D) \rfloor$, where U_n is as in the second step. Then we have for fixed $m \in \mathbb{N}$

$$\frac{D_{m,n}}{n} = \frac{U_n}{n} + \frac{\lfloor n(D_m - D) \rfloor}{n} \to D_m$$

in probability. Let us write

$$A_n S^{s_n,t_n}_{T_n} = A_n S^{s_n,t_n}_{D_{m,k_n}} + A_n \big(S^{s_n,t_n}_{T_n} - S^{s_n,t_n}_{D_{m,k_n}} \big),$$

then by the second step we have as $n \to \infty$ and $m \to \infty$ successively

$$A_n S^{s_n,t_n}_{D_{m,k_n}}(P) \xrightarrow{\text{w}} \int_0^\infty \mu_{rs,rt} \, d\rho_m(r) \xrightarrow{\text{w}} \int_0^\infty \mu_{rs,rt} \, d\rho(r).$$

By an extension of Cramér's theorem for doubly indexed sequences (see Lemma 2 in [4]), it is now sufficient to prove that for all $\varepsilon > 0$

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\Big\{ \big\| A_n \big(S^{s_n,t_n}_{T_n} - S^{s_n,t_n}_{D_{m,k_n}} \big) \big\| > \varepsilon \Big\} = 0. \tag{3.10}$$

For (3.10) it is sufficient to prove for all $\varepsilon > 0$

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\Big\{ \big\| A_n \big(S^{s_n,t_n}_{T_n} - S^{s_n,t_n}_{\lfloor k_n D_m \rfloor} \big) \big\| > \varepsilon \Big\} = 0 \tag{3.11}$$

and

$$\lim_{m\to\infty}\limsup_{n\to\infty} P\Big\{\big\|A_n\big(S^{s_n,t_n}_{D_{m,k_n}} - S^{s_n,t_n}_{\lfloor k_n D_m\rfloor}\big)\big\| > \varepsilon\Big\} = 0. \tag{3.12}$$

As in the second step, (3.11) and (3.12) follow by (3.5) and (3.6) along the subsequence (k_n). □

Remark 3.5. We restricted our considerations in Theorem 3.4 to parameter sequences $0 \le s_n < t_n$ with $s_n \to s$, $t_n \to t$ and $s < t$, but Theorem 3.4 remains valid if $s = t$. Our proof of Anscombe's condition requires $s < t$, hence we have to apply a different method in case $s = t$, as follows.

Since $T_n/k_n \to D > 0$ in probability, every subsequence (n') contains a further subsequence (n'') such that $T_n/k_n \to D$ almost surely along (n''). Especially $T_n \to \infty$ almost surely along (n''). In case $s = t$ by (2.6) we have $B_n S_n^{s_n,t_n} \to 0$ in probability. Hence we can choose a further subsequence (n''') such that $B_{T_n} S_{T_n}^{s_n,t_n} \to 0$ almost surely along (n'''). Then $B_{k_n} B_{T_n}^{-1} \to D^Q$ almost surely along (n''') such that

$$A_n S_{T_n}^{s_n,t_n} = (B_{k_n} B_{T_n}^{-1}) B_{T_n} S_{T_n}^{s_n,t_n} \to 0 \tag{3.13}$$

almost surely along (n'''). Altogether every subsequence (n') contains a further subsequence (n''') such that (3.13) holds almost surely along (n'''), which shows that (3.13) holds in probability. Note that in case $s = t$ the limit in Theorem 3.4 is ε_0.

The stochastic compactness in (2.6) enables us to determine the limiting behavior of random sums if the random sampling sequence proportionally to n instead of k_n converges in probability to a positive random variable, as follows. Since in the stable case we assumed $k_n = n$, we only have to consider domains of attraction of semistable hemigroups in view of Theorem 3.4.

