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DIFFERENTIAL INEQUALITIES FOR GENERAL FLUID MOTIONS BOUNDED BY A FREE SURFACE

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Abstract. We consider a motion of a viscous compressible heat conducting fluid of a fixed mass bounded by a free surface. For a local solution of equations describing such a motion we derive some energy-type inequalities which are necessary to prove the global existence of solutions.

1. Introduction

In this paper we obtain some energy-type inequalities for equations describing motions of a viscous compressible heat-conducting fluid bounded by a free surface. We consider the case when the free surface is not governed by the surface tension. Then the motion of a fluid in a bounded domain $\Omega_t \subset \mathbb{R}^3$ (which depends on time $t \in \mathbb{R}^1_+$) is described by the following system with the boundary and initial conditions (see [3], [4]):

$$\varrho[v_t + (v \cdot \nabla)v] - \operatorname{div} \mathbb{T}(v,p) = \varrho f \qquad \text{in } \tilde{\Omega}^T,$$
$$\varrho_t + \operatorname{div}(\varrho v) = 0 \qquad \text{in } \tilde{\Omega}^T,$$

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$$
\begin{aligned}
&\varrho c_v(\theta_t + v\cdot\nabla\theta) + \theta p_\theta \operatorname{div} v - \varkappa\Delta\theta \\
&\quad - \frac{\mu}{2}\sum_{i,j=1}^{3}(v_{ix_j}+v_{jx_i})^2 - (\nu-\mu)(\operatorname{div} v)^2 = \varrho r \quad \text{in } \tilde\Omega^T, \\
&\mathbb{T}\bar n = -p_0\bar n \quad \text{on } \tilde S^T, \\
&v\cdot\bar n = -\frac{\varphi_t}{|\nabla\varphi|} \quad \text{on } \tilde S^T, \\
&\frac{\partial\theta}{\partial n} = \varkappa_a(\theta_a-\theta) \quad \text{on } \tilde S^T, \\
&\varrho\big|_{t=0} = \varrho_0, \quad v\big|_{t=0} = v_0, \quad \theta\big|_{t=0} = \theta_0 \quad \text{in } \Omega,
\end{aligned} \tag{1.1}
$$

where $\tilde\Omega^T = \bigcup_{t\in(0,T)}\Omega_t\times\{t\}$, $\Omega_0 = \Omega$ is an initial domain, $\tilde S^T = \bigcup_{t\in(0,T)} S_t\times\{t\}$, $S_t = \partial\Omega_t$, $\varphi(x,t)=0$ describes S_t, $\bar n$ is the unit outward vector normal to the boundary, i.e. $\bar n = \nabla\varphi/|\nabla\varphi|$. Moreover, $v = v(x,t)$ is the velocity of the fluid, $\varrho = \varrho(x,t)$ the density, $\theta = \theta(x,t)$ the temperature, $f = f(x,t)$ the external force field per unit mass, $r = r(x,t)$ the heat sources per unit mass, $\theta_a > 0$ the external temperature; $\varkappa_a > 0$ the coefficient of outer heat conductivity, $p = p(\varrho,\theta)$ the pressure, $c_v = c_v(\varrho,\theta)$ the specific heat at constant volume, μ and ν the viscosity coefficients, $\varkappa$ the coefficient of heat conductivity and p_0 the external (constant) pressure. From the thermodynamic considerations we have

$$
\nu > \frac{1}{3}\mu > 0, \qquad \varkappa > 0, \qquad c_v > 0. \tag{1.2}
$$

Finally, $\mathbb{T} = \mathbb{T}(v,p)$ denotes the stress tensor of the form

$$
\mathbb{T}(v,p) = \{T_{ij}\}_{ij=1,2,2} = \{D_{ij}(v) - p\delta_{ij}\}_{i,j=1,2,3},
$$

where

$$
\mathbb{D}(v) = \{D_{ij}(v)\}_{i,j=1,2,2} = \{2\mu S_{ij}(v) + (\nu-\mu)\delta_{ij}\operatorname{div} v\}_{i,j=1,2,3}
$$

and $\mathbb{S}(v) = (1/2)\{v_{ix_j} + v_{jx_i}\}_{i=1,2,3}$ is the velocity deformation tensor.

Assume that the domain Ω is given. Then by $(1.1)_5$, $\Omega_t = \{x\in\mathbb{R}^3 :\ x = x(\xi,t),\ \xi\in\Omega\}$, where $x = x(\xi,t)$ is a solution of the Cauchy problem

$$
\frac{\partial x}{\partial t} = v(x,t), \quad x\big|_{t=0} = \xi\in\Omega, \quad \xi = (\xi_1,\xi_2,\xi_3). \tag{1.3}
$$

Integrating (1.3) we obtain

$$
x = \xi + \int_0^t u(\xi,t')\mathrm{d}t' \equiv X_u(\xi,t), \tag{1.4}
$$

where $u(\xi,t)=v(X_u(\xi,t),t)$ and $x=X_u(\xi,t)$ describes the relation between the Eulerian x and the Lagrangian ξ coordinates. Moreover, by $(1.1)_5$ $S_t=\{x:\ x=x(\xi,t),\ \xi\in S=\partial\Omega\}$.

By the continuity equation and the kinematic condition $(1.1)_5$ the total mass is conserved, i.e.

$$\int\limits_{\Omega_t}\varrho(x,t)\mathrm{d}x=\int\limits_{\Omega}\varrho_0(\xi)\mathrm{d}\xi=M. \tag{1.5}$$

Moreover, taking $f=0$ we get that the momentum is conserved and we assume the second equality

$$\int\limits_{\Omega_t}\varrho v\cdot(a+b\times x)\mathrm{d}x=\int\limits_{\Omega}\varrho_0 v_0\cdot(a+b\times\xi)\mathrm{d}\xi=0, \tag{1.6}$$

where a and b are arbitrary constant vectors (see for example [11]).

Now, assume that $p_\varrho>0$, $p_\theta>0$ for $\varrho,\theta\in\mathbb{R}^1_+$ and consider the equation

$$p(\varrho_e,\theta_a)=p_0. \tag{1.7}$$

Definition 1.1. Let $f=r=\bar\theta=0$. By an equilibrium state we mean a solution $(v,\theta,\varrho,\Omega_t)$ of problem (1.1) such that $v=0$, $\theta=\theta_e$, $\varrho=\varrho_e$, $\Omega_t=\Omega_e$, for $t\ge0$, where $\theta_e=\theta_a$, ϱ_e satisfies (1.7) and $|\Omega_e|=M/\varrho_e$ ($|\Omega_e|=\mathrm{vol}\Omega_e$).

Let

$$f=r=0. \tag{1.8}$$

We consider a motion near the constant state. Let

$$p_\sigma=p-p_0,\qquad \theta_\sigma=\theta-\theta_e,\qquad \varrho_\sigma=\varrho-\varrho_e, \tag{1.9}$$

where θ_e and ϱ_e are introduced in Definition 1.1. Then problem (1.1) takes the form

$$\begin{aligned}
&\varrho[v_t+(v\cdot\nabla)v]-\operatorname{div}\mathbb{T}(v,p_\sigma)=0 && \text{in } \Omega_t,\ t\in[0,T],\\
&\varrho_{\sigma t}+\operatorname{div}(\varrho v)=0 && \text{in } \Omega_t,\ t\in[0,T],\\
&\varrho c_v(\varrho,\theta)(\theta_{\sigma t}+v\cdot\nabla\theta_\sigma)+\theta p_\theta(\varrho,\theta)\operatorname{div}v\\
&\quad-\varkappa\Delta\theta_\sigma-\frac{\mu}{2}\sum_{i,j=1}^3(v_{ix_j}+v_{jx_i})^2-(\nu-\mu)(\operatorname{div}v)^2=0 && \text{in } \Omega_t,\ t\in[0,T],\\
&\mathbb{T}(v,p_\sigma)\bar n=0 && \text{on } S_t,\ t\in[0,T],\\
&\frac{\partial\theta_\sigma}{\partial n}=-\varkappa_a\theta_\sigma && \text{on } S_t,\ t\in[0,T],\\
&\varrho_\sigma\big|_{t=0}=\varrho_{\sigma0}\equiv\varrho_0-\varrho_e,\ v\big|_{t=0}=v_0,
\end{aligned} \tag{1.10}$$

$$\theta_\sigma\big|_{t=0} = \theta_{\sigma 0} \equiv \theta_0 - \theta_e \qquad \text{in } \Omega.$$

The aim of our considerations is to prove the global existence of solutions to problem (1.1) with conditions (1.2), (1.5)–(1.8) which remain close to the equilibrium solution (see Definition 1.1).

Let us introduce the quantities

$$\begin{aligned} &\varphi(t) = |v|^2_{2,0,\Omega_t} + |\theta_\sigma|^2_{2,0,\Omega_t} + |\rho_\sigma|^2_{2,0,\Omega_t}, \\ &\Phi(t) = |v|^2_{3,1,\Omega_t} + |\theta_\sigma|^2_{3,1,\Omega_t} + \|\rho_\sigma\|^2_{2,\Omega_t} + \|\rho_{\sigma t}\|^2_{2,\Omega_t} + \|\rho_{\sigma tt}\|^2_{1,\Omega_t}, \end{aligned} \tag{1.11}$$

where the notation is introduced at the end of this section.

Let us introduce the spaces

$$\mathfrak{N}(t) = \{(v,\rho_\sigma,\theta_\sigma)\colon \varphi(t) < \infty\},$$

$$\mathfrak{M}(t) = \{(v,\rho_\sigma,\theta_\sigma)\colon \sup_{0\le t'\le t} \varphi(t') + \int\limits_0^t \Phi(t')dt' < \infty\}.$$

Theorem 1 (the global existence, see the proof in [9])**.** *Assume (1.2), (1.5)–(1.8), $c_v \in C^2(\mathbb{R}^2)$, $p \in C^3(\mathbb{R}^2)$, $p_\rho > 0$, $p_\theta > 0$ for ρ, θ positive. Let $l > 0$ be a constant such that $\varrho_e - l > 0$, $\theta_e - l > 0$ and $\varrho_1 < \varrho_0 < \varrho_2$, $\theta_1 < \theta_0 < \theta_2$, where $\varrho_1 = \varrho_e - l$, $\varrho_2 = \varrho_e + l$, $\theta_1 = \theta_e - l$, $\theta_2 = \theta_e + l$. Moreover, let compatibility conditions on $t = 0$, and S be satisfied.*
Assume that $(v,\rho_\sigma,\theta_\sigma)|_{t=0} \in \mathfrak{N}(0)$, $S \in H^{5/2}$. There exists $\alpha > 0$ sufficiently small such that if $\varphi(0) \le \alpha$ then there exists a global solution to problem (1.1) close to the equilibrium solution such that $(v,\rho_\sigma,\theta_\sigma) \in \mathfrak{M}(t)$ and $\varphi(t) \le c\alpha$ for any $t \in \mathbb{R}_+$, where $c > 0$ is a constant.

To prove the theorem we prolong a local solution step by step up to infinity. To make such a prolongation possible we need some inequalities for the local solution guaranteeing that the norms determined for the local solution do not increase with time. For this purpose we need two kinds of estimates. In Section 2 we prove some energy type inequalities which are necessary to show differential inequality (1.19) which plays a crucial role in the prolongation of the local solutions.

The inequality tells that $\varphi(0) \le \alpha$ with α sufficiently small implies that $\varphi(t) \le c\alpha$ for any $t \in \mathbb{R}_+$

Local existence of solutions to problem (1.1) is proved by the method of successive approximations using the Lagrangian coordinates (see [8]).

The methods used in Sections 2 and 3 are totally different. In Section 2 we utilize neither Lagrangian coordinates nor local considerations. The estimates follow from energy-type considerations performed in the Eulerian coordinates on the whole domain Ω_t. The main tool is the integration by parts formula connected with such boundary conditions that the appeared boundary integrals vanish. The inequality (2.3) (see Lemma 2.1) follows from multiplying equations (1.10) by $v, \varrho_\sigma, \theta_\sigma$, respectively, and integrating the results over Ω_t. Inequality (2.17) (see Lemma 2.2) follows from the method which implies L_2 estimate for the pressure for the Navier-Stokes equations. Inequalities (2.28) (see Lemma 2.3) and (2.50) (see Lemma 2.5) follow from differentiating equations (1.10) once and twice with respect to time, multiplying by corresponding derivatives of $v, \varrho_\sigma, \theta_\sigma$ with respect to time and integrating the results over Ω_t. Finally after an appropriate adding we obtain (3.74).

From [8] it follows that the local existence of solutions to problem (1.10) can be proved in the space $\mathcal{M}(T)$. Moreover, it follows also that $\mathcal{M}(T)$ is the largest possible space in which the existence of solutions to (1.10) by the energy method and in Sobolev spaces with integer derivatives can be proved. The inequality (1.19) is appropriate for prolongation of local solution step by step. By the energy method and global estimates (without any partition of unity, see Section 2) we show (3.72). Next applying the local considerations we get (3.74) (see Section 3). To show (3.74) we need (1.10) in the Lagrangian coordinates (see (3.4)) and introduce a partition of unity to formulate problem (3.4) in an interior (see (3.10)) and in a boundary (see (3.11)) subdomains. Working with local formulations (3.10) and (3.11) we obtain by the energy method inequality (3.74).

The energy method is the main tool in the proofs of Lemmas 3.1–3.3. In Lemma 3.1 we find an estimate for the second space derivatives of $v, \varrho_\sigma, \theta_\sigma$ (see (3.12)), in Lemma 3.2 — for second space and one time derivatives (see (3.37)) and in Lemma 3.3 for the third space derivatives (see (3.57)). Combining (3.12), (3.37) and (3.57) we obtain (3.74).

We have to underline that in the proofs of Lemmas 3.1–3.3 we obtain the final estimates by summing up estimates of all neighbourhoods of the chosen partition of unity. Moreover in a boundary neighbourhood the flattening technique is used, so we differentiate only along the tangential directions and the normal derivatives are calculated from equations. In the case of compressible fluids there is a special procedure to calculate the normal derivatives (see (3.16), (3.26)–(3.29)).

Finally the main result of this paper, inequality (1.19), follows from (3.72) and (3.74) by appropriate adding.

Using the Taylor formula we can write p_σ as follows:

$$\begin{aligned} p_\sigma &= p(\varrho,\theta) - p(\varrho_e,\theta_e) = p_\varrho(\varrho_e,\theta_e)\varrho_\sigma + p_\theta(\varrho_e,\theta_e)\theta_\sigma + p'_\sigma \\ &\equiv p_1\varrho_\sigma + p_2\theta_\sigma + p'_\sigma, \end{aligned} \tag{1.12}$$

where

$$|p'_\sigma| \le c(\varrho_e,\theta_e)(|\varrho_\sigma|^2 + |\varrho_\sigma||\theta_\sigma| + |\theta_\sigma|^2). \tag{1.13}$$

Now, let $\varrho_1, \varrho_2, \theta_1, \theta_2$ be positive constants such that

$$\varrho_1 < \varrho(x,t) < \varrho_2, \qquad \theta_1 < \theta(x,t) < \theta_2 \quad \text{for } x \in \bar{\Omega}_t,\ t \in [0,T]. \tag{1.14}$$

In Lemmas 2.1–2.5 by ε we denote small constants, by c_i $(i = 1, \dots, 8)$ or c we denote positive constants depending on $\varrho_1, \varrho_2, \theta_1, \theta_2$, $\|S\|_{H^{5/2}}$, on the parameters which guarantee the existence of the inverse transformation to $x = x(\xi, t)$ and also on the constants of the imbeddings theorems and the Korn inequalities. By $\bar{c}_i$ $(i = 1, \dots, 4)$ or c_0 we denote positive constants less or equal to one depending on the same quantities as c_i $(i = 1, \dots, 8)$ and c. We do not distinguish different $\varepsilon's$, $c_0's$ and $c's$.

Since all the below estimates are derived for the local solution obtained in [8] all the quantities $\varrho_1, \varrho_2, \theta_1, \theta_2$, $\|\int_0^t v \mathrm{d}t'\|^2_{3,\Omega_t}$ are estimated by the data functions. Moreover, the existence of the inverse transformation to $x = x(\xi, t)$ is guaranteed by the estimates for the local solution (see [8]).

Moreover, the norm in the Sobolev space $H^k(Q)$, $k \geq 1$, $Q \subset \mathbb{R}^3$ we denote by $\|\cdot\|_{k,Q}$.

In [8] the existence and the uniqueness of a local solution to problem (1.1) has been proved. This solution is such that $u, \vartheta \in \mathcal{A}_{T,\Omega}$, $\eta \in \mathcal{B}_{T,\Omega}$, where u, ϑ, η denote v, θ, ϱ written in the Lagrangian coordinates ξ and

$$\begin{aligned} \mathcal{B}_{T,\Omega} = \{ f \in C([0,T]; H^2(\Omega)) :\ & f_t \in C([0,T]; H^1(\Omega)) \\ \cap L_2(0,T; H^2(\Omega)),\ & f_{tt} \in C([0,T]; L_2(\Omega)) \cap L_2(0,T; H^1(\Omega))\}, \end{aligned} \tag{1.15}$$

$$\mathcal{A}_{T,\Omega} = \mathcal{B}_{T,\Omega} \cap \{ f :\ f \in L_2(0,T; H^3(\Omega))\}. \tag{1.16}$$

In [7] we proved the existence of a global in time solution (v, θ, ϱ) to problem (1.1) such that $\sup_{0 \leq t' \leq t} \phi_1('t) + \int_0^t \Phi_1(t') \mathrm{d}t' < \infty$, where $t \in \mathbb{R}^1_+$, $\phi_1(t) = |v|^2_{3,0,\Omega_t} + |\theta_\sigma|^2_{3,0,\Omega_t} + |\varrho_\sigma|^2_{3,0,\Omega_t}$, $\Phi_1(t) = |v|^2_{4,1,\Omega_t} + |\theta_\sigma|^2_{4,1,\Omega_t} + |\varrho_\sigma|^2_{3,0,\Omega_t}$ and

$$|g|^2_{l,k,\Omega_t} = \sum_{i \leq l-k} \|\partial_t^i g\|^2_{l-i,\Omega_t} \tag{1.17}$$

for $g \in \{v, \theta_\sigma, \varrho_\sigma\}$, $k, l \in \mathbb{N} \cup \{0\}$, $0 \leq k \leq l$. By $\|\cdot\|_{k,\Omega_t}$ we denote the norm in the Sobolev space $H^k(\Omega_t)$ $(k \in \mathbb{N} \cup \{0\})$.

In order to obtain the global existence theorem we assumed in [7] that $\phi_1(0)$ was sufficiently small, and internal energy $e(\varrho, \theta)$ per unit mass had the form:

$$e(\varrho, \theta) = a_0 \varrho^\alpha + h(\varrho, \theta),$$

where $a_0 > 0$, $\alpha > 0$, $h(\varrho, \theta) \geq h_* > 0$ for $\varrho \in [\varrho_*, \varrho^*]$, $\theta \in [\theta_*, \theta^*]$, a_0, α, h_* are constants and

$$\varrho_* = \min_{t \in [0,T]} \min_{\tilde{\Omega}_t} \varrho(x,t), \quad \varrho^* = \max_{t \in [0,T]} \max_{\tilde{\Omega}_t} \varrho(x,t),$$

$$\theta_* = \min_{t \in [0,T]} \min_{\tilde{\Omega}_t} \theta(x,t), \quad \theta^* = \max_{t \in [0,T]} \max_{\tilde{\Omega}_t} \theta(x,t),$$

$T > 0$ is the time of the local existence. Moreover we used in [7] the following differential inequality derived in [6]:

$$\frac{\mathrm{d}\bar{\phi}_1}{\mathrm{d}t} + c_0\Phi_1 \le P(\phi_1)(\phi_1 + \Phi_1) + c_1\Psi, \tag{1.18}$$

where $\rho(\phi_1) = p(\phi_1)\phi_1$,

$$c_2\phi_1 \le \bar{\phi}_1 \le c_3\phi_1,$$

$c_1, c_2, c_3 > 0$ are constants depending on $\varrho_*, \varrho^*, \theta_*, \theta^*$, p is an increasing positive continuous function and $\Psi = \|v\|^2_{0,\Omega_t} + \|p_\sigma\|^2_{0,\Omega_t}$. The term $c_1\Psi$ in (1.18) complicates the proof of the global existence because an additional estimate for Ψ must be obtained.

In the present paper the term $c_1\Psi$ does not occur in inequality (1.19) which makes the proof of the global existence much more simple.

Global existence of solutions for compressible viscous fluid for either barotropic or heat conducting was considered in [10] or [7], respectively.

