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# A GENERALIZED UPPER AND LOWER SOLUTION METHOD FOR SINGULAR DISCRETE BOUNDARY VALUE PROBLEMS FOR THE ONE-DIMENSIONAL $p$-LAPLACIAN

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**Abstract.** This paper presents new existence results for singular discrete boundary value problems for the one-dimension $p$-Laplacian. In particular our nonlinearity may be singular in its dependent variable and is allowed to change sign. Our results are new even for $p = 2$.

## 1. Introduction

An upper and lower solution theory is presented for the singular discrete boundary value problem

$$\begin{cases} \Delta(\phi(\Delta y(i-1))) + q(i)\, f(i, y(i)) = 0, & i \in N = \{1, \dots, T\} \\ y(0) = y(T+1) = 0, \end{cases} \tag{1.1}$$

where $\phi(s) = |s|^{p-2}s,\ p > 1,\ T \in \{1, 2, ...\},\ N^+ = \{0, 1, ..., T+1\}$ and $y\colon N^+ \to \mathbb{R}$. Throughout this paper we will assume $f\colon N \times (0, \infty) \to \mathbb{R}$ is

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continuous. As a result our nonlinearity $f(i,u)$ may be singular at $u=0$ and may change sign.

**Remark 1.1.** Recall that a map $f\colon N\times(0,\infty)\to\mathbb{R}$ is continuous if it is continuous as a map of the topological space $N\times(0,\infty)$ into the topological space $\mathbb{R}$. Throughout this paper the topology on $N$ will be the discrete topology.

We will let $C(N^+,\mathbb{R})$ denote the class of maps $u$ continuous on $N^+$ (discrete topology), with norm $||u||=\max_{i\in N^+}|u(i)|$. By a solution to (1.1) we mean a $y\in C(N^+,\mathbb{R})$ such that $y$ satisfies (1.1) for $i\in N$ and $y$ satisfies the boundary condition.

The literature on the one–dimensional $p$-Laplacian (when the nonlinearity is not singular in its dependent variable) is vast; see [18] and the references therein. Also the existence of solutions to singular boundary value problems in the continuous case have been studied in great detail in the literature (see [6, 7, 8, 10, 13] (when $p=2$) and [14, 15] and the references therein). However, for the discrete case only a few papers have discussed boundary value problems. For example see [4, 5, 11, 12] (when $p=2$) and [16, 17]. In [16] the nonlinearity $f(i,u)$ may be singular at $u=0$ and may change sign, and the approach there is based on an argument initiated by Habets and Zanolin in [10]. In this paper a new approach is given which yields a very general existence theory for (1.1). Our results are new even for $p=2$. Not suprizingly our results improve considerable the results in [2] (when $p=2$) and [16, 17].

## 2. Some preliminary results

In this section we present some results from literature which will be needed in Section 3.

We first state one well known result in [1].

**Lemma 2.1** ([1])**.** *Let $u\in C(N^+,\mathbb{R})$ satisfy $u(i)\geq 0$ for $i\in N^+$. If $y\in C(N^+,\mathbb{R})$ satisfies*

$$\begin{cases}\Delta^2 y(i-1)+u(i)=0, & i\in N=\{1,2,\dots,T\}\\ y(0)=y(T+1)=0,\end{cases}$$

*then*

$$y(i)\geq\mu(i)||y||\ \text{for } i\in N^+;$$

*here*

$$\mu(i)=\min\left\{\frac{T+1-i}{T+1},\frac{i}{T}\right\}.$$

**Lemma 2.2.** *Let* $[a, b] = \{a, a+1, ..., b\} \subset N$. *If* $y \in C(N^+, \mathbb{R})$ *satisfies*

$$\begin{cases} \Delta^2 y(i-1) \leq 0, & i \in [a,b] \\ y(a-1) \geq 0, \ y(b+1) \geq 0, & \end{cases}$$

*then* $y(i) \geq 0$ *for* $i \in [a-1, b+1] = \{a-1, a, ..., b+1\} \subset N^+$.