Theorem 3.6. *Let (U_n) be a sequence of positive integer valued random variables such that $U_n/n \to U$ in probability for some positive random variable U with distribution η. Then for any sequences $0 \le s_n < t_n$ with $s_n \to s$, $t_n \to t$ the sequence of distributions of the random vectors $B_n S_{U_n}^{s_n,t_n}$ is weakly relatively compact, where the set of weak limit points is*

$$\mathrm{LIM}\left\{B_n S_{U_n}^{s_n,t_n}(P)\right\} = \left\{ \int_0^\infty \lambda^{-Q}\mu_{\lambda r s,\lambda r t}\, d\eta(r) \,\middle|\, \lambda \in [1,c) \right\}.$$

Proof. In case $s = t$ we obtain as in (3.13) (see Remark 3.5) that $B_n S_{U_n}^{s_n,t_n} \to 0$ in probability, such that it remains to prove the assertion in case $s < t$. Write $n = \lambda_n k_{p_n}$ with $p_n \in \mathbb{N}$ and $k_{p_n} \le n < k_{p_n+1}$. Thus (λ_n) is relatively compact in $[1,c]$. Let (n') be a subsequence such that $\lambda_n \to \lambda$ along (n'), where $\lambda \in [1,c]$ is an arbitrary limit point. Note that any $\lambda \in [1,c]$ occurs as a limit point taking the subsequence $n' = \lfloor \lambda k_n \rfloor$. Let us write

$$B_n S_{U_n}^{s_n,t_n} = B_n S_{\lfloor \lambda_n U_{k_{p_n}} \rfloor}^{s_n,t_n} + B_n\big(S_{U_n}^{s_n,t_n} - S_{\lfloor \lambda_n U_{k_{p_n}} \rfloor}^{s_n,t_n}\big).$$

Since we assumed $B_{k_n} = A_n$, Proposition 3.4 of [3] together with the proof of Theorem 4.2.13 of [18] shows that the embedding sequence fulfills $B_n = \lambda_n^{-Q} A_{p_n}$. Thus by Theorem 3.4 we have

$$\begin{aligned} B_n S_{\lfloor \lambda_n U_{k_{p_n}} \rfloor}^{s_n,t_n}(P) &= \lambda_n^{-Q} A_{p_n} S_{\lfloor \lambda_n U_{k_{p_n}} \rfloor}^{s_n,t_n}(P) \\ &\xrightarrow{\text{w}} \int_0^\infty \lambda^{-Q} \mu_{\lambda r s, \lambda r t}\, d\eta(r) \end{aligned} \tag{3.14}$$

along the subsequence (n'). Hence it is sufficient to prove

$$B_n\big(S_{U_n}^{s_n,t_n} - S_{\lfloor \lambda_n U_{k_{p_n}} \rfloor}^{s_n,t_n}\big) \to 0 \tag{3.15}$$

in probability. Let V_m be defined as in Lemma 3.3. For (3.15) it is sufficient to prove for every $\varepsilon > 0$

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\Big\{\big\|B_n\big(S_{U_n}^{s_n,t_n} - S_{\lfloor n V_m \rfloor}^{s_n,t_n}\big)\big\| > \varepsilon\Big\} = 0 \tag{3.16}$$

and

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\Big\{\big\|B_n\big(S_{\lfloor \lambda_n U_{k_{p_n}} \rfloor}^{s_n,t_n} - S_{\lfloor n V_m \rfloor}^{s_n,t_n}\big)\big\| > \varepsilon\Big\} = 0. \tag{3.17}$$

Since $U_n/n \to U$ and $\lfloor \lambda_n U_{k_{p_n}} \rfloor / n \to U$ in probability (3.16) and (3.17) follow by (3.5) and (3.6). Note that in view of the scaling relation (2.4), the limits in (3.14) for $\lambda = 1$ and $\lambda = c$ coincide. □

Remark 3.7. We emphasize that the methods of the last section apply also for random sums with random normings, namely under the conditions and with the notations of Theorems 3.4 and 3.6 we have

$$B_{T_n} S_{T_n}^{s_n,t_n}(P) \xrightarrow{\text{w}} \int_0^\infty r^{-Q} \mu_{rs,rt}\, d\rho(r) \tag{3.18}$$

and in the semistable case the sequence of distributions of the random vectors $B_{U_n} S_{U_n}^{s_n,t_n}$ is weakly relatively compact, where the set of weak limit points is

$$\mathrm{LIM}\,\big\{B_{U_n} S_{U_n}^{s_n,t_n}(P)\big\} = \bigg\{ \int_0^\infty (\lambda r)^{-Q} \mu_{\lambda r s, \lambda r t}\, d\eta(r) \,\bigg|\, \lambda \in [1,c) \bigg\}.$$