The case of a free boundary problem for equations of viscous compressible either barotropic or heat conducting capillary fluids where considered in [11] and [5] where the global existence theorems were obtained. Finally we formulate the main result of this paper

Theorem 2. *Let $p \in C^2(\mathbb{R}^1_+ \times \mathbb{R}^1_+)$, $c_v \in C^2(\mathbb{R}^1_+ \times \mathbb{R}^1_+)$ and assume* (1.2), (1.5)–(1.8), (1.14), (2.16). *Then for the local solution (v, θ, ϱ) of problem* (1.1) *such that $u, \vartheta_\sigma \in \mathcal{A}_{T,\Omega}$ and $\eta_\sigma \in \mathcal{B}_{T,\Omega}$ (where $\mathcal{A}_{T,\Omega}$ and $\mathcal{B}_{T,\Omega}$ are given by* (1.15) *and* (1.16)*, respectively; u, ϑ, η denote v, θ, ϱ written in the Lagrangian coordinates ξ) the following differential inequality holds*

$$\frac{\mathrm{d}\bar{\phi}}{\mathrm{d}t} + c_0\Phi \le c\Big[\phi(1+\phi^2) + \Big\|\int_0^t v\,\mathrm{d}t'\Big\|^2_{3,\Omega_t}\Big]\Phi \tag{1.19}$$

for $t \le T$, where

$$\begin{aligned}\bar{\phi}(t) =& \Big[B_0\Big(\psi(t) + \varrho v_{tt}^2 + \frac{p_{\sigma\varrho}}{\varrho}\varrho^2_{\sigma tt} + \frac{\varrho c_v}{\theta}\theta^2_{\sigma tt}\Big) \\ &+ B_1\phi_1(t,\Omega) + B_2\phi_2(t,\Omega) + B_3\phi_3(t,\Omega)\Big],\\ \phi(t) =& |v|^2_{2,0,\Omega_t} + |\theta_\sigma|^2_{2,0,\Omega_t} + |\varrho_\sigma|^2_{2,0,\Omega_t}, \qquad (1.20)\\ \Phi(t) =& |v|^2_{3,1,\Omega_t} + |\theta_\sigma|^2_{3,1,\Omega_t} + \|\varrho_\sigma\|^2_{2,\Omega_t} + \|\varrho_{\sigma t}\|^2_{2,\Omega_t}\\ &+ \|\varrho_{\sigma tt}\|^2_{1,\Omega_t}, \qquad (1.21)\end{aligned}$$

T is the time of the local existence, ψ and ϕ_i $(i = 1, 2, 3)$ are the functions given by (2.47), (3.13), (3.38), (3.58), *respectively; the norms $|g|_{l,k,\Omega_t}$ for $g \in \{v, \theta_\sigma, \varrho_\sigma\}$ are defined by* (1.17); *$c_0 < 1$ and c are positive constants depending on $\varrho_1, \varrho_2, \theta_1, \theta_2$, $\mu, \nu, \varkappa, c_v, p, \|S\|_{H^{5/2}}$ and the constants from the*

imbedding theorems and the Korn inequalities (being also nondecreasing continuous functions of $\| \int_0^T v\mathrm{d}t' \|^2_{3,\Omega_t}$*);* B_i $(0 = 1,2,3)$ *are positive constants depending on the same quantities as* c_0 *and* c*;* $\varrho_\sigma, \theta_\sigma, p_\sigma$ *are given by* (2.3).

2. Energy-type estimates for a local solution

First, we prove

Lemma 2.1. *Let* $p \in C^1(\mathbb{R}^1_+ \times \mathbb{R}^1_+)$, $c_v \in C^1(\mathbb{R}^1_+ \times \mathbb{R}^1_+)$ *and assume* (1.2), (1.8), (1.14). *Moreover, let*

$$v \in H^2(\Omega_t), \quad \varrho_\sigma \in H^1(\Omega_t), \quad \theta_\sigma \in H^1(\Omega_t), \quad \theta_{\sigma t} \in L_2(\Omega_t) \tag{2.1}$$

and

$$\int\limits_\Omega \varrho_0 v_0 \cdot (a + b \times \xi)\mathrm{d}\xi = 0, \tag{2.2}$$

where a *and* b *are arbitrary constant vectors. Let* $(v, \theta_\sigma, \varrho_\sigma)$ *be a solution of* (1.10). *Then*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t} \varrho\left(v^2 + \frac{p_1}{\varrho_e^2}\varrho_\sigma^2 + \frac{p_2 p_3}{p_4}\theta_\sigma^2\right)\mathrm{d}x + \bar{c}_1(\|v\|^2_{1,\Omega_t} + \|\theta_{\sigma x}\|^2_{0,\Omega_t} + \|\theta_\sigma\|^2_{0,S_t})\\ &\le c_1 X_1^2(1 + X_1) + c_2 X_1 \|\theta_{\sigma t}\|^2_{0,\Omega_t},\end{aligned} \tag{2.3}$$

where p_1 *and* p_2 *are defined in* (1.12), p_3 *and* p_4 *are defined by* (2.9) *and*

$$X_1 = \|v\|^2_{2,\Omega_t} + \|\varrho_\sigma\|^2_{1,\Omega_t} + \|\theta_\sigma\|^2_{1,\Omega_t}. \tag{2.4}$$

Proof. Multiplying $(1.10)_1$ by v, integrating over Ω_t and using boundary condition $(1.10)_4$ yield

$$\begin{aligned}&\frac{1}{2}\int\limits_{\Omega_t} \varrho[\partial_t v^2 + (v \cdot \nabla)v^2]\mathrm{d}x + \frac{\mu}{2}E_{\Omega_t}(v) + (\nu - \mu)\int\limits_{\Omega_t} (\operatorname{div} v)^2\mathrm{d}x\\ &- \int\limits_{\Omega_t} p_\sigma \operatorname{div} v\mathrm{d}x = 0,\end{aligned} \tag{2.5}$$

where

$$E_{\Omega_t}(v) = \int\limits_{\Omega_t} (v_{ix_j} + v_{jx_i})^2\mathrm{d}x$$

and the summation convention over the repeated indices is assumed.

Using that $\nu > (1/3)\mu$ and assuming $C_1 = \min\{(3/4)(\nu - (1/3)\mu), \mu/2\}$ we obtain

$$\frac{\mu}{2}E_{\Omega_t}(v) + (\nu - \mu)\|\operatorname{div} v\|^2_{0,\Omega_t} \ge C_1 E_{\Omega_t}(v) \tag{2.6}$$

(see [6]).

Moreover, in view of assumptions (2.1)–(2.2) and (1.4) Lemma 5.2 of [10] yields

$$\|v\|^2_{1,\Omega_t} \le c(E_{\Omega_t}(v) + \|\varrho_\sigma\|^2_{0,\Omega_t}\|v\|^2_{0,\Omega_t}). \tag{2.7}$$

Using in (2.5) relation (1.12), estimates (1.12), (2.6)–(2.7) and equation of continuity $(1.10)_3$ we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t}\varrho v^2\mathrm{d}x + c_0\|v\|^2_{1,\Omega_t} - p_1\int\limits_{\Omega_t}\varrho_\sigma \mathrm{div}\, v\mathrm{d}x - p_2\int\limits_{\Omega_t}\theta_\sigma\mathrm{div}\, v\mathrm{d}x\\ &\le c(\|\varrho_\sigma\|^2_{0,\Omega_t}\|v\|^2_{0,\Omega_t} + \|\varrho_\sigma\|^4_{1,\Omega_t} + \|\theta_\sigma\|^4_{1,\Omega_t}).\end{aligned} \tag{2.8}$$

Now, let

$$p_3 = c_v(\varrho_e,\theta_e), \quad p_4 = \theta_e p_\theta(\varrho_e,\theta_e). \tag{2.9}$$

By the mean value theorem we have

$$c_v(\varrho,\theta) - c_v(\varrho_e,\theta_e) = c_{v1}\varrho_\sigma + c_{v2}\theta_\sigma, \tag{2.10}$$

$$\theta p_\theta(\varrho,\theta) - \theta_e p_\theta(\varrho_e,\theta_e) = b_1\varrho_\sigma + b_2\theta_\sigma, \tag{2.11}$$

where $c_{v1} = c_{v\varrho}(\varrho_e + s(\varrho-\varrho_e),\ \theta_e + s(\theta-\theta_e))$, $c_{v2} = c_{v\vartheta}(\varrho_e + s(\varrho-\varrho_e),$ $\theta_e + s(\theta-\theta_e))$, $b_1 = (\theta p_\theta)_\varrho(\varrho_e + s(\varrho-\varrho_e), \theta_e + s(\theta-\theta_e))$, $b_2 = (\theta p_\theta)_\theta(\varrho_e + s(\varrho-\varrho_e), \theta_e + s(\theta-\theta_e))$, $0 < s < 1$.

Then equation $(1.10)_3$ can be rewritten in the form

$$\begin{aligned}&\varrho p_3(\theta_{\sigma t} + v\cdot\nabla\theta_\sigma) - \varkappa\Delta\theta_\sigma + p_4\mathrm{div}\, v = -(c_{v1}\varrho_\sigma + c_{v2}\theta_\sigma)(\theta_{\sigma t} + v\cdot\nabla\theta_\sigma)\\ &-(b_1\varrho_\sigma + b_2\theta_\sigma)\mathrm{div}\, v + \frac{1}{2}\mu\sum_{i,j=1}^3(v_{ix_j} + v_{jx_i})^2 + (\nu-\mu)(\mathrm{div}\, v)^2.\end{aligned} \tag{2.12}$$

Multiplying (2.12) by θ_σ, integrating over Ω_t, using equation of continuity $(1.10)_2$ and boundary condition $(1.10)_5$ we get

$$\begin{aligned}&\frac{p_3}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t}\varrho\theta_\sigma^2\mathrm{d}x + \varkappa\|\theta_{\sigma x}\|^2_{0,\Omega_t} + \varkappa\varkappa_a\|\theta_\sigma\|^2_{0,S_t} + p_4\int\limits_{\Omega_t}\theta_\sigma\mathrm{div}\, v\mathrm{d}x\\ &\le \varepsilon(\|\theta_{\sigma x}\|^2_{0,\Omega_t} + \|v_x\|^2_{0,\Omega_t})\\ &+ c(\varepsilon)[\|\varrho_\sigma\|^2_{1,\Omega_t}\|\theta_\sigma\|^2_{1,\Omega_t} + \|\theta_\sigma\|^4_{1,\Omega_t})(\|v\|^2_{2,\Omega_t} + 1) + \|\theta_\sigma\|^2_{1,\Omega_t}\|v_x\|^2_{1,\Omega_t}]\\ &+ c(\|\varrho_\sigma\|_{1,\Omega_t}\|\theta_\sigma\|_{1,\Omega_t} + \|\theta_\sigma\|^2_{1,\Omega_t})\|\theta_{\sigma t}\|_{0,\Omega_t}.\end{aligned} \tag{2.13}$$

Next, we write $(1.10)_2$ in the form

$$\varrho_{\sigma t} + v\cdot\nabla\varrho_\sigma + \varrho_e\mathrm{div}\, v = -\varrho_\sigma\mathrm{div}\, v. \tag{2.14}$$

Multiplying (2.14) by $\varrho\varrho_\sigma$, integrating over Ω and using equation $(1.10)_2$ we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t}\varrho\varrho_\sigma^2\mathrm{d}x+\varrho_e^2\int\limits_{\Omega_t}\varrho_\sigma\mathrm{div}\,v\mathrm{d}x\ \le\varepsilon\|v_x\|_{0,\Omega_t}^2+c(\varepsilon)\|\varrho_\sigma\|_{1,\Omega_t}^4. \tag{2.15}$$

Now, multiplying (2.15) by p_1/ϱ_e^2 and (2.13) by p_2/p_4, adding the both estimates to (2.8) and assuming that ε is sufficiently small we obtain (2.3).

This completes the proof of the lemma. □

Now, we obtain estimates for L_2 — norms of ϱ_σ and θ_σ. The following lemma holds.

Lemma 2.2. *Let $p\in C^2(\mathbb{R}_+^1\times\mathbb{R}_+^1)$, $f=r=\bar\theta=0$ and*

$$\begin{aligned}&p_\varrho(\varrho,\theta)\ge a_0>0,\quad p_\theta(\varrho,\theta)\ge a_1>0\\ &\text{for }\varrho\in(\varrho_1,\varrho_2),\qquad \vartheta\in(\theta_1,\theta_2),\end{aligned} \tag{2.16}$$

where $\varrho_1,\varrho_2,\theta_1,\theta_2$ are constants from (1.13) *and a_0,a_1 are constants. Let $(v,\theta_\sigma,\varrho_\sigma)$ be a solution of* (1.10). *Then*

$$\begin{aligned}&\|\varrho_\sigma\|_{0,\Omega_t}^2+\|\theta_\sigma\|_{0,\Omega_t}^2\le c_3(\varepsilon_1)(\|v_t\|_{0,\Omega_t}^2+\|v_x\|_{0,\Omega_t}^2+\|\theta_{\sigma x}\|_{0,\Omega_t}^2+\|\theta_\sigma\|_{0,S_t}^2\\ &+\|v\|_{1,\Omega_t}^4+\|\varrho_\sigma\|_{1,\Omega_t}^4+\|\theta_\sigma\|_{1,\Omega_t}^4)+\varepsilon_1(\|\varrho_{\sigma x}\|_{0,\Omega_t}^2+\|v_{xx}\|_{0,\Omega_t}^2).\end{aligned} \tag{2.17}$$

Proof. Introduce a function ϕ_1 as a solution of the problem

$$\begin{aligned}&\mathrm{div}\,\phi_1=p-\bar p_{\Omega_t}\quad &&\text{in }\Omega_t,\\ &\phi_1=0 &&\text{on }S_t,\end{aligned} \tag{2.18}$$

where $\bar p_{\Omega_t}=(1/|\Omega_t|)\int_{\Omega_t}p\mathrm{d}x$.

In view of Lemma 2.2 of [2] there exists a solution of problem (2.18) such that $\phi_1\in W_2^1(\Omega_t)$ and

$$\|\phi_1\|_{1,\Omega_t}\le c\|p-\bar p_{\Omega_t}\|_{0,\Omega_t}.$$

Now, multiplying $(1.10)_1$ by ϕ_1, integrating over Ω_t and using the above inequality we obtain

$$\|p-\bar p_{\Omega_t}\|_{0,\Omega_t}^2\le c(\|v_t\|_{0,\Omega_t}^2+\|v_x\|_{0,\Omega_t}^2+\|v\|_{1,\Omega_t}^4).$$

Estimating $\|\bar p_{\Omega_t}-p_0\|_{0,\Omega_t}$ in the same way as in [10] and using that

$$\|p_\sigma\|_{0,\Omega_t}\le c(\|p-\bar p_{\Omega_t}\|_{0,\Omega_t}^2+\|\bar p_{\Omega_t}-p_0\|_{0,\Omega_t}^2)$$

we get

$$\begin{aligned}\|p_\sigma\|_{0,\Omega_t}^2&\le\varepsilon(\|\varrho_{\sigma x}\|_{0,\Omega_t}^2+\|\theta_{\sigma x}\|_{0,\Omega_t}^2+\|v_{xx}\|_{0,\Omega_t}^2)\\ &\quad+c(\varepsilon)(\|v_t\|_{0,\Omega_t}^2+\|v\|_{1,\Omega_t}^2+\|v\|_{1,\Omega_t}^4).\end{aligned} \tag{2.19}$$

Next, let us write equation $(1.10)_1$ in the form

$$\varrho[v_t+(v\cdot\nabla)v]-\operatorname{div}\mathbb{D}(v)+p_\varrho\nabla\varrho_\sigma+p_\theta\nabla\theta_\sigma=0 \tag{2.20}$$

and introduce a function ϕ_2 as a solution of the problem

$$\begin{aligned}&\operatorname{div}\phi_2=\varrho-\bar\varrho_{\Omega_t} && \text{in } \Omega_t,\\ &\phi_2=0 && \text{on } S_t.\end{aligned} \tag{2.21}$$

Then there exists $\phi_2\in W_2^1(\Omega_t)$ satisfying (2.21) and

$$\|\phi_2\|_{1,\Omega_t}\le c\|\varrho-\bar\varrho_{\Omega_t}\|_{0,\Omega_t}. \tag{2.22}$$

Multiplying (2.20) by ϕ_2 and integrating over Ω_t we get

$$\begin{aligned}\int\limits_{\Omega_t}p_\varrho\nabla\varrho_\sigma\cdot\phi_2\mathrm{d}x=&-\int\limits_{\Omega_t}\varrho[v_t+(v\cdot\nabla)v]\cdot\phi_2\mathrm{d}x-\int\limits_{\Omega_t}D_{ij}(v)\frac{\partial\phi_{2i}}{\partial x_j}\mathrm{d}x\\ &-\int\limits_{\Omega_t}p_\theta\nabla\theta_\sigma\cdot\phi_2\mathrm{d}x.\end{aligned} \tag{2.23}$$

Next, integrating by parts in the term on the left-hand side of (2.23) we have

$$\int\limits_{\Omega_t}p_\varrho\nabla\varrho_\sigma\cdot\phi_2\mathrm{d}x\;=\;-\int\limits_{\Omega_t}p_\varrho\varrho_\sigma\operatorname{div}\phi_2\mathrm{d}x-\int\limits_{\Omega_t}\varrho_\sigma\nabla p_\varrho\cdot\phi_2\mathrm{d}x. \tag{2.24}$$

Now, in view of (2.16) and (2.21)–(2.24) in obtain

$$\begin{aligned}\|\varrho-\bar\varrho_{\Omega_t}\|^2_{0,\Omega_t}\le& c(\|v_t\|^2_{0,\Omega_t}+\|v_x\|^2_{0,\Omega_t}+\|\theta_{\sigma x}\|^2_{0,\Omega_t})\\ &+c(\|\varrho_\sigma\|^4_{1,\Omega_t}+\|\theta_\sigma\|^4_{1,\Omega_t}+\|v\|^4_{1,\Omega_t}).\end{aligned} \tag{2.25}$$

Now, rewrite the boundary condition $(1.1)_4$ we the form

$$p_\varrho(\bar\varrho_{\Omega_t}-\varrho_e)=\bar n\cdot\mathbb{D}(v)\bar n-p_\varrho(\varrho-\bar\varrho_{\Omega_t})-p_\theta\theta_\sigma.$$

Hence we obtain

$$\begin{aligned}\|\bar\varrho_{\Omega_t}-\varrho_e\|^2_{0,\Omega_t}&\le c\frac{|\Omega_t|}{|S_t|}(\|\varrho-\bar\varrho_{\Omega_t}\|^2_{0,S_t}+\|\theta_\sigma\|^2_{0,S_t}+\|\mathbb{D}(v)\|^2_{0,S_t})\\ &\le\varepsilon(\|\varrho_{\sigma x}\|^2_{0,\Omega_t}+\|v_{xx}\|^2_{0,\Omega_t})+c(\varepsilon)(\|\varrho-\bar\varrho_{\Omega_t}\|^2_{0,\Omega_t}+\|v\|^2_{0,\Omega_t})\\ &\quad+c\|\theta_\sigma\|^2_{0,S_t}.\end{aligned}$$

Therefore by (2.25) we get

$$\begin{aligned}\|\varrho_\sigma\|^2_{0,\Omega_t}\le&\,\varepsilon(\|\varrho_{\sigma x}\|^2_{0,\Omega_t}+\|v_{xx}\|^2_{0,\Omega_t})\\ &+c(\varepsilon)(\|v_t\|^2_{0,\Omega_t}+\|v\|^2_{1,\Omega_t}+\|\theta_{\sigma x}\|^2_{0,\Omega_t}+\|\theta_\sigma\|^2_{0,S_t}\\ &+\|\varrho_\sigma\|^4_{1,\Omega_t}+\|\theta_\sigma\|^4_{1,\Omega_t}+\|v\|^4_{1,\Omega_t}).\end{aligned} \tag{2.26}$$

Finally, the relation

$$p_\sigma = p_\varrho \varrho_\sigma + p_\theta \theta_\sigma,$$

(where the values of p_ϱ and p_θ are taken in a point $(\varrho_e + s(\varrho - \varrho_e),\ \theta_e + s(\theta - \theta_e))$, and $s \in (0,1)$), implies

$$\|\theta_\sigma\|^2_{0,\Omega_t} \le c(\|p_\sigma\|^2_{0,\Omega_t} + \|\varrho_\sigma\|^2_{0,\Omega_t}), \tag{2.27}$$

where we have also used assumption (2.16).

Taking into account (2.19), (2.26) and (2.27) we obtain the assertion of the lemma. □

Now, we obtain an estimate for time-derivatives. The following lemma holds.