**Remark 2.1.** Of course if $b = a$ then Lemma 2.2 is immediate.

**Proof.** Set

$$Q(i) = y(a-1) + \frac{y(b+1) - y(a-1)}{b-a+2}(i+1-a), \quad i \in [a-1, b+1].$$

Let $y(i) = u(i) + Q(i)$. Then $\Delta^2 u(i-1) \leq 0$, $i \in [a,b]$ and $u(a-1) = u(b+1) = 0$. Thus by Lemma 2.1, $u(i) \geq 0$ for $i \in [a-1, b+1]$. Since $Q(i) \geq 0$, then $y(i) \geq 0$ for $i \in [a-1, b+1]$. □

**Lemma 2.3.** *Let* $[a, b] = \{a, a+1, \ldots, b\} \subset N$. *If* $y \in C(N^+, \mathbb{R})$ *satisfies*

$$\begin{cases} \Delta(\phi(\Delta y(i-1))) \leq 0, & i \in [a,b] \\ y(a-1) \geq 0, \quad y(b+1) \geq 0, & \end{cases}$$

*then* $y(i) \geq 0$ *for* $i \in [a-1, b+1] = \{a-1, a, ..., b+1\} \subset N^+$.

**Proof.** Notice $\Delta(\phi(\Delta y(i-1))) \leq 0$ implies $\Delta^2 y(i-1) \leq 0$ for $i \in [a,b]$, so the result follows from Lemma 2.2. □

**Lemma 2.4.** *Let* $[a, b] = \{a, a+1, ..., b\} \subset N$. *If* $u, v \in C(N^+, \mathbb{R})$ *satisfy*

$$\begin{cases} \Delta(\phi(\Delta u(i-1))) \leq \Delta(\phi(\Delta v(i-1))), & i \in [a,b] \\ u(a-1) \geq v(a-1), \quad u(b+1) \geq v(b+1), & \end{cases}$$

*then* $u(i) \geq v(i)$ *for* $i \in [a-1, b+1] = \{a-1, a, ..., b+1\} \subset N^+$.

**Proof.** Suppose $u(i) < v(i)$ for some $i \in [a-1, b+1]$. Since $u(a-1) \geq v(a-1)$, $u(b+1) \geq v(b+1)$, the function $w(i) = u(i) - v(i)$ would have a negative minimum at a point $i_0 \in [a,b]$. Hence $\Delta w(i_0 - 1) \leq 0$, i.e., $\Delta u(i_0 - 1) \leq \Delta v(i_0 - 1)$. Notice that

$$\Delta(\phi(\Delta u(i-1))) \leq \Delta(\phi(\Delta v(i-1))), \quad i \in [a,b].$$

Sum both sides of the above inequality from $i_0$ to $i \in [i_0, b] = \{i_0, ..., b\}$ to get

$$\phi(\Delta u(i)) - \phi(\Delta u(i_0 - 1)) \leq \phi(\Delta v(i)) - \phi(\Delta v(i_0 - 1)), \quad \text{for all } i \in [i_0, b],$$

and so we have

$$\phi(\Delta u(i)) - \phi(\Delta v(i)) \le \phi(\Delta u(i_0 - 1)) - \phi(\Delta v(i_0 - 1)), \quad \text{for all } i \in [i_0, b].$$

As a result

$$\Delta w(i) = \Delta u(i) - \Delta v(i) \le 0, \quad \text{for all } i \in [i_0, b],$$

and so $w(i_0) \ge w(b+1) \ge 0$, a contradiction. □

Consider the discrete boundary value problem

$$\begin{cases} \Delta(\phi(\Delta y(i-1))) + F(i, y(i)) = 0, \quad i \in N = \{1, \dots, T\} \\ y(0) = A, \; y(T+1) = B, \end{cases} \tag{2.1}$$

where $A$ and $B$ are given real numbers, $\phi(s) = |s|^{p-2}s$, $p > 1$. The following existence principle for problem (2.1) was established in [16, 17].