Note that for stable hemigroups the limit in (3.18) is $\mu_{s,t}$. For the proof of these assertions one mainly has to apply an appropriate mixing result and Lemma 3.3, where again, as in the proofs of Theorems 3.4 and 3.6, the random vectors $B_{T_n}S_{T_n}^{s_n,t_n}$ and $B_{U_n}S_{U_n}^{s_n,t_n}$ have to be suitably decomposed. For example, analogously to step 2 of the proof of Theorem 3.4, let us write

$$B_{T_n}S_{T_n}^{s_n,t_n} = \big(B_{T_n}B_{\lfloor k_nD\rfloor}^{-1}\big)\,B_{\lfloor k_nD\rfloor}S_{\lfloor k_nD\rfloor}^{s_n,t_n} + \big(B_{T_n}B_{k_n}^{-1}\big)\,A_n\big(S_{T_n}^{s_n,t_n} - S_{\lfloor k_nD\rfloor}^{s_n,t_n}\big)$$

and note that in view of the regular variation of (B_n) and $T_n/k_n \to D > 0$ in probability, we have $B_{T_n}B_{k_n}^{-1} \to D^{-Q}$ in probability and $B_{T_n}B_{\lfloor k_nD\rfloor}^{-1}$ converges to the identity in probability. Further details are available from the author upon request.

Remark 3.8. Due to our assumptions in the second section, we have chosen the norming operators (A_n) to be embeddable into the regularly varying sequence (B_n) such that $A_n = B_{k_n}$. Note that by Theorems 2.3 and 3.5 of [3], for stable respectively semistable hemigroups, arbitrary norming operators that ensure (1.1) only have to be disturbed by elements of the symmetry group

$$\mathcal{S} = \big\{C \in \mathrm{GL}(\mathbb{R}^d)\big|C\mu_{s,t} = \mu_{s,t} \text{ for all } 0 \le s \le t\big\}$$

to fulfill the embedding property, i.e. arbitrary norming operators are of the form C_nA_n with $C_n \in \mathcal{S}$ and $A_n = B_{k_n}$. Since $\mathcal{S}$ is compact (see [11]), any limit point C of (C_n) belongs to $\mathcal{S}$ and under the conditions of Theorem 3.4 we get along certain subsequences

$$C_nA_nS_{T_n}^{s_n,t_n}(P) \xrightarrow{\text{w}} C\int_0^\infty \mu_{rs,rt}\,d\rho(r) = \int_0^\infty \mu_{rs,rt}\,d\rho(r).$$

This proves that Theorem 3.4 holds for arbitrary norming operators. Note that this result fails for random normings of the previous Remark 3.7, as the following example illustrates.

Example 3.9. Let (X_n) be an i.i.d. sequence of standard normally distributed random variables. Then clearly (X_n) is in the domain of attraction of the stable hemigroup $\mathcal{H} = \{\mathcal{N}_{0,t-s} \mid 0 \le s \le t\}$ of normal distributions. In view of symmetry, we can use $A_n = n^{-1/2}$ as well as $C_n = (-1)^nA_n$ as norming sequences in (1.1). Let

$$T_n = \begin{cases} n & \text{if } S_n > 0 \text{ and } n \text{ is even,} \\ n-1 & \text{if } S_n \ge 0 \text{ and } n \text{ is odd,} \\ n+1 & \text{if } S_n \le 0 \text{ and } n \text{ is even,} \\ n & \text{if } S_n < 0 \text{ and } n \text{ is odd,} \end{cases}$$

then $T_n/n \to 1$ in probability and hence by Remark 3.7 we have

$$A_{T_n}S_{T_n}(P) = A_{T_n}\Big(\sum_{k=1}^{T_n} X_k\Big)(P) \xrightarrow{\text{w}} \mathcal{N}_{0,1}.$$

Since $A_nS_n(P) = \mathcal{N}_{0,1}$ for all $n \in \mathbb{N}$, by spherical symmetry of the product measure $\mathcal{N}_{0,1} \otimes \mathcal{N}_{0,1}$ we compute

$$\begin{aligned} P\big\{S_{n+1} \geq 0, S_n \leq 0\big\} &= P\big\{X_{n+1} \geq -S_n, n^{-1/2}S_n \leq 0\big\} \\ &= P\big\{(n^{-1/2}S_n, X_{n+1}) \in \{(x,y) \mid x \leq 0,\, y \geq -n^{1/2}x\}\big\} \\ &= \mathcal{N}_{0,1} \otimes \mathcal{N}_{0,1}\{(x,y) \mid x \leq 0,\, y \geq -n^{1/2}x\} \\ &= \frac{\arctan n^{-1/2}}{2\pi} \longrightarrow 0. \end{aligned}$$