Lemma 2.3. *Let* $p \in C^1(\mathbb{R}^1_+ \times \mathbb{R}^1_+), c_v \in C^1(\mathbb{R}^1_+ \times \mathbb{R}^1_+)$ *and assume* (1.2), (1.8), (1.14). *Let* $(v, \theta_\sigma, \varrho_\sigma)$ *be a solution of* (1.10). *Then*

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t} \varrho\left(v_t^2 + \frac{p_1}{\varrho_e^2}\varrho_{\sigma t}^2 + \frac{p_2 p_3}{p_4}\theta_{\sigma t}^2\right)\mathrm{d}x + \bar{c}_2(\|v_t\|^2_{1,\Omega_t} + \|\theta_{\sigma t}\|^2_{1,\Omega t}) \\ &\le c_4(\|v_x\|^2_{0,\Omega_t} + \|\theta_{\sigma x}\|^2_{0,\Omega_t}) + c_5[X_2^2(1+X_2) + X_2\|\theta_{\sigma tt}\|^2_{0,\Omega_t}], \end{aligned} \tag{2.28}$$

where p_1 *and* p_2 *are defined in* (1.12), p_3 *and* p_4 *are defined by* (2.9) *and*

$$\begin{aligned} &X_2 = |v|^2_{2,1,\Omega_t} + |\varrho_\sigma|^2_{2,1,\Omega_t} + |\theta_\sigma|^2_{2,1,\Omega_t}, \\ &|g|_{2,1,\Omega_t} \equiv \|g\|^2_{2,\Omega_t} + \|g_t\|^2_{1,\Omega_t}, \quad g \in \{v, \theta_\sigma, \varrho_\sigma\}. \end{aligned} \tag{2.29}$$

Proof. Differentiating $(1.10)_1$ with respect to t we get

$$\begin{aligned} &\varrho[v_{tt} + (v\cdot\nabla)v_t] - \operatorname{div}\mathbb{T}(v_t, p_{\sigma t}) \\ &= -\varrho_{\sigma t}[v_t + (v\cdot\nabla)v] - \varrho(v_t\cdot\nabla)v. \end{aligned} \tag{2.30}$$

Multiplying (2.30) by v_t, integrating over Ω_t and using (2.6) we have

$$\frac{1}{2}\int\limits_{\Omega_t} \varrho[\partial_t v_t^2 + (v\cdot\nabla)v_t^2]\mathrm{d}x + c_1 E_{\Omega_t}(v_t) - \int\limits_{\Omega_t} p_{\sigma t}\operatorname{div} v_t \mathrm{d}x \tag{2.31}$$

$$\begin{aligned} &\le \varepsilon\|v_t\|^2_{1,\Omega_t} + c(\varepsilon)[\|\varrho_{\sigma t}\|^2_{1,\Omega_t}\cdot(\|v_t\|^2_{1,\Omega_t} + \|v\|^2_{1,\Omega_t}\|v\|^2_{2,\Omega_t}) + \|v_t\|^2_{1,\Omega_t}\|v\|^2_{2,\Omega_t} \\ &+ \|\varrho_\sigma\|^2_{2,\Omega_t}\|v\|^2_{1,\Omega_t}\|v_t\|^2_{1,\Omega_t} + (\|v\|^2_{2,\Omega_t} + \|\varrho_\sigma\|^2_{1,\Omega_t} + \|\theta_\sigma\|^2_{1,\Omega_t})\|v\|^2_{2,\Omega_t}] \end{aligned}$$

By Lemma 5.3 of [10] we have

$$\|v_t\|^2_{1,\Omega_t} \le c[E_{\Omega_t}(v_t) + X_2^2(1+X_2)]. \tag{2.32}$$

Using equation of continuity $(1.10)_2$ and estimate (2.32) in (2.31) yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t} \varrho v_t^2 \mathrm{d}x + c_0\|v_t\|^2_{1,\Omega_t} - \int\limits_{\Omega_t} p_{\sigma t}\operatorname{div} v_t \mathrm{d}x \ \le\ cX_2^2(1+X_2). \tag{2.33}$$

Now, we examine the last term on the l.h.s. of (2.33). We have

$$p_{\sigma t} = p_\varrho \varrho_{\sigma t} + p_\theta \theta_{\sigma t} = p_1 \varrho_{\sigma t} + p_2 \theta_{\sigma t} + (p_\varrho(\varrho,\theta) - p_\varrho(\varrho_e,\theta_e))\varrho_{\sigma t} + (p_\theta(\varrho,\theta) - p_\theta(\varrho_e,\theta_e))\theta_{\sigma t} \equiv p_1\varrho_{\sigma t} + p_2\theta_{\sigma t} + p''_\sigma,$$

where

$$|p''_\sigma| \le c(|\varrho_\sigma| + |\theta_\sigma|)(|\varrho_{\sigma t}| + |\theta_{\sigma t}|).$$

Hence

$$-\int_{\Omega_t} p_{\sigma t} \operatorname{div} v_t \mathrm{d}x = -p_1 \int_{\Omega_t} \varrho_{\sigma t} \operatorname{div} v_t \mathrm{d}x - p_2 \int_{\Omega_t} \theta_{\sigma t} \operatorname{div} v_t \mathrm{d}x - \int_{\Omega_t} p''_\sigma \operatorname{div} v_t \mathrm{d}x, \tag{2.34}$$

where

$$\bigg| \int_{\Omega_t} p''_\sigma \operatorname{div} v_t \mathrm{d}x \bigg| \le \varepsilon \|v_t\|^2_{1,\Omega_t} + c(\varepsilon)(|\varrho_\sigma|^2_{2,1,\Omega_t} + |\theta_\sigma|^2_{2,1,\Omega_t})^2. \tag{2.35}$$

Taking into account (2.33), (2.34) and (2.35) we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} \varrho v_t^2 \mathrm{d}x + c_0 \|v_t\|^2_{1,\Omega_t} - p_1 \int_{\Omega_t} \varrho_{\sigma t} \operatorname{div} v_t \mathrm{d}x - p_2 \int_{\Omega_t} \theta_{\sigma t} \operatorname{div} v_t \mathrm{d}x \le cX_2^2(1 + X_2). \tag{2.36}$$

Differentiating (2.14) with respect to t gives

$$\varrho_{\sigma tt} + v \cdot \nabla \varrho_{\sigma t} + \varrho_e \operatorname{div} v_t = -v_t \cdot \nabla \varrho_\sigma - \varrho_{\sigma t} \operatorname{div} v - \varrho_\sigma \operatorname{div} v_t. \tag{2.37}$$

Multiplying (2.37) by $\varrho \varrho_{\sigma t}$, integrating over Ω_t and using equation of continuity $(1.10)_2$ imply

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} \varrho \varrho_{\sigma t}^2 \mathrm{d}x + \varrho_e^2 \int_{\Omega_t} \varrho_{\sigma t} \operatorname{div} v_t \mathrm{d}x \le \varepsilon(\|v_t\|^2_{1,\Omega_t} + \|v_x\|^2_{0,\Omega_t}) + c(\varepsilon)\|\varrho_{\sigma t}\|^2_{1,\Omega_t}(\|\varrho_\sigma\|^2_{1,\Omega_t} + \|\varrho_{\sigma t}\|^2_{1,\Omega_t}). \tag{2.38}$$

Differentiating $(1.10)_3$ with respect to t yields

$$\begin{aligned} &\varrho c_v(\theta_{\sigma tt} + v \cdot \nabla \theta_{\sigma t}) - \varkappa \Delta \theta_{\sigma t} + \theta p_\theta \operatorname{div} v_t \\ &= -[(\varrho c_v)_{,\varrho} \varrho_{\sigma t} + (\varrho c_v)_{,\theta} \theta_{\sigma t}](\theta_{\sigma t} + v \cdot \nabla \theta_\sigma) - [(\theta p_\theta)_{,\varrho} \varrho_{\sigma t} + (\theta p_\theta)_{,\theta} \theta_{\sigma t}] \operatorname{div} v_t \\ &- \mu(v_{ix_j} + v_{jx_i})(v_{itx_j} + v_{jtx_i}) - 2(\nu - \mu) \operatorname{div} v \operatorname{div} v_t \equiv N_1. \end{aligned} \tag{2.39}$$

Repeating considerations leading to (2.12) we have

$$\begin{aligned} &\varrho p_3(\theta_{\sigma tt} + v \cdot \nabla \theta_{\sigma t}) - \varkappa \Delta \theta_{\sigma t} + p_4 \operatorname{div} v_t \\ &= -(c_{v1}\varrho_\sigma + c_{v2}\theta_\sigma)(\theta_{\sigma tt} + v \cdot \nabla \theta_{\sigma t}) - (b_1 \varrho_\sigma + b_2 \theta_\sigma) \operatorname{div} v_t + N_1 \end{aligned}$$

$$\equiv N_2 + N_1, \tag{2.40}$$

where c_{v1}, c_{v2} and b_1, b_2 are given by (2.10) and (2.11), respectively. Multiplying (2.40) by $\theta_{\sigma t}$, integrating over Ω_t and using equation of continuity $(1.1)_2$ imply

$$\begin{aligned}&\frac{p_3}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t}\varrho\theta_{\sigma t}^2\mathrm{d}x + \varkappa\|\theta_{\sigma xt}\|_{0,\Omega_t}^2 + \varkappa\varkappa_a\|\theta_{\sigma t}\|_{0,S_t}^2 + p_4\int\limits_{\Omega_t}\theta_{\sigma t}\mathrm{div}\, v_t\mathrm{d}x\\ &= \int\limits_{\Omega_t}N_1\theta_{\sigma t}\mathrm{d}x + \int\limits_{\Omega_t}N_2\theta_{\sigma t}\mathrm{d}x,\end{aligned} \tag{2.41}$$

where

$$\begin{aligned}&\Big|\int\limits_{\Omega_t}N_1\theta_{\sigma t}\mathrm{d}x\Big| \le \varepsilon(\|v_{tx}\|_{0,\Omega_t}^2 + \|\theta_{\sigma t}\|_{1,\Omega_t}^2)\\ &+ c(\varepsilon)[\|\theta_{\sigma t}\|_{1,\Omega_t}^2(\|\theta_{\sigma t}\|_{1,\Omega_t}^2 + \|\varrho_{\sigma t}\|_{1,\Omega_t}^2 + \|v\|_{2,\Omega_t}^2)\\ &+ \|\theta_{\sigma t}\|_{1,\Omega_t}^4 + \|\varrho_{\sigma t}\|_{1,\Omega_t}^2\|v\|_{1,\Omega_t}^2\|\theta_\sigma\|_{2,\Omega_t}^2]\\ &\le \varepsilon(\|v_{tx}\|_{0,\Omega_t}^2 + \|\theta_{\sigma t}\|_{1,\Omega_t}^2) + c(\varepsilon)X_2^2(1+X_2)\end{aligned} \tag{2.42}$$

and

$$\begin{aligned}&\Big|\int\limits_{\Omega_t}N_2\theta_{\sigma t}\mathrm{d}x\Big| \le \varepsilon\|\theta_{\sigma t}\|_{1,\Omega_t}^2 + c(\varepsilon)(\|\varrho_\sigma\|_{1,\Omega_t}^2\\ &+ \|\theta_\sigma\|_{1,\Omega_t}^2)(\|\theta_{\sigma tt}\|_{0,\Omega_t}^2 + \|v\|_{2,\Omega_t}^2\|\theta_{\sigma xt}\|_{0,\Omega_t}^2)\\ &+ c(\|\varrho_\sigma\|_{1,\Omega_t}^2 + \|\theta_\sigma\|_{1,\Omega_t}^2)\|v_t\|_{1,\Omega_t}^2\\ &\le \varepsilon\|\theta_{\sigma t}\|_{1,\Omega_t}^2 + c(\varepsilon)(X_2^2 + X_2\|\theta_{\sigma tt}\|_{0,\Omega_t}^2).\end{aligned} \tag{2.43}$$

Next, multiplying $(1.10)_1$ by $\theta_{\sigma t}$, integrating over Ω_t and using boundary condition $(1.10)_5$ we get

$$\|\theta_{\sigma t}\|_{0,\Omega_t}^2 \le \varepsilon\|\theta_{\sigma tx}\|_{0,\Omega_t}^2 + c(\|v_x\|_{0,\Omega_t}^2 + \|\theta_{\sigma x}\|_{0,\Omega_t}^2) + cX_1^2, \tag{2.44}$$

where X_1 is given by (2.4). Using (2.42)–(2.43) and (2.44) in (2.41) yields

$$\begin{aligned}&\frac{p_3}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t}\varrho\theta_{\sigma t}^2\mathrm{d}x + \varkappa\|\theta_{\sigma t}\|_{1,\Omega_t}^2 + p_4\int\limits_{\Omega_t}\theta_{\sigma t}\mathrm{div}\, v_t\mathrm{d}x\\ &\le \varepsilon\|v_{tx}\|_{0,\Omega_t}^2 + c(\|v_x\|_{0,\Omega_t}^2 + \|\theta_{\sigma x}\|_{0,\Omega_t}^2)\\ &+ c[X_2^2(1+X_2) + X_2\|\theta_{\sigma tt}\|_{0,\Omega_t}^2].\end{aligned} \tag{2.45}$$

Now, multiplying (2.45) by p_2/p_4 and (2.38) by p_1/ϱ_e^2, and next adding the both mentioned estimates to (2.36) we obtain (2.28).

This concludes the proof. □

Lemmas 2.1 and 2.3 yield

Lemma 2.4. *Let the assumptions of Lemmas* 2.1 *and* 2.3 *be satisfied. Then*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\psi(t)+\bar{c}_3(\|v\|^2_{1,\Omega_t}+\|v_t\|^2_{1,\Omega_t}+\|\theta_{\sigma x}\|^2_{0,\Omega_t}+\|\theta_\sigma\|^2_{0,S_t}+\|\theta_{\sigma t}\|^2_{1,\Omega_t}$$
$$+\|\varrho_{\sigma t}\|^2_{0,\Omega_t})\le c_6[X_2^2(1+X_2)+X_2\|\theta_{\sigma tt}\|^2_{0,\Omega_t}],\tag{2.46}$$

where

$$\psi(t)=\int_{\Omega_t}\varrho\bigg[d_1v^2+v_t^2+\frac{p_1}{\varrho_e^2}(d_1\varrho_\sigma^2+\varrho_{\sigma t}^2)$$
$$+\frac{p_2p_3}{p_4}(d_1\theta_\sigma^2+\theta_{\sigma t}^2)\bigg]\,\mathrm{d}x,\tag{2.47}$$

d_1 *is a constant so large that* $d_1\bar{c}_1/2\ge c_4$ ($\bar{c}_1$ *is the constant from* (2.3) *and* c_4 *is the constant from* (2.28)); p_1,p_2 *are defined in* (1.12); p_3,p_4 *are given by* (2.9), X_2 *is determined by* (2.29).

Proof. First multiply estimate (2.3) by a constant d_1 so large that $d_1\bar{c}_1/2>c_4$. Adding the obtained inequality to (2.28) we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\psi(t)+\frac{d_1\bar{c}_1}{2}(\|v\|^2_{1,\Omega_t}+\|\theta_{\sigma x}\|^2_{0,\Omega_t})+\bar{c}_2(\|v_t\|^2_{1,\Omega_t}+\|\theta_{\sigma t}\|^2_{1,\Omega_t})$$
$$\le c_1d_1X_1^2+c_2d_1X_1\|\theta_{\sigma t}\|^2_{0,\Omega_t}$$
$$+c_5[X_2^2(1+X_2)+X_2\|\theta_{\sigma tt}\|^2_{0,\Omega_t}].\tag{2.48}$$

Next, from equation $(1.10)_2$ we obtain

$$\|\varrho_{\sigma t}\|^2_{0,\Omega_t}\le c\|v\|^2_{1,\Omega_t}+cX_2^2.\tag{2.49}$$

Adding (2.49) to (2.48) and using that $X_1\le X_2$ and $\|\theta_{\sigma t}\|^2_{0,\Omega_t}\le cX_2$ we get (2.46) with $\bar{c}_3\le\min\{d_1\bar{c}_1/4,1\}$.

This completes the proof. □

Next, we prove

Lemma 2.5. *Let* $p\in C^2(\mathbb{R}^1_+\times\mathbb{R}^1_+)$, $c_v\in C^2(\mathbb{R}^1_+\times\mathbb{R}^1_+)$ *and assume* (1.2), (1.8), (1.14). *Let* $(v,\theta_\sigma,\varrho_\sigma)$ *be a solution of* (1.10). *Then*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\left(\varrho v_{tt}^2+\frac{\varrho_\varrho}{\varrho}\varrho_{\sigma tt}^2+\frac{\varrho c_v}{\theta}\theta_{\sigma tt}^2\right)\mathrm{d}x$$
$$+\bar{c}_4(\|v_{tt}\|^2_{1,\Omega_t}+\|\theta_{\sigma tt}\|^2_{1,\Omega_t}+\|\varrho_{\sigma tt}\|^2_{0,\Omega_t})\tag{2.50}$$
$$\le c_7(\|v\|^2_{1,\Omega_t}+\|v_t\|^2_{1,\Omega_t}+\|\theta_{\sigma t}\|^2_{1,\Omega_t}+c_8\bar{X}_1\bar{X}_2(1+\bar{X}_1^2)),$$

where

$$\bar{X}_1 = |v|^2_{2,0,\Omega_t} + |\theta_\sigma|^2_{2,0,\Omega_t} + |\varrho_\sigma|^2_{2,0,\Omega_t}, \tag{2.51}$$

$$\bar{X}_2 = |v|^2_{3,1,\Omega_t} + |\theta_\sigma|^2_{3,1,\Omega_t} + \|\varrho_\sigma\|^2_{2,\Omega_t} + \|\varrho_{\sigma t}\|^2_{2,\Omega_t} + \|\varrho_{\sigma tt}\|^2_{1,\Omega_t},$$

$$|g|_{l,k,\Omega_t} \equiv \sum_{i\le l-k} \|\partial_t^i g\|_{l-i,\Omega_t}, \quad g\in\{v,\theta_\sigma,\varrho_\sigma\}. \tag{2.52}$$

Proof. Differentiating $(1.10)_1$ twice with respect to t we get

$$\begin{aligned}
&\varrho[v_{ttt} + (v\cdot\nabla)v_{tt}] - \operatorname{div}\mathbb{T}(v_{tt},p_{\sigma tt})\\
&= -\varrho_{\sigma tt}[v_t + (v\cdot\nabla)v] - 2\varrho_{\sigma t}[v_{tt} + (v_t\cdot\nabla)v + (v\cdot\nabla)v_t]\\
&\quad - \varrho[2(v_t\cdot\nabla)v_t + (v_{tt}\cdot\nabla)v].
\end{aligned} \tag{2.53}$$

Multiplying (2.53) by v_{tt} and integrating over Ω_t we obtain

$$\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\varrho v_{tt}^2\mathrm{d}x + \frac{\mu}{2}E_{\Omega_t}(v_t) + (\nu-\mu)\|\operatorname{div} v_{tt}\|^2_{0,\Omega_t}\\
&-\int_{\Omega_t} p_{\sigma tt}\operatorname{div} v_{tt}\mathrm{d}x - \int_{S_t}\mathbb{T}(v_{tt},p_{\sigma tt})\bar{n}\cdot v_{tt}\mathrm{d}x\\
&\le \varepsilon\|v_{tt}\|^2_{1,\Omega_t} + c(\varepsilon)[\|\varrho_{\sigma tt}\|^2_{0,\Omega_t}(\|v_t\|^2_{1,\Omega_t} + \|v\|^4_{2,\Omega_t})\\
&+\|\varrho_{\sigma t}\|^2_{1,\Omega_t}(\|v_{tt}\|^2_{1,\Omega_t} + \|v_t\|^2_{1,\Omega_t}\|v\|^2_{2,\Omega_t})\\
&+\|v_t\|^4_{1,\Omega} + \|v\|^2_{2,\Omega_t}\|\varrho_\sigma\|^2_{2,\Omega_t} + \|v\|^2_{2,\Omega_t}\|v_{tt}\|^2_{1,\Omega_t}].
\end{aligned} \tag{2.54}$$

Using boundary condition $(1.10)_4$ we have

$$\int_{S_t}\mathbb{T}(v_{tt},p_{\sigma tt})\bar{n}\cdot v_{tt}\mathrm{d}x = -2\int_{S_t}\mathbb{T}(v_t,p_{\sigma t})\bar{n}_t\cdot v_{tt}\mathrm{d}s - \int_{S_t}\mathbb{T}(v,p_\sigma)\bar{n}_{tt}\cdot v_{tt}\mathrm{d}s,$$

where $p_{\sigma t} = p_\varrho\varrho_{\sigma t} + p_\theta\theta_{\sigma t}$. Hence we have

$$\begin{aligned}
&\Big|\int_{S_t}\mathbb{T}(v_{tt},p_{\sigma tt})\bar{n}\cdot v_{tt}\mathrm{d}s\Big| \\
&\le \varepsilon\|v_{tt}\|^2_{1,\Omega_t} + c(\varepsilon)[\|v\|^2_{2,\Omega_t}(\|v_t\|^2_{2,\Omega_t} + \|\varrho_{\sigma t}\|^2_{1,\Omega_t} + \|\theta_{\sigma t}\|^2_{1,\Omega_t} + \|v\|^2_{2,\Omega_t}\\
&+\|v\|^2_{2,\Omega_t}\|\varrho_\sigma\|^2_{1,\Omega_t} + \|v\|^2_{2,\Omega_t}\|\theta_\sigma\|^2_{1,\Omega_t}) + \|v_t\|^2_{1,\Omega_t}(\|\varrho_\sigma\|^2_{1,\Omega_t} + \|\theta_\sigma\|^2_{1,\Omega_t})].
\end{aligned} \tag{2.55}$$