**Lemma 2.5.** *Suppose that $F(i,u)\colon N \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exists $h \in C(N, [0,\infty))$ with $|F(i,u)| \le h(i)$ for $i \in N$. Then (2.1) has a solution $y \in C(N^+, \mathbb{R})$.*

## 3. Existence theory

In this section we combine the ideas in [9] (when $p = 2$) and [16] to obtain new results for the singular discrete boundary value problem

$$\begin{cases} \Delta(\phi(\Delta y(i-1))) + q(i)\, f(i, y(i)) = 0, \quad i \in N = \{1, \dots, T\} \\ y(0) = y(T+1) = 0, \end{cases} \tag{3.1}$$

where our nonlinearity $f$ may change sign. Our main result can be stated immediately.

**Theorem 3.1.** *Let $n_0 \in \{1, 2, ....\}$ be fixed and suppose the following conditions are satisfied:*

$$f\colon N \times (0,\infty) \to \mathbb{R} \text{ is continuous} \tag{3.2}$$

$$q \in C(N, (0,\infty)) \tag{3.3}$$

$$\begin{cases} \text{there exists a function } \alpha \in C(N^+, \mathbb{R}) \\ \text{with } \alpha(0) = \alpha(T+1) = 0, \; \alpha > 0 \text{ on } N \text{ such} \\ \text{that } q(i)\, f(i, \alpha(i)) \ge -\Delta(\phi(\Delta\alpha(i-1))) \text{ for } i \in N \end{cases} \tag{3.4}$$

*and*

$$\begin{cases} \text{there exists a function } \beta \in C(N^+, \mathbb{R}) \text{ with} \\ \beta(i) \ge \alpha(i) \text{ and } \beta(i) \ge 1/n_0 \text{ for } i \in N^+ \text{ with} \\ q(i)\, f(i, \beta(i)) \le -\Delta(\phi(\Delta\beta(i-1))) \text{ for } i \in N. \end{cases} \tag{3.5}$$

*Then* (3.1) *has a solution* $y \in C(N^+, \mathbb{R})$ *with* $y(i) \geq \alpha(i)$ *for* $i \in N^+$.

**Proof.** We begin with the discrete boundary value problem

$$\begin{cases} -\Delta(\phi(\Delta y(i-1))) = q(i)\, f^\star_{n_0}(i, y(i)), & i \in N \\ y(0) = y(T+1) = \dfrac{1}{n_0}; \end{cases} \tag{3.6}$$

here

$$f^\star_{n_0}(i,y) = \begin{cases} f(i,\alpha(i)), & y \leq \alpha(i) \\ f(i,y), & \alpha(i) \leq y \leq \beta(i) \\ f(i,\beta(i)), & y \geq \beta(i). \end{cases}$$

From Lemma 2.5 we know that (3.6) has a solution $y_{n_0} \in C(N^+, \mathbb{R})$. We first show

$$y_{n_0}(i) \geq \alpha(i), \quad i \in N^+. \tag{3.7}$$

Suppose (3.7) is not true. Since $y_{n_0}(0) > \alpha(0) = 0$, $y_{n_0}(T+1) > \alpha(T+1) = 0$, then there exists $[a,b] = \{a, a+1, \ldots, b\} \subset N$ such that

$$y_{n_0}(i) < \alpha(i) \text{ on } [a,b],\ y_{n_0}(a-1) \geq \alpha(a-1),\ y_{n_0}(b+1) \geq \alpha(b+1).$$

Thus for $i \in [a,b]$, we have

$$\begin{aligned} -\Delta(\phi(\Delta y_{n_0}(i-1))) =& q(i)\, f^\star_{n_0}(i, y_{n_0}(i)) = q(i)\, f(i,\alpha(i)) \\ \geq& -\Delta(\phi(\Delta(\alpha(i-1))). \end{aligned}$$

Since $y_{n_0}(a-1) \geq \alpha(a-1)$, $y_{n_0}(b+1) \geq \alpha(b+1)$, it follows from Lemma 2.4 that $y_{n_0}(i) \geq \alpha(i)$ for $i \in [a-1, b+1] = \{a-1, a, \ldots, b+1\} \subset N^+$, a contradiction.