If we use the norming sequence (C_n) instead of (A_n) we get

$$\begin{aligned} P\{C_{T_n}S_{T_n} \leq 0\} &= \begin{cases} P\{C_nS_n \leq 0, S_n > 0\} + P\{C_{n+1}S_{n+1} \leq 0,\ S_n \leq 0\} \\ \hfill \text{if } n \text{ is even,} \\ P\{C_{n-1}S_{n-1} \leq 0, S_n \geq 0\} + P\{C_nS_n \leq 0,\ S_n < 0\} \\ \hfill \text{if } n \text{ is odd,} \end{cases} \\ &= \begin{cases} P\{S_{n+1} \geq 0, S_n \leq 0\} & \text{if } n \text{ is even,} \\ P\{S_{n-1} \leq 0, S_n \geq 0\} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Altogether we obtain

$$\lim_{n\to\infty} P\{C_{T_n}S_{T_n} \leq 0\} = 0 \neq 1/2 = \lim_{n\to\infty} P\{A_{T_n}S_{T_n} \leq 0\},$$

which shows that we cannot use arbitrary norming sequences if we are interested in limits for random sums with random normings as in (3.18). However, if we switch to a regularly varying norming sequence, which we can always do for domains of attraction of stable and semistable hemigroups, the limiting behaviour is as pointed out in Remark 3.7.

4. Special case of identically distributed random vectors

Due to the absence of an assumption on identical distribution of (X_n), we avoided additional shift-terms in (1.1), since shifted sequences of partial sums can be treated telescopic for fulfillment of (1.1). Throughout this section we will additionally assume that (X_n) is an i.i.d. sequence with common distribution $\nu = X_1(P)$. Then in general we will need additional shift-terms to achieve that ν is in the domain of attraction of an operator stable or semistable law. For details on operator stable and semistable distributions and their domains of attraction we refer to [18]. We will now argue that affine normalized sequences of partial sums of i.i.d. random vectors are within a hemigroup setting. Assume the existence of linear operators A_n

and shifts $a_n \in \mathbb{R}^d$ such that

$$A_n\big(S_{k_n} - a_n\big)(P) = A_n\Big(\sum_{k=1}^{k_n} X_k - a_n\Big)(P) = A_n\big(\nu^{k_n} * \varepsilon_{-a_n}\big) \xrightarrow{\mathrm{w}} \mu, \tag{4.1}$$

where μ is a full probability measure.

In the stable case $k_n = n$ the limit μ in (4.1) fulfills $\mu^t = t^Q\mu * \varepsilon_{u(t)}$ for any $t > 0$, some exponent $Q \in \mathrm{GL}(\mathbb{R}^d)$, whose eigenvalues belong to $\{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) \geq 1/2\}$, and some continuous drift-function $t \mapsto u(t) \in \mathbb{R}^d$ with $u(t) \to 0$ as $t \to 0$. Then ν is said to belong to the domain of attraction of the operator stable distribution μ and the norming sequence (A_n) can be chosen to vary regularly with index $-Q$.

Let $a_0 = 0$ and define $Y_n = X_n - a_n + a_{n-1}$ then Theorem 4.1 of [3] shows that $(Y_n) \in \mathrm{DOA}(\mathcal{H})$, where $\mathcal{H} = \{\mu_{s,t} \mid 0 \leq s \leq t\}$ is the stable hemigroup with

$$\mu_{s,t} = \mu^{t-s} * \varepsilon_{u(s)-u(t)}. \tag{4.2}$$

For consistency with the notation of the following semistable case, we will rename the centering shifts as $c_n = a_n$.