Moreover

$$p_{\sigma tt} = p_{\varrho\varrho}\varrho^2_{\sigma t} + 2p_{\varrho\theta}\varrho_{\sigma t}\theta_{\sigma t} + p_{\theta\theta}\theta^2_{\sigma t} + p_\varrho\varrho_{\sigma tt} + p_\theta\theta_{\sigma tt},$$

so

$$-\int_{\Omega_t} p_{\sigma tt}\operatorname{div} v_{tt}\mathrm{d}x = -\int_{\Omega_t} p_\varrho\varrho_{\sigma tt}\operatorname{div} v_{tt}\mathrm{d}x - \int_{\Omega_t} p_\theta\theta_{\sigma tt}\operatorname{div} v_{tt}\mathrm{d}x + J, \tag{2.56}$$

where

$$|J| \le \varepsilon\|v_{tt}\|^2_{1,\Omega_t} + c(\varepsilon)(\|\varrho_{\sigma t}\|^4_{1,\Omega_t} + \|\theta_{\sigma t}\|^4_{1,\Omega_t}). \tag{2.57}$$

In view of (2.55)–(2.57) and (2.6) from (2.54) it follows the estimate

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t}\varrho v^2_{tt}\mathrm{d}x + c_1E_{\Omega_t}(v_{tt}) - \int\limits_{\Omega_t}p_\varrho\varrho_{\sigma tt}\mathrm{div}\, v_{tt}\mathrm{d}x - \int\limits_{\Omega_t}p_\theta\theta_{\sigma tt}\mathrm{div}\, v_{tt}\mathrm{d}x\\ &\le \varepsilon\|v_{tt}\|^2_{1,\Omega_t} + c\bar{X}_1\bar{X}_2(1+\bar{X}_1^2).\end{aligned} \tag{2.58}$$

By Lemma 5.4 of [10] we have

$$\|v_{tt}\|^2_{1,\Omega_t} \le c[E_{\Omega_t}(v_{tt}) + \bar{X}_1\bar{X}_2(1+\bar{X}_1^2)]. \tag{2.59}$$

From (2.58)–(2.59) it follows the estimate

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t}\varrho v^2_{tt}\mathrm{d}x + c_0\|v_{tt}\|^2_{1,\Omega_t} - \int\limits_{\Omega_t}p_\varrho\varrho_{\sigma tt}\mathrm{div}\, v_{tt}\mathrm{d}x - \int\limits_{\Omega_t}p_\theta\theta_{\sigma tt}\mathrm{div}\, v_{tt}\mathrm{d}x\\ &\le c[\|v_t\|^2_{1,\Omega_t} + \|\varrho_{\sigma t}\|^2_{0,\Omega_t} + \|\theta_{\sigma t}\|^2_{0,\Omega_t} + \bar{X}_1\bar{X}_2(1+\bar{X}_1^2)],\end{aligned} \tag{2.60}$$

where we used that ε in (2.58) is sufficiently small.

Next, dividing $(1.10)_3$ by θ, differentiating twice with respect to t, multiplying by $\theta_{\sigma tt}$ and integrating over Ω_t we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t}\frac{\varrho c_v}{\theta}\theta^2_{\sigma tt}\mathrm{d}x + \frac{\varkappa}{\theta_2}\|\theta_{\sigma ttx}\|^2_{0,\Omega_t} + \int\limits_{\Omega_t}p_\theta\theta_{\sigma tt}\mathrm{div}\, v_{tt}\mathrm{d}x\\ &-\varkappa\int\limits_{S_t}\frac{1}{\theta}\nabla\theta_{\sigma tt}\cdot\bar{n}\theta_{\sigma tt}\mathrm{d}s \le \varepsilon\|\theta_{\sigma tt}\|^2_{1,\Omega_t} + c\bar{X}_1\bar{X}_2(1+\bar{X}_1^2).\end{aligned} \tag{2.61}$$

Using boundary condition $(1.10)_5$ we have

$$\begin{aligned}\int\limits_{S_t}\frac{1}{\theta}\nabla\theta_{\sigma tt}\cdot\bar{n}\theta_{\sigma tt}\mathrm{d}s &= -\int\limits_{S_t}\frac{1}{\theta}\nabla\theta_\sigma\cdot\bar{n}_{tt}\theta_{\sigma tt}\mathrm{d}s - 2\int\limits_{S_t}\frac{1}{\theta}\nabla\theta_{\sigma t}\cdot\bar{n}_t\theta_{\sigma tt}ds\\ &\quad-\varkappa_a\int\limits_{S_t}\frac{1}{\theta}\theta^2_{\sigma tt}ds.\end{aligned} \tag{2.62}$$

Hence

$$\begin{aligned}\Big|\int\limits_{S_t}\nabla\theta_{\sigma tt}\cdot\bar{n}\theta_{\sigma tt}ds\Big| &\le \varepsilon\|\theta_{\sigma tt}\|^2_{1,\Omega_t} + c(\varepsilon)[\|\theta_{\sigma t}\|^2_{2,\Omega_t}\|v\|^2_{2,\Omega_t}\\ &+\|\theta_\sigma\|^2_{2,\Omega_t}(\|v_t\|^2_{2,\Omega_t} + \|v\|^4_{2,\Omega_t}) + \|\theta_{\sigma tt}\|^2_{0,\Omega_t}].\end{aligned} \tag{2.63}$$

From equation (2.39) we get the following estimate for $\|\theta_{\sigma tt}\|^2_{0,\Omega_t}$

$$\|\theta_{\sigma tt}\|^2_{0,\Omega_t} \le \varepsilon\|\theta_{\sigma ttx}\|^2_{0,\Omega_t}$$

$$+ c(\varepsilon)[\|v_t\|^2_{1,\Omega_t} + \|\theta_{\sigma t}\|^2_{1,\Omega_t} + \bar{X}_1\bar{X}_2(1+\bar{X}_1^2)]. \tag{2.64}$$

Taking into account (2.60)–(2.64) we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t}\Big(\varrho v_{tt}^2 + \frac{\varrho c_v}{\theta}\theta_{\sigma tt}^2\Big)\mathrm{d}x + c_0(\|v_{tt}\|^2_{1,\Omega_t} + \|\theta_{\sigma tt}\|^2_{1,\Omega_t}) - \int\limits_{\Omega_t} p_\varrho \varrho_{\sigma tt}\mathrm{div}\, v_{tt}\mathrm{d}x\\ &\le c(\|v_t\|^2_{1,\Omega_t} + \|\theta_{\sigma t}\|^2_{1,\Omega_t} + \|\varrho_{\sigma t}\|^2_{0,\Omega_t} + \bar{X}_1\bar{X}_2(1+\bar{X}_1^2)].\end{aligned} \tag{2.65}$$

Now, using continuity equation $(1.10)_2$ (twice differentiated with respect to t) yields

$$\begin{aligned}&-\int\limits_{\Omega_t} p_\varrho \varrho_{\sigma tt}\mathrm{div}\, v_{tt}\mathrm{d}x = \int\limits_{\Omega_t}\Big(\frac{p_\varrho \varrho_{\sigma tt}}{\varrho}\varrho_{\sigma ttt} + \frac{p_\varrho \varrho_{\sigma tt}^2}{\varrho}\mathrm{div}\, v + \frac{2p_\varrho \varrho_{\sigma t}\varrho_{\sigma tt}}{\varrho}\mathrm{div}\, v_t\\ &+ \frac{p_\varrho \varrho_{\sigma tt}}{\varrho}v_{tt}\cdot\nabla\varrho_\sigma + \frac{2p_\varrho \varrho_{\sigma tt}}{\varrho}v_t\cdot\nabla\varrho_{\sigma t} + \frac{p_\varrho \varrho_{\sigma tt}}{\varrho}v\cdot\nabla\varrho_{\sigma tt}\Big)\mathrm{d}x\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t}\frac{p_\varrho}{\varrho}\varrho_{\sigma tt}^2\mathrm{d}x + I,\end{aligned} \tag{2.66}$$

where

$$\begin{aligned}|I| &\le \varepsilon(\|v_{tt}\|^2_{1,\Omega_t} + \|\varrho_{\sigma tt}\|^2_{0,\Omega_t})\\ &+ c(\varepsilon)[\|\varrho_{\sigma tt}\|^2_{0,\Omega_t}(\|\varrho_\sigma\|^2_{2,\Omega_t} + \|\varrho_{\sigma t}\|^2_{2,\Omega_t} + \|\theta_{\sigma t}\|^2_{2,\Omega_t} + \|v\|^2_{2,\Omega_t}\|\theta_\sigma\|^2_{3,\Omega_t})\\ &+ \|v_t\|^2_{2,\Omega_t}\|\varrho_{\sigma t}\|^2_{1,\Omega_t} + \|\varrho_{\sigma tt}\|^2_{1,\Omega_t}\|\varrho_\sigma\|^2_{2,\Omega_t}\|v\|^2_{2,\Omega_t} + \|v_t\|^2_{1,\Omega_t}].\end{aligned} \tag{2.67}$$

In order to find an estimate for $\|\varrho_{\sigma tt}\|^2_{0,\Omega_t}$ we use continuity equation $(1.10)_1$ which we differentiate with respect to t, i.e. the following equation

$$\varrho_{\sigma tt} + \varrho_{\sigma t}\mathrm{div}\, v + \varrho\mathrm{div}\, v_t + \nabla\varrho_{\sigma t}\cdot v + \nabla\varrho_\sigma\cdot v_t = 0. \tag{2.68}$$

Equation (2.68) gives

$$\|\varrho_{\sigma tt}\|^2_{0,\Omega_t} \le c(\|v_t\|^2_{1,\Omega_t} + \|\varrho_{\sigma t}\|^2_{1,\Omega_t}\|v\|^2_{2,\Omega_t} + \|v_t\|^2_{1,\Omega_t}\|\varrho_\sigma\|^2_{2,\Omega_t}). \tag{2.69}$$

Estimates (2.65)–(2.67), (2.69) and (2.49) give inequality (2.50).

This completes the proof of the lemma. □

3. Differential inequality

In Section 2 we derived some energy-type inequalities for a local solution (u,ϑ,η) to problem (1.1) such that $u,\vartheta\in\mathcal{A}_{T,\Omega}$, $\mu\in\mathcal{B}_{T,\Omega}$, where u,ϑ,η denote v,θ,ϱ written in the Lagrangian coordinates ξ and $\mathcal{B}_{T,\Omega}$, $\mathcal{A}_{T,\Omega}$ are defined by (1.15) and (1.16). The existence and uniqueness of such a solution is proved in [8].

Now, we derive the remaining energy-type inequalities which together with inequalities proved in Section 2 are used to obtain differential inequality (1.19). For this purpose we need local considerations.

To do this we consider the motion near the equilibrium state. Let

$$p_\sigma = p - p_0, \qquad \theta_\sigma = \theta - \theta_e, \qquad \varrho_\sigma = \varrho - \varrho_e, \tag{3.1}$$

where θ_e and ϱ_e are introduced in Definition 1.1.

Moreover, assume

$$f = r = \bar{\theta} = 0 \tag{3.2}$$

and

$$\varrho_1 < \varrho(x,t) < \varrho_2, \quad \theta_1 < \theta(x,t) < \theta_2 \quad \text{for } x \in \bar{\Omega}_t \ t \in [0,T], \tag{3.3}$$

where $0 < \varrho_1 < \varrho_2$, $0 < \theta_1 < \theta_2$ are constants.

Using the Lagrangian coordinates and (3.1)–(3.2) we write (1.1) in the form

$$\begin{aligned}
&\eta u_{it} - \nabla_{u_j} T_{uij}(u,p_\sigma) = 0 \ (i=1,2,3) && \text{in } \Omega^T \equiv \Omega \times (0,T),\\
&\eta_{\sigma t} + \eta \operatorname{div}_u u = 0 && \text{in } \Omega^T,\\
&\eta c_v(\eta,\vartheta)\vartheta_{\sigma t} - \varkappa \nabla_u^2 \vartheta_\sigma + \vartheta p_\vartheta(\eta,\vartheta) \operatorname{div}_u u \\
&-\frac{\mu}{2}\sum_{i,j=1}^3 (\xi_{x_i}\cdot\nabla_\xi u_j + \xi_{x_j}\cdot\nabla_\xi u_i)^2 - (\nu-\mu)(\operatorname{div}_u u)^2 = 0 && \text{in } \Omega^T,\\
&\mathbb{T}_u(u,p_\sigma)\bar{n}_u = 0 && \text{on } S^T \equiv S \times (0,T),\\
&\bar{n}_u \cdot \nabla_u \vartheta_\sigma = -\varkappa_a \vartheta_\sigma && \text{on } S^T,\\
&u\big|_{t=0} = v_0, \ \eta_\sigma\big|_{t=0} = \varrho_{\sigma 0}, \ \vartheta_\sigma\big|_{t=0} = \theta_{\sigma 0} && \text{in } \Omega,
\end{aligned} \tag{3.4}$$

where $\eta(\xi,t) = \varrho(X_u(\xi,t),t)$, $u(\xi,t) = v(X_u(\xi,t),t)$, $\vartheta(\xi,t) = \theta(X_u(\xi,t),t)$ (X_u is given by (1.4)), $\eta_\sigma = \eta - \varrho_e$, $\vartheta_\sigma = \vartheta - \theta_e$, $\varrho_{\sigma 0} = \varrho_0 - \varrho_e$, $\theta_{\sigma 0} = \theta_0 - \theta_e$, $\mathbb{T}_u(u,p_\sigma) = \{T_{uij}(u,p_\sigma)\}_{i,j=1,2,3} = \{-p_\sigma \delta_{ij} + \mu(\partial_{x_i}\xi_k \partial_{\xi_k} u_j + \partial_{x_j}\xi_k \partial_{\xi_k} u_i) + (\nu-\mu)\delta_{ij} \operatorname{div}_u u\}_{i,j=1,2,3}$, $\operatorname{div}_u u = \nabla_u \cdot u = \partial_{x_i}\xi_k \partial_{\xi_k} u_i$, $\nabla_u = (\xi_{kx_i}\partial_{\xi_k})_{i=1,2,3}$, $\nabla_{u_j} = \xi_{kx_j}\partial_{\xi_k}$, $\partial_{x_i}\xi_k$ are the elements of the matrix ξ_x which is inverse to $x_\xi = I + \int_0^t u_\xi(\xi,t')\mathrm{d}t'$, $I = \{\delta_{ij}\}_{i,j=1,2,3}$ is the unit matrix,

$$\bar{n}_u = \bar{n}(X_u(\xi,t),t) = \frac{\nabla_x \varphi(x,t)}{|\nabla_x \varphi(x,t)|}\bigg|_{x = X_u(\xi,t)}$$

(S_t is determined at least locally by the equation $\varphi(x,t) = 0$) and the summation over repeated indices is assumed.

Using the Taylor formula we can write p_σ as follows:

$$\begin{aligned}
p_\sigma = p(\eta,\vartheta) - p(\varrho_e,\theta_e) &= p_\eta(\varrho_e,\theta_e)\eta_\sigma + p_\vartheta(\varrho_e,\theta_e)\vartheta_\sigma + p'_\sigma \\
&\equiv p_1\eta_\sigma + p_2\vartheta_\sigma + p'_\sigma,
\end{aligned} \tag{3.5}$$

where

$$|p'_\sigma| \le c(\varrho_e,\theta_e)(|\eta_\sigma|^2 + |\eta_\sigma||\vartheta_\sigma| + |\vartheta_\sigma|^2). \tag{3.6}$$

Now, introduce a partition of unity $(\{\tilde\Omega_i\},\{\zeta_i\})$, $\Omega \subset \bigcup_i \tilde\Omega_i$ such that $\tilde\Omega_i$ for $i \in \mathcal{M}$ is an interior subdomain and $\tilde\Omega_i$ for $i \in \mathcal{N}$ is a boundary subdomain. Let $\tilde\Omega$ be one of the $\tilde\Omega_i's$ and $\zeta(\xi) = \zeta_i(\xi)$ be the corresponding function. If $\tilde\Omega$ is an interior subdomain then let $\tilde\omega$ be a set such that $\bar{\tilde\omega} \subset \tilde\Omega$ and $\zeta(\xi) = 1$ for $\xi \in \tilde\omega$. Otherwise we assume that $\tilde\Omega \cap S \ne \varnothing$, $\overline{\tilde\omega} \cap S \ne \varnothing$, $\overline{\tilde\omega} \subset \tilde\Omega$. Take any $\beta \in \overline{\tilde\omega} \cap S \subset \tilde S$, $\tilde S = S \cap \tilde\Omega$ and introduce local coordinates $\{y\}$ associated with $\{\xi\}$ by

$$y_k = \alpha_{kl}(\xi_l - \beta_l), \quad \alpha_{3k} = n_k(\beta), \quad k = 1,2,3, \tag{3.7}$$

where $\{\alpha_{kl}\}$ is a constant orthogonal matrix such that $\tilde S$ is determined by the equation $y_3 = F(y_1,y_2)$, $F \in H^{5/2}$ and

$$\begin{aligned}\tilde\Omega \cap \Omega = \{y\colon |y_i| < d,\ i = 1,2,\ F(y') < y_3 < F(y') + d,\\ y' = (y_1,y_2)\}.\end{aligned} \tag{3.8}$$

Next, we introduce functions $u',\eta',\vartheta',\eta'_\sigma,\vartheta'_\sigma$ by

$$\begin{aligned}
&u'_i(y) = \alpha_{ij}u_j(\xi)\big|_{\xi=\xi(y)}\ (i = 1,2,3),\\
&\eta'(y) = \eta(\xi)\big|_{\xi=\xi(y)}, && \vartheta'(y) = \vartheta(\xi)\big|_{\xi=\xi(y)},\\
&\eta'_\sigma(y) = \eta_\sigma(\xi)\big|_{\xi=\xi(y)}, && \vartheta'_\sigma(y) = \vartheta_\sigma(\xi)\big|_{\xi=\xi(y)},
\end{aligned}$$

where $\xi = \xi(y)$ is the inverse transformation to (3.7).

Let us introduce new variables given by

$$z_i = y_i\ (i = 1,2), \quad z_3 = y_3 - \tilde F(y), \quad y \in \tilde\Omega \cap \Omega,$$

which will be denoted by $z = \Phi(y)$ (where $\tilde F \in H^3$ is an extension of F). Let

$$\begin{aligned}&\hat\Omega = \Phi(\tilde\Omega \cap \Omega) = \{z\colon |z_i| < d,\ i = 1,2,\ 0 < z_3 < d\} \quad \text{and}\\ &\hat S = \Phi(\tilde S).\end{aligned} \tag{3.9}$$

Next define

$$\begin{aligned}
&\hat u(z) = u'(y)\big|_{y=\Phi^{-1}(z)}, \quad \hat\eta(z) = \eta'(y)\big|_{y=\Phi^{-1}(z)},\\
&\hat\vartheta(z) = \vartheta'(y)\big|_{y=\Phi^{-1}(z)}, \quad \hat\eta_\sigma(z) = \hat\eta(z) - \varrho_e, \qquad \hat\vartheta_\sigma(z) = \hat\vartheta(z) - \theta_e.
\end{aligned}$$

Set $\hat\nabla_k = \xi_{lx_k} z_{i,\xi_k} \nabla_{z_i}\big|_{\chi=x^{-1}(z)}$, where

$$\chi(\xi) = \Phi(\psi(\xi))$$

and $y = \psi(\xi)$ is described by (3.7). We also introduce the following notation: $\tilde u(\xi) = u(\xi)\zeta(\xi)$, $\tilde\eta(\xi) = \eta(\xi)\zeta(\xi)$, $\tilde\vartheta(\xi) = \vartheta(\xi)\zeta(\xi)$, $\tilde\eta_\sigma(\xi) = \eta_\sigma(\xi)\zeta(\xi)$,

$\tilde{\vartheta}_\sigma(\xi) = \vartheta_\sigma(\xi)\zeta(\xi)$, for $\xi \in \tilde{\Omega}$, $\tilde{\Omega} \cap S = \varnothing$ and $\tilde{u}(z) = \hat{u}(z)\hat{\zeta}(z)$, $\tilde{\eta}(z) = \hat{\eta}(z)\hat{\zeta}(z)$, $\tilde{\vartheta}(z) = \hat{\vartheta}(z)\hat{\zeta}(z)$, $\tilde{\eta}_\sigma(z) = \hat{\eta}_\sigma(z)\hat{\zeta}(z)$, $\tilde{\vartheta}_\sigma(z) = \hat{\vartheta}_\sigma(z)\hat{\zeta}(z)$ for $z \in \hat{\Omega} = \Phi(\tilde{\Omega} \cap \Omega)$, $\bar{\tilde{\Omega}} \cap S \neq \varnothing$, where $\hat{\zeta}(z) = \zeta(\xi)\big|_{\xi=\chi^{-1}(z)}$.