A similar argument shows

$$y_{n_0}(i) \leq \beta(i) \quad \text{for } i \in N^+. \tag{3.8}$$

Thus

$$\alpha(i) \leq y_{n_0}(i) \leq \beta(i) \quad \text{for } i \in N^+. \tag{3.9}$$

Now proceed inductively to construct $y_{n_0+1}, y_{n_0+2}, y_{n_0+3}, \ldots$ as follows. Suppose we have $y_k$ for some $k \in \{n_0, n_0+1, n_0+2, \ldots\}$ with $\alpha(i) \leq y_k(i) \leq y_{k-1}(i)$ for $i \in N^+$ (here $y_{n_0-1} = \beta$). Then consider the discrete boundary value problem

$$\begin{cases} -\Delta(\phi(\Delta y(i-1))) = q(i)\, f^\star_{k+1}(i, y(i)), & i \in N \\ y(0) = \dfrac{1}{k+1}; \end{cases} \tag{3.10}$$

here

$$f^\star_{k+1}(i,y) = \begin{cases} f(i,\alpha(i)), & y \leq \alpha(i) \\ f(i,y), & \alpha(i) \leq y \leq y_k(i) \\ f(i,y_k(i)), & y \geq y_k(i). \end{cases}$$

Now Lemma 2.5 guarantees that (3.10) has a solution $y_{k+1} \in C(N^+, \mathbb{R})$, and essentially the same reasoning as above yields

$$\alpha(i) \le y_{k+1}(i) \le y_k(i) \text{ for } i \in N^+. \tag{3.11}$$

Thus for each $n \in \{n_0, n_0+1, \dots\}$ we have

$$\alpha(i) \le y_n(i) \le y_{n-1}(i) \le \dots \le y_{n_0}(i) \le \beta(i) \text{ for } i \in N^+. \tag{3.12}$$

Bolzano's theorem guarantees the existence of a subsequence $Z_{n_0}$ of integers and a function $y$ with $y_n$ converging to $y$ on $N^+$ as $n \to \infty$ through $Z_{n_0}$. Also $y(0) = y(T+1) = 0$. Now $y_n$, $n \in Z_{n_0}$, satisfies $y_n(i) \ge \alpha(i) > 0$ for $i \in N$. Fix $i \in N$, and we obtain

$$\begin{aligned}\Delta(\phi(\Delta y_n(i-1))) &= \phi(\Delta y_n(i)) - \phi(\Delta y_n(i-1)) \\ &= \phi(y_n(i+1) - y_n(i)) - \phi(y_n(i) - y_n(i-1)) \\ &\to \Delta(\phi(\Delta y(i-1))),\ i \in N,\ n \in Z_{n_0},\ n \to \infty,\end{aligned}$$

and

$$f(i, y_n(i)) \to f(i, y(i)),\ i \in N,\ n \in Z_{n_0},\ n \to \infty.$$

Thus $\Delta(\phi(\Delta y(i-1))) + q(i)f(i, y(i)) = 0$ for $i \in N$, $y(0) = y(T+1) = 0$. As a result $y \in C(N^+, \mathbb{R})$ is a solution to (3.1) and also we have $\alpha(i) \le y(i) \le \beta(i)$, $i \in N^+$. □

Suppose (3.2)–(3.4) hold, and in addition assume the following conditions are satisfied:

$$\begin{cases} q(i)\, f(i, y) \ge -\Delta(\phi(\Delta\alpha(i-1))) \\ \text{for } (i, y) \in N \times \{y \in (0, \infty) \colon y < \alpha(i)\} \end{cases} \tag{3.13}$$

and

$$\begin{cases} \text{there exists a function } \beta \in C(N^+, \mathbb{R}) \text{ with} \\ \beta(i) \ge \dfrac{1}{n_0} & \text{for } i \in N^+ \text{ with} \\ q(i)\, f(i, \beta(i)) \le -\Delta(\phi(\Delta\beta(i-1))) & \text{for } i \in N. \end{cases} \tag{3.14}$$