In the semistable case $k_{n+1}/k_n \to c > 1$ the limit distribution μ in (4.1) fulfills $\mu^c = c^Q\mu*\varepsilon_u$ for some $u \in \mathbb{R}^d$ and some exponent $Q \in \mathrm{GL}(\mathbb{R}^d)$, whose eigenvalues belong to $\{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) \geq 1/2\}$. Then ν is said to belong to the generalized domain of attraction of the operator semistable distribution μ. Eventually we have to switch to the sampling sequence (k_{2n}) to achieve the existence of an exponent, but since the generalized domains of attraction of μ for c and c^2 coincide, assuming the existence of an exponent is no restriction. The norming sequence (A_n) can be chosen to be embeddable into a regularly varying sequence (B_n) with index $-Q$ such that $B_{k_n} = A_n$. By the spectral decomposition for generalized domains of attraction we will restrict our considerations to the case, where all eigenvalues of Q belong to $\{\lambda \in \mathbb{C} \mid 1/2 \leq \mathrm{Re}(\lambda) \leq 1\}$, since in case of an exponent, where every real part of an eigenvalue is bigger than 1, we can choose $a_n = 0$ in (4.1); see [19] for details.

Let us write $n = \alpha_n k_{p_n}$ with $p_n \in \mathbb{N}$, $k_{p_n} \leq n < k_{p_n+1}$ and define $c_0 = 0$ and

$$c_n = \alpha_n a_{p_n} + \alpha_n c^{-1} A_{p_n}^{-1} F(\alpha_n) F(c)^{-1} u,$$

where

$$F(t) = \sum_{k=0}^{\infty} \frac{(\log t)^{k+1}}{(k+1)!}(Q-I)^k$$

and note that we have $c_{k_n} = a_n$ for the embedding shifts. Let us further define $Y_n = X_n - c_n + c_{n-1}$, then Theorems 4.2 and 4.3 of [3] show that

$(Y_n) \in \mathrm{DOSA}(\mathcal{H}, c)$, where $\mathcal{H} = \{\mu_{s,t} \mid 0 \leq s \leq t\}$ is the semistable hemigroup defined by (4.2) with redefined drift-function $u(t) = tc^{-1}F(t)F(c)^{-1}u$. Again the drift-function is continuous and fulfills $u(t) \to 0$ as $t \to 0$.

The above hemigroup embedding enables us to apply Theorem 3.4 to obtain weak limits of appropriately centered random sums for domains of attraction of operator stable and semistable laws. Note that known results on random summation cited in the introduction only frequently allow a treatment of arbitrary centering shifts. Now for random samples T_n with $T_n/k_n \to D$ in probability for some positive random variable D with distribution ρ an application of Theorem 3.4 yields

$$\begin{aligned} A_n\big(S_{T_n} - c_{T_n}\big)(P) =& A_n\Big(\sum_{k=1}^{T_n} Y_k\Big)(P) \xrightarrow{\mathrm{w}} \int_0^\infty \mu_{0,r}\, d\rho(r) \\ =& \int_0^\infty \mu^r * \varepsilon_{-u(r)}\, d\rho(r). \end{aligned} \tag{4.3}$$

By Theorem 2.6 of [2], random centering in (4.3) is in general necessary, since for nonrandom centering shifts we must have $T_n/k_n \to d$ in probability for some positive constant $d > 0$. In view of random centering, (4.3) differs from an earlier result, Theorem 2.4 in [2], derived with similar methods than in the second section. By Theorem 2.4 of [2] for random samples T_n with $T_n/k_n \to D$ in probability for some positive random variable D with distribution ρ we have

$$A_n\big(S_{T_n} - (T_n/k_n)\, a_n\big)(P) \xrightarrow{\mathrm{w}} \int_0^\infty \mu^r\, d\rho(r). \tag{4.4}$$

Note that if the real parts of the eigenvalues of the exponent Q are all less than 1, the expectation $m = \mathbb{E}(X_1)$ exists and the centering shifts can be chosen as $a_n = k_n m$. If the real parts of the eigenvalues of Q all exceed 1, the shifts can be chosen as $a_n = 0$. Due to the spectral decomposition in [19], we see that if none of the eigenvalues of Q has real part equal to 1, the shifts can be chosen as $a_n = k_n a$ for some $a \in \mathbb{R}^d$ and hence in the stable situation, where $a_n = c_n$, we have

$$(T_n/k_n)\, a_n = T_n \cdot a = c_{T_n}.$$

This shows that in the stable situation, under the above restriction on the exponent, the random centerings in (4.3) and (4.4) coincide for this specially chosen centerings. However, (4.3) and (4.4) hold for arbitrary centering shifts and in general give different results for different random centerings.

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