Using the above notation we rewrite problem (3.4) in the following form in an interior subdomain:

$$
\begin{aligned}
&\eta\tilde{u}_{it} - \nabla_{u_j} T_{uij}(\tilde{u}, \tilde{p}_\sigma) = -\nabla_{u_j} B_{uij}(u,\zeta) - T_{uij}(u,p_\sigma)\nabla_{u_j}\zeta \\
&\equiv k_{1i}, \quad i = 1,2,3, \\
&\tilde{\eta}_{\sigma t} + \eta\nabla_u \cdot \tilde{u} = \eta u \cdot \nabla_u \zeta \equiv k_2 \\
&\eta c_v(\eta,\vartheta)\tilde{\vartheta}_{\sigma t} - \varkappa\nabla_u^2\tilde{\vartheta}_\sigma + \vartheta p_\vartheta(\eta,\vartheta)\nabla_u \cdot \tilde{u} \qquad (3.10) \\
&= \frac{\mu}{2}\Big[\sum_{i,j=1}^{3}(\xi_{kx_i}\partial_{\xi_k}u_j + \xi_{kx_j}\partial_{\xi_k}u_i)^2 + (\nu-\mu)(\nabla_u \cdot u)^2\Big]\zeta \\
&+ \vartheta p_\vartheta(\eta,\vartheta) u \cdot \nabla_u\zeta - \varkappa(\nabla_u^2\zeta\vartheta_\sigma + 2\nabla_u\zeta \cdot \nabla_u\vartheta_\sigma) \equiv k_3,
\end{aligned}
$$

where $\tilde{p}_\sigma = p_\sigma\zeta$ and $\mathbb{B}_u(u,\zeta) = \{B_{uij}(u,\zeta)\}_{i,j=1,2,3} = \{\mu(u_i\nabla_{u_j}\zeta + u_j\nabla_{u_i}\zeta) + (\nu-\mu)\delta_{ij}u\cdot\nabla_u\zeta\}_{i,j=1,2,3}$.

In boundary subdomains we have

$$
\begin{aligned}
&\hat{\eta}\tilde{u}_{it} - \hat{\nabla}_j\hat{T}_{ij}(\tilde{u},\tilde{p}_\sigma) = -\hat{\nabla}_j\hat{B}_{ij}(\hat{u},\hat{\zeta}) - \hat{T}_{ij}(\hat{u},p_\sigma)\hat{\nabla}_j\hat{\zeta} \\
&\equiv k_{4i}, \quad i = 1,2,3, \\
&\tilde{\eta}_{\sigma t} + \hat{\eta}\hat{\nabla}\cdot\tilde{u} = \hat{\eta}\hat{u}\cdot\hat{\nabla}\hat{\zeta} \equiv k_5, \\
&\hat{\eta}c_v(\hat{\eta},\hat{\vartheta})\tilde{\vartheta}_{\sigma t} - \varkappa\hat{\nabla}^2\hat{\vartheta}_\sigma + \hat{\vartheta}p_{\hat{\vartheta}}(\hat{\eta},\hat{\vartheta})\hat{\nabla}\cdot\tilde{u} \\
&= \frac{\mu}{2}\Big[\sum_{i,j=1}^{3}(\hat{\nabla}_i\hat{u}_j + \hat{\nabla}_j\hat{u}_i)^2 + (\nu-\mu)(\hat{\nabla}\cdot\hat{u})^2\Big]\hat{\zeta} \qquad (3.11) \\
&+ \hat{\vartheta}p_{\hat{\vartheta}}(\hat{\eta},\hat{\vartheta})\hat{u}\cdot\hat{\nabla}\hat{\zeta} - \varkappa(\hat{\nabla}^2\hat{\zeta}\hat{\vartheta}_\sigma + 2\hat{\nabla}\hat{\zeta}\cdot\hat{\nabla}\hat{\vartheta}_\sigma) \equiv k_6 \\
&\hat{T}_{ij}(\tilde{u},\tilde{p}_\sigma)\hat{n}_j = \hat{B}_{ij}(\hat{n},\hat{\zeta})\hat{n}_j \equiv k_{7i}, \quad i = 1,2,3 \\
&\hat{n}\cdot\hat{\nabla}\tilde{\vartheta}_\sigma = \hat{n}\cdot\hat{\nabla}\hat{\zeta}\tilde{\vartheta}_\sigma - \varkappa_a\tilde{\vartheta}_\sigma \equiv k_8 - \varkappa_a\tilde{\vartheta}_\sigma,
\end{aligned}
$$

where $\hat{\nabla} = (\hat{\nabla}_j)_{j=1,2,3}$, $\hat{\mathbb{T}}$ and $\hat{\mathbb{B}}$ indicate that operator ∇_u is replaced by $\hat{\nabla}$.

In the below Lemmas 3.1–3.3 by ε we denote small constants, by C_i $(i = 1,\dots,8)$ or c we denote positive constants depending on $\varrho_1, \varrho_2, \theta_1, \theta_2$, $\|\int_0^t v\mathrm{d}t'\|^2_{3,\Omega_t}$, $\|S\|_{H^{5/2}}$, on the parameters which guarantee the existence of the inverse transformation to $x = x(\xi,t)$ and also on the constants of the imbeddings theorems and the Korn inequalities. By $\bar{C}_i$ $(i = 1,2,3)$ or c_0 we denote positive constants less or equal to one depending on the same quantities as C_i $(i = 1,\dots,8)$ and c. We do not distinguish different $\varepsilon's$,

$c_0's$ and $c's$. Moreover, by τ and n we denote z_1, z_2, i.e. $\tau = (z_1, z_2)$ and z_3, respectively.

The existence of the inverse transformation to $x = x(\xi, t)$ is quaranteed by the estimates for the local solution (see [8]).

First, we prove.

Lemma 3.1. *Let* $p \in C^1(\mathbb{R}^1_+ \times \mathbb{R}^1_+)$, $c_v \in C^1(\mathbb{R}^1_+ \times \mathbb{R}^1_+)$ *and assume* (1.2), (3.2), (3.3). *Let* (v, θ, ϱ) *be a solution of* (1.1). *Then*

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\phi_1(t,\Omega) + \bar{C}_1(\|v\|^2_{2,\Omega_t} + \|\varrho_\sigma\|^2_{1,\Omega_t} + \|\theta_\sigma\|^2_{2,\Omega_t})\\
&\le C_1(\|v\|^2_{1,\Omega_t} + \|v_t\|^2_{1,\Omega_t} + \|\theta_\sigma\|^2_{1,\Omega_t} + \|\theta_{\sigma t}\|^2_{1,\Omega_t}\\
&+ \|\varrho_{\sigma t}\|^2_{0,\Omega_t} + \|\varrho_\sigma\|^2_{0,\Omega_t}) + C_2\bar{X}_1\Big(\bar{X}_1 + \Big\|\int_0^t v\mathrm{d}t'\Big\|^2_{3,\Omega_t}\Big)\cdot(1+\bar{X}_1),
\end{aligned}\tag{3.12}$$

where

$$\begin{aligned}
\phi_1(t,\Omega) &= \sum_{i\in\mathcal{M}}\int_{\tilde{\Omega}_i}\left(\eta\tilde{u}^2_\xi + \frac{p_\eta}{\eta}\tilde{\eta}^2_{\sigma\xi} + \frac{\eta c_v}{\vartheta}\tilde{\vartheta}^2_{\sigma\xi}\right)A\mathrm{d}\xi\\
&+\sum_{i\in\mathcal{N}}\int_{\hat{\Omega}_i}\left[D_2\left(\hat{\eta}\tilde{u}^2_\tau + \frac{p_{\hat{\eta}}}{\hat{\eta}}\tilde{\eta}^2_{\sigma\tau} + \frac{\hat{\eta}c_v}{\hat{\vartheta}}\tilde{\vartheta}^2_{\sigma\tau}\right) + D_1\frac{p_{\sigma\hat{\eta}}}{\hat{\eta}}\tilde{\eta}^2_{\sigma n} + \hat{\eta}\tilde{u}^2_{3n} + \frac{\hat{\eta}c_v}{\hat{\vartheta}}\tilde{\vartheta}^2_{\sigma n}\right]J\mathrm{d}z\\
&+\int_\Omega \eta u^2_\xi A\mathrm{d}\xi,
\end{aligned}\tag{3.13}$$

$$\bar{X}_1 = |v|^2_{2,0,\Omega_t} + |\theta_\sigma|^2_{2,0,\Omega_t} + |\varrho_\sigma|^2_{2,0,\Omega_t},\tag{3.14}$$

A and J are the Jacobians of transformations $x = x(\xi)$ and $x = x(z)$, respectively; $k^2_\xi = \sum_{i,j=1}^3 k^2_{i\xi_j}$, $k \in \{\tilde{u}, u\}$, $g^2_\xi = \sum_{i=1}^3 g^2_{\xi_i}$, $g \in \{\tilde{\eta}_\sigma, \tilde{\vartheta}_\sigma\}$, $h^2_\tau = \sum_{i=1}^2 h^2_{\tau_i}$, $h \in \{\tilde{\eta}_\sigma, \tilde{\vartheta}_\sigma\}$, $\tilde{u}^2_\tau = \sum_{i=1}^3\sum_{j=1}^2 \tilde{u}^2_{i\tau_j}$; D_1, D_2 are constants depending on the same quantitees as $\bar{C}_1, C_1$ and C_2.

Proof. First we consider the following elliptic problem

$$\begin{aligned}
&\mu\nabla^2_u u + \nu\nabla_u\nabla.u - p_{\sigma\eta}\nabla_u\eta_\sigma = \eta u_t + p_{\sigma\vartheta}\nabla_u\vartheta_\sigma && \text{in } \Omega,\\
&\operatorname{div}_u u = \operatorname{div}_u u && \text{in } \Omega,\\
&\mathbb{T}_u(u, p_\sigma)\bar{n}_u = 0 && \text{on } S.
\end{aligned}\tag{3.15}$$

Since the complementary condition to $(3.15)_3$ is satisfied we can apply to problem (3.15) the Agmon-Douglis-Nirenberg theory (see [1]). Thus, we get

$$\|u\|^2_{2,\Omega} + \|\eta_\sigma\|^2_{1,\Omega} \le c(\|\eta u_t\|^2_{0,\Omega} + \|\vartheta_{\sigma\xi}\|^2_{0,\Omega} + \|\operatorname{div}_u u\|^2_{1,\Omega} + \|\vartheta_\sigma\bar{n}_u\|^2_{1/2,S})$$

$$\leq c\big(\|u_t\|_{0,\Omega}^2+\|\vartheta_\sigma\|_{1,\Omega}^2+\|\operatorname{div}u\|_{1,\Omega}^2+\|\vartheta_\sigma\|_{1,\Omega}^2\Big\|\int\limits_0^t u\mathrm{d}t'\Big\|_{3,\Omega}^2\big), \tag{3.16}$$

where we have used that $\|\operatorname{div}_u u-\operatorname{div}u\|_{1,\Omega}^2\leq\varepsilon\|u\|_{2,\Omega}^2$ ($\varepsilon>0$ is sufficiently small).

In view of (3.16) we see that in order to obtain inequality (3.12) it remains to get appropriate estimates for $\|\operatorname{div}u\|_{1,\Omega}^2$ and for

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\phi_1(t,\Omega)+\bar{C}_1\|\theta_{\sigma xx}\|_{0,\Omega_t}^2.$$

To do this first consider boundary subdomains $\hat{\Omega}_i$, $i\in\mathcal{N}$. Differentiate $(3.11)_1$ with respect to τ, multiply the result by $\tilde{u}_\tau J$ and integrate over $\hat{\Omega}$ (which is one of $\hat{\Omega}_i{}'s$). Next, divide $(3.11)_3$ by $\hat{\vartheta}$, differentiate the result with respect to τ, multiply by $\tilde{\vartheta}_{\sigma\tau}J$ and integrate over $\hat{\Omega}$. Using the Korn inequality and equation $(3.11)_2$ we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\hat{\Omega}}\left(\hat{\eta}\tilde{u}_\tau^2+\frac{\hat{\eta}c_v}{\hat{\vartheta}}\tilde{\vartheta}_{\sigma\tau}^2\right)J\mathrm{d}z+c_0(\|\tilde{u}_\tau\|_{1,\hat{\Omega}}^2+\|\tilde{\vartheta}_{\sigma\tau z}^2\|_{0,\hat{\Omega}}^2)\\
&-\int\limits_{\hat{S}}(\hat{\mathbb{T}}(\tilde{u},\tilde{p}_\sigma)\hat{n})_{,\tau}\tilde{u}_\tau J\mathrm{d}z_{\hat{S}}-\varkappa\int\limits_{\hat{S}}\frac{1}{\hat{\vartheta}}(\hat{n}\cdot\hat{\nabla}\hat{\vartheta}_\sigma)_{,\tau}\hat{\vartheta}_{\sigma\tau}J\mathrm{d}z_{\hat{S}}\\
&-\int\limits_{\hat{\Omega}}\tilde{p}_{\sigma\tau}\hat{\nabla}\cdot\tilde{u}_\tau J\mathrm{d}z+\int\limits_{\hat{S}}p_{\sigma\hat{\vartheta}}\hat{\vartheta}_{\sigma\tau}\hat{\nabla}\cdot\tilde{u}_\tau J\mathrm{d}z_{\hat{S}}\\
&\leq\varepsilon(\|\hat{\eta}_{\sigma\tau}\|_{0,\hat{\Omega}}^2+\|\tilde{u}_\tau\|_{1,\hat{\Omega}}^2+\|\tilde{\vartheta}_{\sigma\tau z}\|_{0,\hat{\Omega}}^2)+c[\|\hat{u}\|_{1,\hat{\Omega}}^2+\|\hat{\vartheta}_\sigma\|_{1,\hat{\Omega}}^2+\|\hat{\eta}_\sigma\|_{0,\hat{\Omega}}^2\\
&+\bar{X}_1(\hat{\Omega})(\bar{X}_1(\hat{\Omega})+\Big\|\int\limits_0^t\hat{u}\mathrm{d}t'\Big\|_{3,\hat{\Omega}}^2)(1+\bar{X}_1(\hat{\Omega}))],
\end{aligned}\tag{3.17}$$

where

$$\bar{X}_1(\hat{\Omega})=|\hat{u}|_{2,0,\hat{\Omega}}^2+|\hat{\vartheta}_\sigma|_{2,0,\hat{\Omega}}^2+|\hat{\eta}_\sigma|_{2,0,\hat{\Omega}}^2. \tag{3.18}$$

Using boundary conditions $(3.11)_4$–$(3.11)_5$ we have

$$\begin{aligned}
&-\int\limits_{\hat{S}}(\hat{\mathbb{T}}(\tilde{u},\tilde{p}_\sigma)\hat{n})_{,\tau}\tilde{u}_\tau J\mathrm{d}z_{\hat{S}}=-\int\limits_{\hat{S}}(\hat{B}_{ij}(\hat{u},\hat{\zeta})\hat{n}_j)_{,\tau}\tilde{u}_{i\tau}J\mathrm{d}z_{\hat{S}}\\
&=\bar{c}\int\limits_{\hat{S}}\partial_\tau^{1/2}(\hat{B}_{ij}(\hat{u},\hat{\zeta})\hat{n}_j)\partial_\tau^{1/2}(\tilde{u}_{i\tau}J)\mathrm{d}z_{\hat{S}}
\end{aligned}$$

$$\leq \varepsilon\|\tilde{u}_\tau\|^2_{1,\hat{\Omega}} + c(\|\hat{u}\|^2_{1,\hat{\Omega}} + \bar{X}_1(\hat{\Omega})\Big\|\int_0^t \hat{u}\mathrm{d}t'\Big\|^2_{2,\hat{\Omega}}) \tag{3.19}$$

and

$$-\int_{\hat{S}} \frac{1}{\hat{\vartheta}}(\hat{n}\cdot\hat{\nabla}\hat{\vartheta}_\sigma)_{,\tau}\tilde{\vartheta}_{\sigma\tau}J\mathrm{d}z_{\hat{S}} = -\int_{\hat{S}} \frac{1}{\hat{\vartheta}}(\hat{n}\cdot\hat{\nabla}\hat{\zeta}\hat{\vartheta}_\sigma)_{,\tau}\tilde{\vartheta}_{\sigma\tau}J\mathrm{d}z_{\hat{S}} + \varkappa_a\int_{\hat{S}}\frac{1}{\hat{\vartheta}}\tilde{\vartheta}^2_{\sigma\tau}Jdz$$

$$\equiv I + \varkappa_a\int_{\hat{S}}\frac{1}{\hat{\vartheta}}\tilde{\vartheta}^2_{\sigma\tau}Jdz,$$

$$I = \bar{c}\int_{\hat{S}} \partial_\tau^{1/2}(\hat{n}\cdot\hat{\nabla}\hat{\zeta}\hat{\vartheta}_\sigma)\partial_\tau^{1/2}\left(\frac{1}{\hat{\vartheta}}\tilde{\vartheta}_{\sigma\tau}J\right)\mathrm{d}z_{\hat{S}} \tag{3.20}$$

$$\leq \varepsilon\|\tilde{\vartheta}_{\sigma\tau}\|^2_{1,\hat{\Omega}} + c(\|\hat{\vartheta}_\sigma\|^2_{1,\hat{\Omega}} + \bar{X}_1(\hat{\Omega})\Big\|\int_0^t \hat{u}\mathrm{d}t'\Big\|^2_{2,\hat{\Omega}}),$$

where to use derivative $\partial_\tau^{1/2}$ we have to apply the Fourier transformation and $\bar{c} > 0$ is a constant. Next,

$$-\int_{\hat{\Omega}} \tilde{p}_{\sigma\tau}\hat{\nabla}\cdot\tilde{u}_\tau J\mathrm{d}z = -\int_{\hat{\Omega}} p_{\hat{\eta}}\tilde{\eta}_{\sigma\tau}\hat{\nabla}\cdot\tilde{u}_\tau J\mathrm{d}z - \int_{\hat{\Omega}} p_{\hat{\vartheta}}\tilde{\vartheta}_{\sigma\tau}\hat{\nabla}\cdot\tilde{u}_\tau J\mathrm{d}z + I_1, \tag{3.21}$$

where

$$|I_1| \leq \varepsilon\|\tilde{u}_\tau\|^2_{1,\hat{\Omega}} + c(\|\hat{\eta}_\sigma\|^2_{0,\hat{\Omega}} + \|\hat{\vartheta}_\sigma\|^2_{0,\hat{\Omega}}) \tag{3.22}$$

and

$$-\int_{\hat{\Omega}} p_{\hat{\eta}}\tilde{\eta}_{\sigma\tau}\hat{\nabla}\cdot\tilde{u}_\tau J\mathrm{d}z = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\hat{\Omega}}\frac{p_{\hat{\eta}}}{\hat{\eta}}\tilde{\eta}^2_{\sigma\tau}J\mathrm{d}z + I_2, \tag{3.23}$$

where

$$|I_2| \leq \varepsilon\|\tilde{\eta}_{\sigma\tau}\|^2_{0,\hat{\Omega}} + c\Big[\|\hat{u}\|^2_{1,\hat{\Omega}} + \bar{X}_1(\hat{\Omega})(\bar{X}_1(\hat{\Omega}) + \Big\|\int_0^t \hat{u}\mathrm{d}t'\Big\|^2_{2,\hat{\Omega}})\Big]. \tag{3.24}$$