Then the result in Theorem 3.1 is again true. This follows immediately from Theorem 3.1 once we show $\beta(i) \ge \alpha(i)$ for $i \in N^+$. Suppose it is false. Since $\beta(0) > \alpha(0) = 0, \beta(T+1) > \alpha(T+1) = 0$, then there exists $[a, b] = \{a, a+1, \dots, b\} \subset N$ such that

$$\beta(i) < \alpha(i) \text{ on } [a, b],\ \beta(a-1) \ge \alpha(a-1),\ \beta(b+1) \ge \alpha(b+1).$$

Thus for $i \in [a, b]$, we have

$$q(i)\, f(i, \beta(i)) \ge -\Delta(\phi(\Delta\alpha(i-1))),$$

and therefore

$$-\Delta(\phi(\Delta\beta(i-1))) \ge -\Delta(\phi(\Delta(\alpha(i-1))), \quad i \in [a, b].$$

Since $\beta(a-1) \geq \alpha(a-1)$, $\beta(b+1) \geq \alpha(b+1)$, it follows from Lemma 2.4 that $\beta(i) \geq \alpha(i)$ for $i \in [a-1, b+1] = \{a-1, a, \dots, b+1\} \subset N^+$, a contradiction. Thus we have

**Corollary 3.1.** *Let $n_0 \in \{1, 2, \dots\}$ be fixed and suppose (3.2)–(3.4), (3.13) and (3.14) hold. Then (3.1) has a solution $y \in C(N^+, \mathbb{R})$ with $y(i) \geq \alpha(i)$ for $i \in N^+$.*

Next we discuss how to construct the lower solution $\alpha$ in (3.4) and in (3.13). Suppose the following condition is satisfied:

$$\begin{cases} \text{let } n \in \{n_0, n_0+1, \dots\} \text{ and associated with each } n \\ \text{there exists a constant } k_0 > 0 \text{ such that for } i \in N \\ \text{and } 0 < y \leq \dfrac{1}{n} \text{ we have } q(i)\, f(i, y) \geq k_0. \end{cases} \tag{3.15}$$

Let $\alpha(i) = kv(i)$, $i \in N^+$, where $v \in C(N^+, [0, \infty))$ is the solution of

$$\begin{cases} \Delta(\phi(\Delta v(i-1))) + 1 = 0, \quad i \in N = \{1, \dots, T\} \\ v(0) = v(T+1) = 0; \end{cases} \tag{3.16}$$

here

$$0 < k < \min\left\{ [k_0]^{1/(p-1)},\, \frac{1}{n_0||v||} \right\}.$$

Since $\Delta(\phi(\Delta v(i-1))) < 0$ implies $\Delta^2 v(i-1) < 0$ for $i \in N$, it follows from Lemma 2.1 that $v(i) \geq \mu(i)||v||$ for $i \in N^+$. Thus, $\alpha(i) \leq 1/n_0$, $-\Delta(\phi(\Delta\alpha(i-1))) = k^{p-1} \leq k_0$, $\alpha(0) = \alpha(T+1) = 0$, $\alpha > 0$ for $i \in N$, so (3.4) and (3.13) hold, since

$$q(i)f(i, y) \geq k_0 \geq -\Delta(\phi(\Delta\alpha(i-1))), \quad \text{for } i \in N,\ 0 < y < \alpha(i),$$

and

$$q(i)f(i, \alpha(i)) \geq k_0 \geq -\Delta(\phi(\Delta\alpha(i-1))), \quad i \in N.$$

We combine this with Corollary 3.2 to obtain our next result.