Taking into account (3.17), (3.19)–(3.24) and assuming that ε is sufficiently small we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\hat{\Omega}}\left(\hat{\eta}\tilde{u}^2_\tau + \frac{p_{\hat{\eta}}}{\hat{\eta}}\tilde{\eta}^2_{\sigma\tau} + \frac{\hat{\eta}c_v}{\hat{\vartheta}}\tilde{\vartheta}^2_{\sigma\tau}\right)J\mathrm{d}z + c_0(\|\tilde{u}_\tau\|^2_{1,\hat{\Omega}} + \|\tilde{\vartheta}_{\sigma\tau z}\|^2_{0,\hat{\Omega}} + \|\tilde{\vartheta}_{\sigma\tau}\|^2_{0,\hat{S}})$$

$$\le \varepsilon\|\hat\eta_{\sigma\tau}\|^2_{0,\hat\Omega}+c\Big[\|\hat u\|^2_{1,\hat\Omega}+\|\hat\vartheta_\sigma\|^2_{1,\hat\Omega}+\|\hat\eta_\sigma\|^2_{0,\hat\Omega}$$

$$+\bar X_1(\hat\Omega)(\bar X_1(\hat\Omega)+\Big\|\int\limits_0^t\hat u\mathrm{d}t'\Big\|^2_{3,\hat\Omega})(1+\bar X_1(\hat\Omega))\Big]. \tag{3.25}$$

Now, applying the operator $(\mu+\nu)\nabla_{z_i}$ to $(3.11)_2$, dividing the result by $\hat\eta$, adding to $(3.11)_1$ and multiplying the both sides of the result by $p_{\hat\eta}$ gives

$$\begin{aligned}&\frac{(\mu+\nu)}{\hat\eta}p_{\hat\eta}\nabla_{z_i}\tilde\eta_{\sigma t}+p^2_{\hat\eta}\nabla_{z_i}\tilde\eta_\sigma=p^2_{\hat\eta}\hat\eta_\sigma\nabla_{z_i}\hat\zeta\\&-p_{\hat\eta}p_{\hat\vartheta}\nabla_{z_i}\tilde\vartheta_\sigma+p_{\hat\eta}p_{\hat\vartheta}\hat\vartheta_\sigma\nabla_{z_i}\hat\zeta-p_\sigma p_{\hat\eta}\hat\eta_\sigma\nabla_{z_i}\hat\zeta\\&-p_\sigma p_{\hat\eta}\hat\vartheta_\sigma\nabla_{z_i}\hat\zeta+p_{\hat\eta}k_{4i}+\mu p_{\hat\eta}(\hat\nabla^2\tilde u_i-\hat\nabla_i\hat\nabla\cdot\tilde u)\\&+(\mu+\nu)p_{\hat\eta}(\hat\nabla_i-\nabla_{z_i})\hat\nabla\cdot\tilde u+\frac{(\mu+\nu)}{\hat\eta}p_{\hat\eta}\nabla_{z_i}(\hat\eta\hat u\cdot\hat\nabla\hat\zeta)\\&-p_{\hat\eta}\hat\eta\tilde u_{it}-\frac{(\mu+\nu)}{\hat\eta}p_{\hat\eta}\nabla_{z_i}\hat\eta_\sigma\hat\nabla\cdot\tilde u,\quad i=1,2,3.\end{aligned} \tag{3.26}$$

Multiplying the normal component of (3.26) by $\tilde\eta_{\sigma n}J$ and integrating over $\hat\Omega$ we obtain

$$\begin{aligned}&\frac12\frac{\mathrm d}{\mathrm dt}\int\limits_{\hat\Omega}\frac{p_{\hat\eta}}{\hat\eta}\tilde\eta^2_{\sigma n}J\mathrm dz+c_0\|\tilde\eta_{\sigma n}\|^2_{0,\hat\Omega}\le cd\|\tilde u_{nn}\|^2_{0,\hat\Omega}+\varepsilon\|\tilde\eta_{\sigma n}\|^2_{0,\hat\Omega}\\&+c\Big[\|\tilde u_{z\tau}\|^2_{0,\hat\Omega}+\|\hat u\|^2_{1,\hat\Omega}+\|\tilde u_t\|^2_{0,\hat\Omega}+\|\hat\eta_\sigma\|^2_{0,\hat\Omega}+\|\hat\vartheta_\sigma\|^2_{1,\hat\Omega}+(\bar X_1(\hat\Omega)\\&+\Big\|\int\limits_0^t\hat u\mathrm dt'\Big\|^2_{3,\Omega})\bar X_1(\hat\Omega)\Big],\end{aligned} \tag{3.27}$$

where d is from formulas (3.8) and (3.9).

Now, we write $(3.11)_1$ in the form

$$\hat\eta\tilde u_{it}-\mu\Delta\tilde u_i-\nu\nabla_{z_i}\nabla\cdot\tilde u=\hat\nabla_i\tilde p_\sigma+k_{4i}-k_{9i}, \tag{3.28}$$

where $k_{9i}=(\mu\Delta\tilde u_i+\nu\nabla_{z_i}\nabla\cdot\tilde u)-(\mu\hat\nabla^2\tilde u_i+\nu\hat\nabla_i\hat\nabla\cdot\tilde u)$. Multiplying the third component of (3.28) by $\tilde u_{3nn}J$ and integrating over $\hat\Omega$ yields

$$\begin{aligned}&\frac12\frac{\mathrm d}{\mathrm dt}\int\limits_{\hat\Omega}\hat\eta\tilde u^2_{3n}J\mathrm dz+c_0\|\tilde u_{3nn}\|^2_{0,\hat\Omega}\le(\varepsilon+cd)\|\tilde u_{nn}\|^2_{0,\hat\Omega}\\&+c\Big[\|\tilde u_{z\tau}\|^2_{0,\hat\Omega}+\|\hat u\|^2_{1,\hat\Omega}+\|\tilde u_t\|^2_{1,\hat\Omega}+\|\hat\eta_\sigma\|^2_{0,\hat\Omega}+\|\tilde\eta_{\sigma n}\|^2_{0,\hat\Omega}\\&+\|\hat\vartheta_\sigma\|^2_{1,\hat\Omega}+(\bar X_1(\hat\Omega)+\Big\|\int\limits_0^t\hat u\mathrm dt'\Big\|^2_{3,\hat\Omega})\bar X_1(\hat\Omega)\Big].\end{aligned} \tag{3.29}$$

In order to estimate $\tilde{\vartheta}_{\sigma nn}$ rewrite equation $(3.11)_3$ in the form

$$\hat{\eta} c_v \tilde{\vartheta}_{\sigma t} - \varkappa \Delta \tilde{\vartheta}_\sigma = \varkappa(\hat{\nabla}^2 \tilde{\vartheta}_\sigma - \Delta \tilde{\vartheta}_\sigma) - \hat{\vartheta} p_{\hat{\vartheta}}(\hat{\eta}, \hat{\vartheta}) \hat{\nabla} \cdot \tilde{u} + k_6. \tag{3.30}$$

Multiplying equation (3.30) by $\tilde{\vartheta}_{\sigma nn} J$, dividing by $\hat{\vartheta}$ and integrating over $\hat{\Omega}$ we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\hat{\Omega}} \frac{\hat{\eta} c_v}{\hat{\vartheta}} \tilde{\vartheta}^2_{\sigma n} J \mathrm{d}z + c_0 \|\tilde{\vartheta}_{\sigma nn}\|^2_{0,\hat{\Omega}} \le cd \|\tilde{\vartheta}_{\sigma nn}\|^2_{0,\hat{\Omega}} + c[\|\hat{u}\|^2_{1,\hat{\Omega}} + \|\hat{\vartheta}_{\sigma z\tau}\|^2_{0,\hat{\Omega}} \\
&+ \|\hat{\vartheta}_\sigma\|^2_{1,\hat{\Omega}} + \|\hat{\vartheta}_{\sigma t}\|^2_{1,\hat{\Omega}} + (\bar{X}_1(\hat{\Omega}) + \Big\| \int\limits_0^t \hat{u} \mathrm{d}t' \Big\|^2_{3,\hat{\Omega}}) \bar{X}_1(\hat{\Omega})].
\end{aligned} \tag{3.31}$$

For an interior subdomain $\tilde{\Omega}$ (which is one of $\tilde{\Omega}_i' s$, $i \in \mathcal{M}$) the following estimate is obtained in the same way as (3.25)

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\tilde{\Omega}} \left(\eta \tilde{u}_\xi^2 + \frac{p_\eta}{\eta} \tilde{\eta}^2_{\sigma\xi} + \frac{\eta c_v}{\vartheta} \tilde{\vartheta}^2_{\sigma\xi} \right) A \mathrm{d}\xi + + c_0(\|\tilde{u}\|^2_{2,\tilde{\Omega}} + \|\tilde{\vartheta}_{\sigma\xi\xi}\|^2_{0,\tilde{\Omega}} \\
&\le \varepsilon(\|\tilde{\eta}_{\sigma\xi}\|^2_{0,\tilde{\Omega}} + \|\tilde{u}_{\xi\xi}\|^2_{0,\tilde{\Omega}} + \|\tilde{\vartheta}_{\sigma\xi\xi}\|^2_{0,\tilde{\Omega}}) + c[\|u\|^2_{1,\tilde{\Omega}} + \|\vartheta_\sigma\|^2_{1,\tilde{\Omega}} \\
&+ \|\eta_\sigma\|^2_{0,\tilde{\Omega}} + \bar{X}_1(\tilde{\Omega})(\bar{X}_1(\tilde{\Omega}) + \Big\| \int\limits_0^t \hat{u} \mathrm{d}t' \Big\|^2_{3,\hat{\Omega}})(1 + \bar{X}_1(\tilde{\Omega}))],
\end{aligned} \tag{3.32}$$

where $\tilde{\Omega}$ is one of the interior subdomains $\tilde{\Omega}_i$, $i \in \mathcal{M}$ and

$$\bar{X}_1(\tilde{\Omega}) = |u|^2_{2,0,\tilde{\Omega}} + |\vartheta_\sigma|^2_{2,0,\tilde{\Omega}} + |\eta_\sigma|^2_{2,0,\tilde{\Omega}}. \tag{3.33}$$

Finally, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega} \eta u_\xi^2 A \mathrm{d}\xi \le c(\|u\|^2_{1,\Omega} + \|u_t\|^2_{1,\Omega}), \tag{3.34}$$

where we have used $(3.4)_2$.

Now, assume that ε is sufficiently small and multiply estimate (3.27) by a sufficiently large constant D_1. Adding the result to inequality (3.29) we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\hat{\Omega}} \left(D_1 \frac{p_{\hat{\eta}}}{\hat{\eta}} \tilde{\eta}^2_{\sigma n} + \hat{\eta} \tilde{u}^2_{3n} \right) J \mathrm{d}z + c_0(\|\tilde{\eta}_{\sigma n}\|^2_{0,\hat{\Omega}} + \|\tilde{u}_{3nn}\|^2_{0,\hat{\Omega}}) \\
&\le cd \|\tilde{u}_{nn}\|^2_{0,\hat{\Omega}} + c\Big[\|\tilde{u}_{z\tau}\|^2_{0,\hat{\Omega}} + \|\hat{u}\|^2_{1,\hat{\Omega}} + \|\tilde{u}_t\|^2_{1,\hat{\Omega}} + \|\hat{\eta}_\sigma\|^2_{0,\hat{\Omega}} + \|\hat{\vartheta}_\sigma\|^2_{1,\hat{\Omega}} \\
&+ (\bar{X}_1(\hat{\Omega}) + \Big\| \int\limits_0^t \hat{u} \mathrm{d}t' \Big\|^2_{3,\hat{\Omega}}) \bar{X}_1(\hat{\Omega}) \Big].
\end{aligned} \tag{3.35}$$

Next, multiply inequality (3.25) by a sufficiently large constant D_2. Adding the obtained result to (3.35) and (3.31) we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\hat{\Omega}}\Big[D_2\left(\hat{\eta}\tilde{u}_\tau^2+\frac{p_{\hat{\eta}}}{\hat{\eta}}\tilde{\eta}_{\sigma\tau}^2+\frac{\hat{\eta}c_v}{\hat{\vartheta}}\tilde{\vartheta}_{\sigma\tau}^2\right)+D_1\frac{p_{\sigma\hat{\eta}}}{\tilde{\eta}}\tilde{\eta}_{\sigma n}^2+\hat{\eta}\tilde{u}_{3n}^2\Big]J\mathrm{d}z$$
$$+c_0(\|\tilde{u}_\tau\|_{1,\hat{\Omega}}^2+\|\tilde{\vartheta}_{\sigma zz}\|_{0,\hat{\Omega}}^2+\|\tilde{\eta}_{\sigma n}\|_{0,\hat{\Omega}}^2+\|\tilde{u}_{3nn}\|_{0,\hat{\Omega}}^2)$$
$$\leq\varepsilon\|\hat{\eta}_{\sigma z}\|_{0,\hat{\Omega}}^2+cd\|\tilde{u}_{nn}\|_{0,\hat{\Omega}}^2+c\Big[\|\hat{u}\|_{1,\hat{\Omega}}^2$$
$$+\|\hat{\vartheta}_\sigma\|_{1,\hat{\Omega}}^2+\|\hat{\eta}_\sigma\|_{0,\hat{\Omega}}^2+\|\tilde{u}_t\|_{1,\hat{\Omega}}^2+\|\hat{\vartheta}_{\sigma t}\|_{1,\hat{\Omega}}^2$$
$$+\bar{X}_1(\hat{\Omega})(\bar{X}_1(\hat{\Omega})+\Big\|\int\limits_0^t\hat{u}\mathrm{d}t'\Big\|_{3,\hat{\Omega}}^2)(1+\bar{X}_1(\hat{\Omega}))\Big].\tag{3.36}$$

Finally, summing up (3.36) over all $i\in\mathcal{N}$,adding the result to (3.34), (3.32) (for all $i\in\mathcal{M}$) and (3.16) multiplied by a sufficiently small constant ε_1, and then returning to variables x we obtain estimate (3.12).

This completes the proof. □

Next, we have

Lemma 3.2. *Let* $p\in C^2(\mathbb{R}_+^1\times\mathbb{R}_+^1)$*,* $c_v\in C^2(\mathbb{R}_+^1\times\mathbb{R}_+^1)$ *and assume* (1.2)*,* (3.2)*,* (3.3)*. Let* (v,θ,ϱ) *be a solution of* (1.1)*. Then*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\phi_2(t,\Omega)+\bar{C}_2(\|v_t\|_{2,\Omega_t}^2+\|\varrho_{\sigma t}\|_{1,\Omega_t}^2+\|\theta_{\sigma txx}\|_{0,\Omega_t}^2)$$
$$\leq C_3(\|v\|_{1,\Omega_t}^2+\|v_t\|_{1,\Omega_t}^2+\|v_{tt}\|_{1,\Omega_t}^2+\|\theta_\sigma\|_{2,\Omega_t}^2+\|\theta_{\sigma t}\|_{1,\Omega_t}^2+\|\theta_{\sigma tt}\|_{1,\Omega_t}^2$$
$$+\|\varrho_{\sigma t}\|_{0,\Omega_t}^2+\|\varrho_\sigma\|_{0,\Omega_t}^2)$$
$$+C_4\bar{X}_1\bar{X}_2(1+\bar{X}_1^2)+C_5\bar{X}_2\Big\|\int\limits_0^t v\mathrm{d}t'\Big\|_{3,\Omega_t}^2,\tag{3.37}$$

where

$$\phi_2(t,\Omega)=\sum_{i\in\mathcal{M}}\int\limits_{\tilde{\Omega}_i}\left(\eta\tilde{u}_{t\xi}^2+\frac{p_\eta}{\eta}\tilde{\eta}_{\sigma t\xi}^2+\frac{\eta c_v}{\vartheta}\tilde{\vartheta}_{\sigma t\xi}^2\right)A\mathrm{d}\xi$$
$$+\sum_{i\in\mathcal{N}}\int\limits_{\hat{\Omega}_i}\Big[D_4\left(\hat{\eta}\tilde{u}_{t\tau}^2+\frac{p_{\hat{\eta}}}{\tilde{\eta}}\tilde{\eta}_{\sigma t\tau}^2+\frac{\hat{\eta}c_v}{\hat{\vartheta}}\tilde{\vartheta}_{\sigma t\tau}^2\right)+$$
$$+D_3\frac{p_{\sigma\hat{\eta}}}{\hat{\eta}}\tilde{\eta}_{\sigma nt}^2+\hat{\eta}\tilde{u}_{3nt}^2+\frac{\hat{\eta}c_v}{\hat{\vartheta}}\tilde{\vartheta}_{\sigma nt}^2\Big]J\mathrm{d}z+\int\limits_\Omega\eta u_{t\xi}^2A\mathrm{d}\xi,\tag{3.38}$$

$$\bar{X}_2=|v|_{3,1,\Omega_t}^2+|\theta_\sigma|_{3,1,\Omega_t}^2+\|\varrho_\sigma\|_{2,\Omega_t}^2+\|\varrho_{\sigma t}\|_{2,\Omega_t}^2+\|\varrho_{\sigma tt}\|_{1,\Omega_t}^2,\tag{3.39}$$

D_3, D_4 are constants depending on the same quantities as $\bar C_2, C_3, C_4, C_5$; $\bar X_1$ is given by (3.14).

Proof. Differentiating problem (3.15) with respect to t we get the following elliptic problem

$$\begin{aligned}
&\mu\nabla_u^2 u_t + \nu\nabla_u\nabla_u\cdot u_t - p_{\sigma\eta}\nabla_u\eta_{\sigma t}\\
&= \eta_{\sigma t}u_t + \eta u_{tt} + p_{\sigma\vartheta\eta}\eta_{\sigma t}\nabla_u\vartheta_\sigma + p_{\sigma\vartheta\vartheta}\vartheta_{\sigma t}\nabla_u\vartheta_\sigma\\
&+ p_{\sigma\vartheta}(\nabla_u)_{,t}\vartheta_\sigma + p_{\sigma\vartheta}\nabla_u\vartheta_{\sigma t} + p_{\sigma\eta\eta}\eta_{\sigma t}\nabla_u\eta_\sigma\\
&+ p_{\sigma\eta\vartheta}\vartheta_{\sigma t}\nabla_u\eta_\sigma + p_{\sigma\eta}(\nabla_u)_{,t}\eta_\sigma - \nu(\nabla_u\nabla_u)_{,t}\cdot u - \mu(\nabla_u^2)_{,t}u \equiv K_1 \quad \text{in } \Omega,\\
&\operatorname{div}_u u_t = \operatorname{div}_u u_t \quad \text{in } \Omega,\\
&\mathbb{T}_u(u_t, p_{\sigma t})\bar n_u = -(\bar n_u\mathbb{T}_u)_{,t}(u, p_\sigma) \equiv K_2 \quad \text{on } S.
\end{aligned}$$

By the Agmon-Douglis-Nirenberg theory (see [1]) we have the estimate

$$\begin{aligned}
&\|u_t\|_{2,\Omega}^2 + \|\eta_{\sigma t}\|_{1,\Omega}^2 \qquad (3.40)\\
&\le c(\|K_1\|_{0,\Omega}^2 + \|K_2\|_{1/2,S}^2 + \|p_{\sigma\vartheta}\vartheta_{\sigma t}\bar n_u\|_{1/2,S}^2 + \|\operatorname{div} u_t\|_{1,\Omega}^2),
\end{aligned}$$

where

$$\begin{aligned}
&\|K_1\|_{0,\Omega}^2 + \|K_2\|_{1/2,S}^2 + \|p_{\sigma\vartheta}\vartheta_{\sigma t}\bar n_u\|_{1/2,S}^2 \le c(\|\eta_\sigma\|_{1,\Omega}^2 + \|\vartheta_\sigma\|_{1,\Omega}^2\\
&+ \|\vartheta_{\sigma t}\|_{1,\Omega}^2 + \|u_{tt}\|_{0,\Omega}^2) + c\bar X_1(\Omega)\Big(\bar X_2(\Omega) + \Big\|\int_0^t u\,\mathrm{d}t'\Big\|_{3,\Omega}^2\Big),\\
&\bar X_1(\Omega) = |u|_{2,0,\Omega}^2 + |\eta_\sigma|_{2,0,\Omega}^2 + |\vartheta_\sigma|_{2,0,\Omega}^2,\\
&\bar X_2(\Omega) = |u|_{3,1,\Omega}^2 + |\vartheta_\sigma|_{3,1,\Omega}^2 + \|\eta_\sigma\|_{2,\Omega}^2 + \|\eta_{\sigma t}\|_{2,\Omega}^2 + \|\eta_{\sigma tt}\|_{1,\Omega}^2.
\end{aligned}$$

In view of (3.40) it remains to obtain estimates for $\|\operatorname{div} u_t\|_{1,\Omega}^2$ and for

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\phi_2(t,\Omega) + \bar C_2\|\theta_{\sigma txx}\|_{0,\Omega_t}^2.$$

Consider first boundary subdomains $\hat\Omega_i$, $i \in \mathcal{N}$. Differentiate $(3.11)_1$ with respect to t and τ, multiply the result by $\tilde u_{t\tau}J$ and integrate over $\hat\Omega$ (which is one of $\hat\Omega_i's$). Next, divide $(3.11)_3$ by $\hat\vartheta$, differentiate the result by t and τ, multiply by $\tilde\vartheta_{\sigma t\tau}J$ and integrate over $\hat\Omega$. Using the Korn inequality and equation $(3.11)_2$ we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\hat\Omega}\Big(\hat\eta\tilde u_{t\tau}^2 + \frac{\hat\eta c_v}{\hat\vartheta}\tilde\vartheta_{\sigma t\tau}^2\Big)J\mathrm{d}z + c_0(\|\tilde u_{t\tau}\|_{1,\hat\Omega}^2 + \|\tilde\vartheta_{\sigma t\tau z}\|_{0,\hat\Omega}^2)\\
&- \int_{\hat S}(\hat{\mathbb{T}}(\tilde u, \tilde p_\sigma)\hat n)_{,t\tau}\tilde u_{t\tau}J\mathrm{d}z_{\hat S} - \varkappa\int_{\hat S}\frac{1}{\hat\vartheta}(\hat n\cdot\hat\nabla\hat\vartheta_\sigma)_{,t\tau}\tilde\vartheta_{\sigma t\tau}J\mathrm{d}z_{\hat S}
\end{aligned}$$

$$
\begin{aligned}
&-\int_{\hat\Omega}\tilde p_{\sigma t\tau}\hat\nabla\cdot\tilde u_{t\tau}J\mathrm{d}z+\int_{\hat\Omega}\tilde p_{\sigma\hat\vartheta}\tilde\vartheta_{\sigma t\tau}\hat\nabla\cdot\tilde u_{t\tau}J\mathrm{d}z\\
&\le\varepsilon(\|\tilde u_{t\tau}\|^2_{1,\hat\Omega}+\|\hat\eta_{\sigma t\tau}\|^2_{0,\hat\Omega}+\|\hat\vartheta_{\sigma t\tau z}\|^2_{0,\hat\Omega})\\
&+c\Big[\|\hat u_t\|^2_{1,\hat\Omega}+\|\hat u\|^2_{2,\hat\Omega}+\|\hat\eta_\sigma\|^2_{1,\hat\Omega}+\|\hat\vartheta_\sigma\|^2_{2,\hat\Omega}+\|\hat\vartheta_{\sigma t}\|^2_{1,\hat\Omega}+\\
&\bar X_1(\hat\Omega)\bar X_2(\hat\Omega)(1+\bar X_1^2(\hat\Omega))+\bar X_2(\hat\Omega)\Big\|\int_0^t\hat u\mathrm{d}t'\Big\|^2_{3,\hat\Omega}\Big],
\end{aligned}\tag{3.41}
$$

where $\bar X_1(\hat\Omega)$ is defined by (3.18) and

$$
\bar X_2(\hat\Omega)\;=\;|\hat u|^2_{3,1,\hat\Omega}+|\hat\vartheta_\sigma|^2_{3,1,\hat\Omega}+\|\hat\eta_\sigma\|^2_{2,\hat\Omega}+\|\hat\eta_{\sigma t}\|^2_{2,\hat\Omega}+\|\hat\eta_{\sigma tt}\|^2_{1,\hat\Omega}.\tag{3.42}
$$

Using boundary conditions $(3.11)_4$–$(3.11)_5$ we obtain

$$
\begin{aligned}
&-\int_{\hat S}(\hat{\mathbb T}(\tilde u,\tilde p_\sigma)\hat n)_{,t\tau}\tilde u_{t\tau}J\mathrm{d}z_{\hat S}=-\int_{\hat S}(\hat B_{ij}(\hat u,\hat\zeta)\hat n_j)_{,t\tau}\tilde u_{it\tau}J\mathrm{d}z_{\hat S}\\
&=\bar c\int_{\hat S}\partial_\tau^{1/2}((\hat B_{ij}(\hat u,\hat\zeta)\hat n_j)_{,t})\partial_\tau^{1/2}(\tilde u_{it\tau}J)\mathrm{d}z_{\hat S}\\
&\le\varepsilon\|\tilde u_{t\tau}\|^2_{1,\hat\Omega}+c\Big[\|\hat u_t\|^2_{1,\hat\Omega}+\bar X_2(\hat\Omega)(\bar X_1(\hat\Omega)+\Big\|\int_0^t\hat u\mathrm{d}t'\Big\|^2_{3,\hat\Omega})\Big]
\end{aligned}\tag{3.43}
$$

and

$$
\begin{aligned}
&-\int_{\hat S}\frac{1}{\hat\vartheta}(\hat n\cdot\hat\nabla\hat\vartheta_\sigma)_{,t\tau}\tilde\vartheta_{\sigma t\tau}J\mathrm{d}z_{\hat S}\\
&=-\int_{\hat S}\frac{1}{\hat\vartheta}(\hat n\cdot\hat\nabla\hat\zeta\hat\vartheta_\sigma)_{,t\tau}\tilde\vartheta_{\sigma t\tau}J\mathrm{d}z_{\hat S}+\varkappa_a\int_{\hat S}\frac{1}{\hat\vartheta}\tilde\vartheta^2_{\sigma t\tau}Jdx\\
&\equiv I+\varkappa_a\int_{\hat S}\frac{1}{\hat\vartheta}\tilde\vartheta^2_{\sigma t\tau}Jds,
\end{aligned}\tag{3.44}
$$

$$
\begin{aligned}
I&=\bar c\int_{\hat S}\partial_\tau^{1/2}((\hat n\cdot\hat\nabla\hat\zeta\hat\vartheta_\sigma)_{,t})\cdot\partial_\tau^{1/2}\left(\frac{1}{\hat\vartheta}\tilde\vartheta_{\sigma t\tau}J\right)\mathrm{d}z_{\hat S}\\
&\le\varepsilon\|\tilde\vartheta_{\sigma t\tau}\|^2_{1,\hat\Omega}+c\Big[\|\hat\vartheta_{\sigma t}\|^2_{1,\hat\Omega}+\bar X_2(\hat\Omega)(\bar X_1(\hat\Omega)+\Big\|\int_0^t\hat u\mathrm{d}t'\Big\|^2_{3,\hat\Omega})\Big].
\end{aligned}
$$

Moreover, we have

$$-\int_{\hat\Omega}\tilde p_{\sigma t\tau}\hat\nabla\cdot\tilde u_{t\tau}J\mathrm{d}z=-\int_{\hat\Omega}p_{\hat\eta}\tilde\eta_{\sigma t\tau}\hat\nabla\cdot\tilde u_{t\tau}J\mathrm{d}z-\int_{\hat\Omega}p_{\hat\vartheta}\tilde\vartheta_{\sigma t\tau}\hat\nabla\cdot\tilde u_{t\tau}J\mathrm{d}z+I_1,\tag{3.45}$$

where

$$|I_1|\le\varepsilon\|\tilde u_{t\tau}\|^2_{1,\hat\Omega}+c\Big[\|\hat\eta_\sigma\|^2_{0,\hat\Omega}+\|\hat\vartheta_\sigma\|^2_{0,\hat\Omega}+\bar X_1^2(\hat\Omega)\Big]\tag{3.46}$$

and

$$-\int_{\hat\Omega}p_{\sigma\hat\eta}\tilde\eta_{\sigma t\tau}\hat\nabla\cdot\tilde u_{t\tau}J\mathrm{d}z=\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\hat\Omega}\frac{p_{\sigma\hat\eta}}{\hat\eta}\tilde\eta^2_{\sigma t\tau}J\mathrm{d}z+I_2,\tag{3.47}$$

where

$$|I_2|\le\varepsilon\|\tilde\eta_{\sigma t\tau}\|^2_{0,\hat\Omega}+c\Big[\|\hat u\|^2_{1,\hat\Omega}+\|\hat u_t\|^2_{1,\hat\Omega}+\bar X_1(\hat\Omega)\bar X_2(\hat\Omega)(1+\bar X_1^2(\hat\Omega))+\bar X_2(\hat\Omega)\Big\|\int_0^t\hat u\mathrm{d}t'\Big\|^2_{3,\hat\Omega}\Big].\tag{3.48}$$

Summarizing inequalities (3.41), (3.43)–(3.48) and assuming that ε is sufficiently small we get

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\hat\Omega}\left(\hat\eta\tilde u^2_{t\tau}+\frac{p_{\hat\eta}}{\hat\eta}\tilde\eta^2_{\sigma t\tau}+\frac{\hat\eta c_v}{\hat\vartheta}\tilde\vartheta^2_{\sigma t\tau}\right)J\mathrm{d}z\\&+c_0(\|\tilde u_{t\tau}\|^2_{1,\hat\Omega}+\|\tilde\vartheta_{\sigma t\tau z}\|^2_{0,\hat\Omega}+\|\tilde\vartheta_{\sigma t\tau}\|^2_{0,\hat S})\\&\le\varepsilon\|\hat\eta_{\sigma t\tau}\|^2_{0,\hat\Omega}+c\Big[\|\hat u_t\|^2_{1,\hat\Omega}+\|\hat u\|^2_{2,\hat\Omega}+\|\hat\eta_\sigma\|^2_{1,\hat\Omega}+\|\hat\vartheta_{\sigma t}\|^2_{1,\hat\Omega}+\|\hat\vartheta_\sigma\|^2_{2,\hat\Omega}\\&+\bar X_1(\hat\Omega)\bar X_2(\hat\Omega)(1+\bar X_1^2(\hat\Omega))+\bar X_2(\hat\Omega)\Big\|\int_0^t\hat u\mathrm{d}t'\Big\|^2_{3,\hat\Omega}\Big].\end{aligned}\tag{3.49}$$

Next, differentiate the third component of (3.26) with respect to t, multiply the result by $\tilde\eta_{\sigma tn}J$ and integrate over $\hat\Omega$. We obtain

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\hat\Omega}\frac{p_{\hat\eta}}{\hat\eta}\tilde\eta^2_{\sigma tn}J\mathrm{d}z+c_0\|\tilde\eta_{\sigma tn}\|^2_{0,\hat\Omega}\\&\le\varepsilon\|\tilde\eta_{\sigma tn}\|^2_{0,\hat\Omega}+cd\|\tilde u_{nnt}\|^2_{0,\hat\Omega}+c\Big[\|\hat\eta_{\sigma t}\|^2_{0,\hat\Omega}+\|\hat\eta_\sigma\|^2_{0,\hat\Omega}\\&+\|\hat\vartheta_{\sigma t}\|^2_{1,\hat\Omega}+\|\hat\vartheta_\sigma\|^2_{0,\hat\Omega}+\|\hat u\|^2_{2,\hat\Omega}+\|\hat u_t\|^2_{1,\hat\Omega}\end{aligned}$$

$$+ \|\tilde{u}_{tt}\|^2_{0,\hat{\Omega}} + \|\tilde{u}_{t\tau z}\|^2_{0,\hat{\Omega}}$$

$$+ \bar{X}_1(\hat{\Omega})\bar{X}_2(\hat{\Omega})(1+\bar{X}_1(\hat{\Omega})) + \bar{X}_2(\hat{\Omega})\bigg\|\int\limits_0^t \hat{u}\mathrm{d}t'\bigg\|^2_{3,\hat{\Omega}}\bigg], \tag{3.50}$$

where d is from formulas (3.8) and (3.9).

Now, differentiating the third component of (3.28) with respect to t we obtain the equation

$$\hat{\eta}\tilde{u}_{3tt} - \mu\Delta\tilde{u}_{3t} - \nabla_{z_3}\nabla\cdot\tilde{u}_t = -\hat{\eta}_{\sigma t}\tilde{u}_{3t} + \hat{\nabla}_3\tilde{p}_{\sigma t} + k_{43t} - k_{93t}. \tag{3.51}$$

Multiplying (3.51) by $\tilde{u}_{3nnt}J$ and integrating over $\hat{\Omega}$ gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\hat{\Omega}} \hat{\eta}\tilde{u}^2_{3nt}J\mathrm{d}z + c_0\|\tilde{u}_{3nnt}\|^2_{0,\hat{\Omega}} \le (\varepsilon + cd)\|\tilde{u}_{nnt}\|^2_{0,\hat{\Omega}}$$

$$+ c\Big[\|\hat{\eta}_{\sigma t}\|^2_{0,\hat{\Omega}} + \|\tilde{\eta}_{\sigma tn}\|^2_{0,\hat{\Omega}} + \|\hat{\vartheta}_{\sigma t}\|^2_{0,\hat{\Omega}} + \|\tilde{u}_{tt}\|^2_{1,\hat{\Omega}} + \|\tilde{u}_t\|^2_{1,\hat{\Omega}} + \|\tilde{u}_{t\tau z}\|^2_{0,\hat{\Omega}}$$

$$+ \bar{X}_1(\hat{\Omega})(\bar{X}_2(\hat{\Omega}) + \bar{X}_1^2(\hat{\Omega})) + \bar{X}_2(\hat{\Omega})\bigg\|\int\limits_0^t \hat{u}\mathrm{d}t'\bigg\|^2_{3,\hat{\Omega}}\bigg]. \tag{3.52}$$

Now, we estimate $\tilde{\vartheta}_{\sigma nnt}$. To do this differentiate (3.30) with respect to t, multiply by $\tilde{\vartheta}_{\sigma nnt}J$, divide by $\hat{\vartheta}$ and integrate over $\hat{\Omega}$. We get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\hat{\Omega}} \frac{\hat{\eta}c_v}{\hat{\vartheta}}\tilde{\vartheta}^2_{\sigma nt}J\mathrm{d}z + c_0\|\tilde{\vartheta}_{\sigma nnt}\|^2_{0,\hat{\Omega}}$$

$$\le (\varepsilon + cd)\|\tilde{\vartheta}_{\sigma nnt}\|^2_{0,\hat{\Omega}} + c\Big[\|\hat{\vartheta}_{\sigma t\tau z}\|^2_{0,\hat{\Omega}} + \|\hat{\vartheta}_{\sigma tt}\|^2_{1,\hat{\Omega}} + \|\hat{\vartheta}_{\sigma t}\|^2_{1,\hat{\Omega}} + \|\hat{\vartheta}_{\sigma}\|^2_{1,\hat{\Omega}}$$

$$+ \bar{X}_1(\hat{\Omega})\bar{X}_2(\hat{\Omega})(1+\bar{X}_1(\hat{\Omega})) + \bar{X}_2(\hat{\Omega})\bigg\|\int\limits_0^t \hat{u}\mathrm{d}t'\bigg\|^2_{3,\hat{\Omega}}\bigg]. \tag{3.53}$$

For an interior subdomain the following estimate holds

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\tilde{\Omega}} \left(\eta\tilde{u}^2_{t\xi} + \frac{p_\eta}{\eta}\tilde{\eta}^2_{\sigma t\xi} + \frac{\eta c_v}{\vartheta}\tilde{\vartheta}^2_{\sigma t\xi}\right) A\mathrm{d}\xi + c_0(\|\tilde{u}_t\|^2_{2,\tilde{\Omega}} + \|\tilde{\vartheta}_{\sigma t\xi\xi}\|^2_{0,\tilde{\Omega}})$$

$$\le \varepsilon(\|\tilde{u}_{t\xi\xi}\|^2_{0,\tilde{\Omega}} + \|\tilde{\vartheta}_{\sigma t\xi\xi}\|^2_{0,\tilde{\Omega}} + \|\tilde{\eta}_{\sigma t\xi}\|^2_{0,\tilde{\Omega}}) + c\Big[\|u_t\|^2_{1,\tilde{\Omega}} + \|u\|^2_{2,\tilde{\Omega}}$$

$$+ \|\eta_\sigma\|^2_{1,\tilde{\Omega}} + \|\vartheta_\sigma\|^2_{2,\tilde{\Omega}} + \|\vartheta_{\sigma t}\|^2_{1,\tilde{\Omega}} + \bar{X}_1(\tilde{\Omega})$$

$$\bar{X}_2(\tilde{\Omega})(1+\bar{X}_1^2(\tilde{\Omega})) + \bar{X}_2(\tilde{\Omega})\bigg\|\int\limits_0^t \hat{u}\mathrm{d}t'\bigg\|^2_{3,\hat{\Omega}}\bigg], \tag{3.54}$$

where $\bar{X}_1(\tilde{\Omega})$ is defined by (3.33) and

$$\bar{X}_2(\tilde{\Omega}) = |u|^2_{3,1,\tilde{\Omega}} + |\vartheta_\sigma|^2_{3,1,\tilde{\Omega}} + \|\eta_\sigma\|^2_{2,\tilde{\Omega}} + \|\eta_{\sigma t}\|^2_{2,\tilde{\Omega}} + \|\eta_{\sigma tt}\|^2_{1,\tilde{\Omega}}. \tag{3.55}$$

Finally, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega \eta u^2_{t\xi} A\mathrm{d}\xi \le \varepsilon\|u_{tt}\|^2_{1,\Omega} + c\|\eta_{\sigma t}\|^2_{1,\Omega}\|u_t\|^2_{2,\Omega} + c\|u_t\|^2_{1,\Omega}. \tag{3.56}$$

Taking into account estimates (3.40), (3.49), (3.50), (3.52)–(3.54), (3.56) and repeating the argument from the last part of the proof of Lemma 3.1 we get (3.37).

This completes the proof. □

Finally, we prove

Lemma 3.3. *Let* $p \in C^2(\mathbb{R}^1_+ \times \mathbb{R}^1_+)$, $c_v \in C^2(\mathbb{R}^1_+ \times \mathbb{R}^1_+)$ *and assume* (1.2), (3.2), (3.3). *Let* (v, θ, ϱ) *be a solution of* (1.1). *Then*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\phi_3(t,\Omega) + \bar{C}_3(\|v\|^2_{3,\Omega_t} + \|\varrho_\sigma\|^2_{2,\Omega_t} + \|\theta_{\sigma xxx}\|^2_{0,\Omega_t})\\ &\le C_6(\|v\|^2_{2,\Omega_t} + \|v_t\|^2_{2,\Omega_t} + \|\theta_\sigma\|^2_{2,\Omega_t} + \|\theta_{\sigma t}\|^2_{2,\Omega_t} + \|\varrho_\sigma\|^2_{1,\Omega_t})\\ &\quad + C_7\bar{X}_1\bar{X}_2(1+\bar{X}_1^2) + C_8\bar{X}_2\Big\|\int\limits_0^t v\mathrm{d}t'\Big\|^2_{3,\Omega_t},\end{aligned} \tag{3.57}$$

where

$$\begin{aligned}\phi_3(t,\Omega) &= \sum_{i\in\mathcal{M}}\int\limits_{\tilde{\Omega}_i}\left(\eta\tilde{u}^2_{\xi\xi} + \frac{p_\eta}{\eta}\tilde{\eta}^2_{\sigma\xi\xi} + \frac{\eta c_v}{\vartheta}\tilde{\vartheta}^2_{\sigma\xi\xi}\right)A\mathrm{d}\xi\\ &+ \sum_{i\in\mathcal{N}}\int\limits_{\hat{\Omega}_i}\Big[D_7\left(\hat{\eta}\tilde{u}^2_{\tau\tau} + \frac{p_{\hat{\eta}}}{\hat{\eta}}\tilde{\eta}^2_{\sigma\tau\tau} + \frac{\hat{\eta}c_v}{\hat{\vartheta}}\tilde{\vartheta}^2_{\sigma\tau\tau}\right)\\ &+ D_6\left(D_5\frac{p_{\hat{\eta}}}{\hat{\eta}}\tilde{\eta}^2_{\sigma n\tau} + \hat{\eta}\tilde{u}^2_{3n\tau}\right) + \frac{p_{\hat{\eta}}}{\hat{\eta}}\tilde{\eta}^2_{\sigma nn} + \frac{\hat{\eta}c_v}{\hat{\vartheta}}(\tilde{\vartheta}^2_{\sigma n\tau} + \tilde{\vartheta}^2_{\sigma nn})\Big]J\mathrm{d}z\\ &+ \int\limits_\Omega \eta u^2_{\xi\xi}A\mathrm{d}\xi;\end{aligned} \tag{3.58}$$

D_i $(i = 5,6,7)$ *are constants depending on the same quantities as* $\bar{C}_3$ *and* C_i $(i = 6,7,8)$, $d^2_{\xi\xi} = \sum_{i,j,k=1}^3 d^2_{i\xi_j\xi_k}$, $d \in \{\tilde{u}, u\}$, $g^2_{\xi\xi} = \sum_{j,k=1}^3 g^2_{\xi_j\xi_k}$, $g \in \{\tilde{\eta}_\sigma, \tilde{\vartheta}_\sigma\}$, $h^2_{\tau\tau} = \sum_{j,k=1}^2 h^2_{\tau_j\tau_k}$, $h \in \{\tilde{\eta}_\sigma, \tilde{\vartheta}_\sigma\}$, $\tilde{u}^2_{\tau\tau} = \sum_{i=1}^3\sum_{j,k=1}^2 \tilde{u}^2_{i\tau_j\tau_k}$; $\bar{X}_1$ *and* $\bar{X}_2$ *are given by* (3.14) *and* (3.39), *respectively.*

Proof. First we consider problem (3.15). By the Agmon-Douglis-Nirenberg theory (see [1]) we have

$$\|u\|^2_{3,\Omega}+\|\eta_\sigma\|^2_{2,\Omega}\le c\Big[\|u_t\|^2_{1,\Omega}+\|\vartheta_\sigma\|^2_{2,\Omega}+\|\operatorname{div}u\|^2_{2,\Omega}$$
$$+\|\eta_\sigma\|^2_{2,\Omega}\|u_t\|^2_{1,\Omega}+\|\vartheta_\sigma\|^2_{2,\Omega}\Big\|\int_0^t u\mathrm{d}t'\Big\|^2_{3,\Omega}\Big]. \tag{3.59}$$

Thus, to obtain (3.57) we have to estimate $\|\operatorname{div}u\|^2_{2,\Omega}$ and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\phi_3(t,\Omega)+\bar C_3\|\theta_{\sigma xxx}\|^2_{0,\Omega_t}.$$

To do this consider first boundary subdomains $\hat\Omega_i$, $i\in\mathcal{N}$. Differentiate $(3.11)_1$ twice with respect to τ, multiply the result by $\tilde u_{\tau\tau}J$ and integrate over $\hat\Omega$. Next, divide $(3.11)_3$ by $\hat\vartheta$, differentiate the result twice with respect to τ, multiply by $\tilde\vartheta_{\sigma\tau\tau}J$ and integrate over $\hat\Omega$ (which is one of $\hat\Omega_i's$). Using the Korn inequality, continuity equation $(3.11)_2$ and boundary conditions $(3.11)_4$–$(3.11)_5$ we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\hat\Omega}\left(\hat\eta\tilde u^2_{\tau\tau}+\frac{p_{\hat\eta}}{\hat\eta}\tilde\eta^2_{\sigma\tau\tau}+\frac{\hat\eta c_v}{\hat\vartheta}\tilde\vartheta^2_{\sigma\tau\tau}\right)J\mathrm{d}z$$
$$+c_0(\|\tilde u_{\tau\tau}\|^2_{1,\hat\Omega}+\|\tilde\vartheta_{\sigma\tau\tau z}\|^2_{0,\hat\Omega}+\varkappa_a\|\tilde\vartheta_{\sigma\tau\tau}\|^2_{0,\hat S})$$
$$\le\varepsilon(\|\hat\eta_{\sigma\tau\tau}\|^2_{0,\hat\Omega}+\|\tilde u_{\tau\tau}\|^2_{1,\hat\Omega}+\|\tilde\vartheta_{\sigma\tau\tau z}\|^2_{0,\hat\Omega})+c\Big[\|\hat u\|^2_{2,\hat\Omega}+\|\hat\vartheta_\sigma\|^2_{2,\hat\Omega}+\|\hat\eta_\sigma\|^2_{1,\hat\Omega}$$
$$+\bar X_1(\hat\Omega)\bar X_2(\hat\Omega)(1+\bar X_1^2(\hat\Omega))+\bar X_2(\hat\Omega)\Big\|\int_0^t\hat u\mathrm{d}t'\Big\|^2_{3,\hat\Omega}\Big], \tag{3.60}$$

where $\bar X_1(\hat\Omega)$ and $\bar X_2(\hat\Omega)$ are given by (3.18) and (3.42) respectively.

In the same way we obtain the following estimate in an interior subdomain $\tilde\Omega$ (which is one of $\tilde\Omega_i's$)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\tilde\Omega}\left(\eta\tilde u^2_{\xi\xi}+\frac{p_\eta}{\eta}\tilde\eta^2_{\sigma\xi\xi}+\frac{\eta c_v}{\vartheta}\tilde\vartheta^2_{\sigma\xi\xi}\right)A\mathrm{d}\xi+c_0(\|\tilde u\|^2_{3,\tilde\Omega}+\|\tilde\vartheta_\sigma\|^2_{3,\tilde\Omega})$$
$$\le\varepsilon(\|\tilde\eta_{\sigma\xi\xi}\|^2_{0,\tilde\Omega}+\|\tilde u_{\xi\xi\xi}\|^2_{0,\tilde\Omega}+\|\tilde\vartheta_{\sigma\xi\xi\xi}\|^2_{0,\tilde\Omega})+c\Big[\|u\|^2_{2,\tilde\Omega}+\|\vartheta_\sigma\|^2_{2,\tilde\Omega}+\|\eta_\sigma\|^2_{1,\tilde\Omega}$$
$$+\bar X_1(\tilde\Omega)\bar X_2(\tilde\Omega)(1+\bar X_1^2(\tilde\Omega))+\bar X_2(\tilde\Omega)\Big\|\int_0^t u\mathrm{d}t'\Big\|^2_{3,\tilde\Omega}\Big], \tag{3.61}$$

where $\bar X_1(\tilde\Omega)$ and $\bar X_2(\tilde\Omega)$ are given by (3.33) and (3.55), respectively.

Now, differentiate the third component of (3.26) with respect to τ, multiply the result by $\tilde{\eta}_{\sigma n\tau}J$ and integrate over $\hat{\Omega}$. We get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\hat{\Omega}}\frac{p_{\hat{\eta}}}{\hat{\eta}}\tilde{\eta}^2_{\sigma n\tau}J\mathrm{d}z + c_0\|\tilde{\eta}_{\sigma n\tau}\|^2_{0,\hat{\Omega}} \le \varepsilon\|\tilde{\eta}_{\sigma n\tau}\|^2_{0,\hat{\Omega}}\\
&+ cd\|\hat{u}\|^2_{3,\hat{\Omega}} + c\Big[\|\hat{u}\|^2_{2,\hat{\Omega}} + \|\hat{u}_t\|^2_{1,\hat{\Omega}} + \|\hat{\vartheta}_\sigma\|^2_{2,\hat{\Omega}} + \|\hat{\eta}_\sigma\|^2_{1,\hat{\Omega}} + \|\tilde{u}_{z\tau\tau}\|^2_{0,\hat{\Omega}}\\
&+ \bar{X}_1(\hat{\Omega})(\bar{X}_2(\hat{\Omega}) + \bar{X}_1^2(\hat{\Omega})) + \bar{X}_2(\hat{\Omega})\Big\|\int\limits_0^t\hat{u}\mathrm{d}t'\Big\|^2_{3,\hat{\Omega}}\Big]
\end{aligned}\tag{3.62}$$

where d is from formulas (3.8) and (3.9).

In the same way we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\hat{\Omega}}\frac{p_{\hat{\eta}}}{\hat{\eta}}\hat{\eta}^2_{\sigma nn}J\mathrm{d}z + c_0\|\tilde{\eta}_{\sigma nn}\|^2_{0,\hat{\Omega}} \le \varepsilon\|\tilde{\eta}_{\sigma nn}\|^2_{0,\hat{\Omega}} + cd\|\tilde{u}\|^2_{3,\hat{\Omega}}\\
&+ c\Big[\|\hat{u}\|^2_{2,\hat{\Omega}} + \|\hat{u}_t\|^2_{0,\hat{\Omega}} + \|\hat{\vartheta}_\sigma\|^2_{2,\hat{\Omega}} + \|\hat{\eta}_\sigma\|^2_{1,\hat{\Omega}} + \|\tilde{u}_{zn\tau}\|^2_{0\hat{\Omega}}\\
&+ \bar{X}_1(\hat{\Omega})(\bar{X}_2(\hat{\Omega}) + \bar{X}_1^2(\hat{\Omega})) + \bar{X}_2(\hat{\Omega})\Big\|\int\limits_0^t\hat{u}\mathrm{d}t'\Big\|^2_{3,\hat{\Omega}}\Big].
\end{aligned}\tag{3.63}$$

Next, differentiating the third component of (3.28) with respect to τ, multiplying by $\tilde{u}_{3nn\tau}J$ and integrating over $\hat{\Omega}$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\hat{\Omega}}\hat{\eta}\tilde{u}^2_{3n\tau}J\mathrm{d}z + c_0\|\tilde{u}_{3nn\tau}\|^2_{0,\hat{\Omega}}\\
&\le \varepsilon\|\tilde{u}_{3nn\tau}\|^2_{0,\hat{\Omega}} + cd\|\hat{u}\|^2_{3,\hat{\Omega}} + c\Big[\|\tilde{u}\|^2_{2,\hat{\Omega}} + \|\tilde{u}_t\|^2_{2,\hat{\Omega}}\\
&+ \|\tilde{u}_{z\tau\tau}\|^2_{0,\hat{\Omega}} + \|\tilde{\eta}_{\sigma n\tau}\|^2_{0,\hat{\Omega}} + \|\hat{\vartheta}_\sigma\|^2_{1,\hat{\Omega}} + \|\hat{\eta}_\sigma\|^2_{1,\hat{\Omega}}\\
&+ \bar{X}_2(\hat{\Omega})(\bar{X}_1(\hat{\Omega}) + \Big\|\int\limits_0^t\hat{u}\mathrm{d}t'\Big\|^2_{3,\hat{\Omega}})\Big].
\end{aligned}\tag{3.64}$$

To estimate $\tilde{\vartheta}_{\sigma nn\tau}$ differentiate (3.30) with respect to τ, multiply the result by $\tilde{\vartheta}_{\sigma nn\tau}J$, divide by $\hat{\vartheta}$ and integrate over $\hat{\Omega}$. We obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\hat{\Omega}}\frac{\hat{\eta}c_v}{\hat{\vartheta}}\tilde{\vartheta}^2_{\sigma n\tau}J\mathrm{d}z + c_0\|\tilde{\vartheta}_{\sigma nn\tau}\|^2_{0,\hat{\Omega}}\\
&\le \varepsilon\|\tilde{\vartheta}_{\sigma n\tau t}\|^2_{0,\hat{\Omega}} + cd\|\tilde{\vartheta}_{\sigma zzz}\|^2_{0,\hat{\Omega}} + c\Big[\|\tilde{\vartheta}_{\sigma z\tau\tau}\|^2_{0,\hat{\Omega}} + \|\hat{\vartheta}_\sigma\|^2_{2,\hat{\Omega}} + \|\hat{\vartheta}_{\sigma t}\|^2_{2,\hat{\Omega}}
\end{aligned}$$

$$+\bar{X}_2(\hat{\Omega})(\bar{X}_1(\hat{\Omega})+\Big\|\int_0^t\hat{u}\mathrm{d}t'\Big\|^2_{3,\hat{\Omega}})\Big]. \tag{3.65}$$

Differentiating (3.30) with respect to n, multiplying the result by $\tilde{\vartheta}_{\sigma nnn}J$, dividing by $\hat{\vartheta}$ and integrating over $\hat{\Omega}$ yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\hat{\Omega}}\frac{\hat{\eta}c_v}{\hat{\vartheta}}\tilde{\vartheta}^2_{\sigma nn}J\mathrm{d}z+c_0\|\tilde{\vartheta}_{\sigma nnn}\|^2_{0,\hat{\Omega}}\\&\le\varepsilon\|\tilde{\vartheta}_{\sigma nnn}\|^2_{0,\hat{\Omega}}+cd\|\tilde{\vartheta}_{\sigma zzz}\|^2_{0,\hat{\Omega}}+c\Big[\|\tilde{\vartheta}_{\sigma z\tau\tau}\|^2_{0,\hat{\Omega}}+\|\hat{\vartheta}_\sigma\|^2_{2,\hat{\Omega}}+\|\hat{\vartheta}_{\sigma t}\|^2_{2,\hat{\Omega}}\\&+\bar{X}_2(\hat{\Omega})(\bar{X}_1(\hat{\Omega})+\Big\|\int_0^t\hat{u}\mathrm{d}t'\Big\|^2_{3,\hat{\Omega}})\Big].\end{aligned} \tag{3.66}$$

In order to estimate $\|(\operatorname{div}\tilde{u})_{,nn}\|^2_{0,\hat{\Omega}}$ rewrite equation $(3.11)_1$ in the form

$$\begin{aligned}&(\nu+\mu)\nabla_{z_i}\operatorname{div}\tilde{u}=-\mu(\Delta\tilde{u}_i-\nabla_{z_i}\operatorname{div}\tilde{u})+\hat{\eta}\tilde{u}_{it}\\&-k_{4i}+(\mu\Delta\tilde{u}_i+\nu\nabla_{z_i}\operatorname{div}\tilde{u}-\mu\hat{\nabla}^2\tilde{u}_i-\nu\hat{\nabla}_i\hat{\nabla}\cdot\tilde{u})\\&+p_5\hat{\eta}_\sigma\hat{\nabla}_i\hat{\zeta}+p_6\hat{\vartheta}_\sigma\hat{\nabla}_i\hat{\zeta}+\hat{\zeta}p_{\sigma\hat{\eta}}\hat{\nabla}_i\hat{\eta}_\sigma+\hat{\zeta}p_{\sigma\hat{\vartheta}}\hat{\nabla}_i\hat{\vartheta}_\sigma,\quad i=1,2,3,\end{aligned} \tag{3.67}$$

where p_5,p_6 are defined by $\hat{p}_\sigma=p(\hat{\varrho}_e,\theta_e)-p(\varrho_e,\theta_e)=p_5\hat{\eta}_\sigma+p_6\hat{\vartheta}_\sigma$. Differentiating the third component of (3.67) with respect to n gives

$$\begin{aligned}&\|(\operatorname{div}\tilde{u})_{,nn}\|^2_{0,\hat{\Omega}}\le cd\|\tilde{u}_{nnn}\|^2_{0,\hat{\Omega}}+c\Big[\|\tilde{u}_\tau\|^2_{2,\hat{\Omega}}+\|\hat{u}\|^2_{2,\hat{\Omega}}\\&+\|\tilde{u}_t\|^2_{1,\hat{\Omega}}+\|\hat{\vartheta}_\sigma\|^2_{2,\hat{\Omega}}+\|\tilde{\eta}_{\sigma n}\|^2_{1,\hat{\Omega}}+\|\hat{\eta}_\sigma\|^2_{1,\hat{\Omega}}\\&+\bar{X}_2(\hat{\Omega})(\bar{X}_1(\hat{\Omega})+\Big\|\int_0^t\hat{u}\mathrm{d}t'\Big\|^2_{3,\hat{\Omega}})\Big].\end{aligned} \tag{3.68}$$

To obtain an estimate for $\|\tilde{u}_\tau\|^2_{2,\hat{\Omega}}$ consider the following elliptic problem

$$\begin{aligned}&\mu\hat{\nabla}^2\tilde{u}+\nu\hat{\nabla}\hat{\nabla}\cdot\tilde{u}-p_{\sigma\hat{\eta}}\hat{\nabla}\tilde{\eta}_\sigma=\hat{\eta}\tilde{u}_t\\&+(p_5-p_{\sigma\hat{\eta}})\hat{\eta}_\sigma\hat{\nabla}\hat{\zeta}+(p_6-p_{\sigma\hat{\vartheta}})\hat{\vartheta}_\sigma\hat{\nabla}\hat{\zeta}\\&+p_{\sigma\hat{\vartheta}}\hat{\zeta}\hat{\nabla}\hat{\vartheta}_\sigma+\hat{\nabla}\cdot\hat{\mathbb{B}}(\hat{u},\hat{\zeta})+\hat{\mathbb{T}}(\hat{u},p_\sigma)\cdot\hat{\nabla}\hat{\zeta}\quad\text{in }\hat{\Omega},\\&\hat{\nabla}\cdot\tilde{u}=\tilde{\hat{\nabla}}\cdot\tilde{u}\quad\text{in }\hat{\Omega},\\&\hat{\mathbb{T}}(\tilde{u},\tilde{p}_\sigma)\hat{n}=k_7\quad\text{on }\hat{S},\end{aligned} \tag{3.69}$$

where

$$\hat{\nabla}\cdot\hat{\mathbb{B}}(\hat{u},\hat{\zeta})=\{\hat{\nabla}_j\hat{B}_{ij}(\hat{u},\hat{\zeta})\}_{i=1,2,3},\quad\hat{\mathbb{T}}(\hat{u},p_\sigma)\cdot\hat{\nabla}\hat{\zeta}=\{\hat{T}_{ij}(\hat{u},p_\sigma)\hat{\nabla}_j\hat{\zeta}\}_{i=1,2,3}.$$

Differentiating (3.69) with respect to τ and next using the Agmon-Douglis-Nirenberg theory [1] we get

$$\|\tilde{u}_\tau\|^2_{2,\hat{\Omega}} + \|\tilde{\eta}_{\sigma\tau}\|^2_{1,\hat{\Omega}} \le c\Big[\|\tilde{u}_{\tau\tau}\|^2_{1,\hat{\Omega}} + \|\tilde{u}_{3nn\tau}\|^2_{0,\hat{\Omega}} + \|\hat{u}\|^2_{2,\hat{\Omega}} + \|\tilde{u}_t\|^2_{1,\hat{\Omega}} + \|\hat{\vartheta}_\sigma\|^2_{2,\hat{\Omega}} + \|\hat{\eta}_\sigma\|^2_{1,\hat{\Omega}} + \bar{X}_1^2(\hat{\Omega}) + \bar{X}_2(\hat{\Omega})\Big\|\int_0^t \hat{u}\mathrm{d}t'\Big\|^2_{3,\hat{\Omega}}\Big]. \tag{3.70}$$

Finally, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta u^2_{\xi\xi} A\mathrm{d}\xi \le c(\|u\|^2_{2,\Omega} + \|u_t\|^2_{2,\Omega}) + c\bar{X}_2(\Omega). \tag{3.71}$$

Now, applying to inequalities (3.59)–(3.66), (3.68), (3.70) and (3.71) the same argument as in the last part of the proof of Lemma 3.1 yields (3.57).

This concludes the proof. □

Lemmas 3.1–3.3 and Lemmas 2.4, 2.5 imply Theorem 2 which is the main result of the paper.

Proof of Theorem 2. Multiplying inequality (2.46) by a sufficiently large constant and adding to (2.50) yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\phi_4(t) + \bar{C}_6(\|v\|^2_{1,\Omega_t} + \|v_t\|^2_{1,\Omega_t} + \|v_{tt}\|^2_{1,\Omega_t} + \|\theta_{\sigma x}\|^2_{0,\Omega_t}\\ &+ \|\theta_\sigma\|^2_{0,S_t} + \|\theta_{\sigma t}\|^2_{1,\Omega_t} + \|\theta_{\sigma tt}\|^2_{1,\Omega_t} + \|\varrho_{\sigma t}\|^2_{0,\Omega_t} + \|\varrho_{\sigma tt}\|^2_{0,\Omega_t})\\ &\le C_{10}\bar{X}_1\bar{X}_2(1+\bar{X}_1^2) + C_{11}\bar{X}_1(1+\bar{X}_1)\Big\|\int_0^t v\mathrm{d}t'\Big\|^2_{3,\Omega_t},\end{aligned} \tag{3.72}$$

where

$$\phi_4(t) = \psi(t) + \int_{\Omega_t}\left(\varrho v^2_{tt} + \frac{p_\varrho}{\varrho}\varrho^2_{\sigma tt} + \frac{\varrho c_v}{\theta}\theta^2_{\sigma tt}\right)\mathrm{d}x. \tag{3.73}$$

Using in (3.12) estimate (2.17) and then adding appropriately (3.12), (3.37) and (3.57) we obtain

$$\begin{aligned}&\frac{d}{dt}(B_1\phi_1(t,\Omega) + B_2\phi_2(t,\Omega) + B_3\phi_3(t,\Omega))\\ &+ C\Big[\|v\|^2_{3,\Omega_t} + \|v_t\|^2_{2,\Omega_t} + \|\varrho_\sigma\|^2_{2,\Omega_t} + \|\varrho_{\sigma t}\|^2_{1,\Omega_t} + \|\theta_\sigma\|^2_{3,\Omega_t} + \|\theta_{\sigma t}\|^2_{2,\Omega_t}\Big]\\ &\le c\Big[\|v\|^2_{1,\Omega_t} + \|v_t\|^2_{1,\Omega_t} + \|v_{tt}\|^2_{1,\Omega_t} + \|\theta_\sigma\|^2_{1,\Omega_t} + \|\theta_{\sigma t}\|^2_{1,\Omega_t}\\ &+ \|\theta_{\sigma tt}\|^2_{1,\Omega_t} + \|\varrho_\sigma\|^2_{0,\Omega_t} + \|\varrho_{\sigma t}\|^2_{1,\Omega_t} + \|\varrho_{\sigma tt}\|^2_{1,\Omega_t}\Big]\end{aligned}$$

$$+ c\bar{X}_1\bar{X}_2(1+\bar{X}_1^2) + c\bar{X}_1(1+\bar{X}_1)\Big\|\int_0^t v\mathrm{d}t'\Big\|^2_{3,\Omega_t}. \tag{3.74}$$

Adding appropriately (3.72) multiplied by a sufficiently large constant and (3.74) we obtain (1.19). This concludes the proof. □

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