**Theorem 3.2.** *Let $n_0 \in \{1, 2, ....\}$ be fixed and suppose (3.2), (3.3), (3.14), and (3.15) hold. Then (3.1) has a solution $y \in C(N^+, \mathbb{R})$ with $y(i) > 0$ for $i \in N$.*

Looking at Theorem 3.3 we see that the main difficulty when discussing examples is the construction of the $\beta$ in (3.14). Our next result replaces (3.14) with a growth condition which is natural from an application viewpoint and easy to check in practice. We first present the result in its full generality.

**Theorem 3.3.** *Let* $n_0 \in \{1, 2, \dots\}$ *be fixed and suppose* (3.2)–(3.4) *hold. Also assume the following condition is satisfied:*

$$\begin{cases} |f(i,y)| \leq g(y) + h(y) \text{ on } N \times (0,\infty) \text{ with} \\ g > 0 \text{ continuous and nonincreasing on } (0,\infty) \\ \text{and } h \geq 0 \text{ continuous on } [0,\infty) \\ \dfrac{h}{g} \text{ nondecreasing on } (0,\infty). \end{cases} \tag{3.17}$$

*Also suppose there exists a constant* $M > \sup_{i \in N^+} \alpha(i)$ *with*

$$b_0 < \frac{1}{\phi^{-1}\left(1 + \dfrac{h(M)}{g(M)}\right)} \int_0^M \frac{dy}{\phi^{-1}(g(y))} \tag{3.18}$$

*holding; here*

$$b_0 = \max_{i \in N} \left( \sum_{j=1}^{i} \phi^{-1}(\sum_{z=j}^{i} q(z)),\ \sum_{j=i}^{T} \phi^{-1}(\sum_{z=i}^{j} q(z)) \right).$$

*Then* (3.1) *has a solution* $y \in C(N^+, \mathbb{R})$ *with* $y(i) \geq \alpha(i)$ *for* $i \in N^+$.

**Proof.** Choose $\varepsilon > 0$, $\varepsilon < M$, with

$$\frac{1}{\phi^{-1}\left(1 + \dfrac{h(M)}{g(M)}\right)} \int_\varepsilon^M \frac{dy}{\phi^{-1}(g(y))} > b_0. \tag{3.19}$$

Without loss of generality assume $1/n_0 < \varepsilon$. We consider the discrete boundary value problem

$$\begin{cases} \Delta(\phi(\Delta y(i-1))) + q(i)g(y(i)) \left(1 + \dfrac{h(M)}{g(M)}\right) = 0,\ i \in N, \\ y(0) = y(T+1) = \dfrac{1}{n_0}. \end{cases} \tag{3.20}$$

First we consider the modified discrete boundary value problem

$$\begin{cases} \Delta(\phi(\Delta y(i-1))) + q(i)g^*(y(i)) \left(1 + \dfrac{h(M)}{g(M)}\right) = 0,\ i \in N, \\ y(0) = y(T+1) = \dfrac{1}{n_0}; \end{cases} \tag{3.21}$$

here

$$g^\star(y) = \begin{cases} g\left(\dfrac{1}{n_0}\right), & y \leq \dfrac{1}{n_0} \\ g(y), & y \geq \dfrac{1}{n_0}. \end{cases}$$

Now $|g^\star(y)| = g^\star(y) \leq g(1/n_0)$ for $y \in \mathbb{R}$, so Lemma 2.5 guarantees that (3.21) has a solution $\beta \in C(N^+, \mathbb{R})$. Let $u(i) = \beta(i) - 1/n_0$ for $i \in N^+$. Then $\Delta(\phi(\Delta u(i-1))) = \Delta(\phi(\Delta\beta(i-1))) \leq 0$ for $i \in N$, and $u(0) = u(T+1) = 0$. Lemma 2.3 guarantees that $u(i) \geq 0$, and so $\beta(i) \geq 1/n_0$ for $i \in N^+$. Then $\beta$ is a solution to problem (3.20) also.

Now we claim that $\alpha(i) \leq \beta(i) \leq M$, $i \in N^+$. First we show

$$\beta(i) \geq \alpha(i), \quad i \in N^+. \tag{3.22}$$

Suppose (3.22) is false. Since $\beta(0) = \beta(T+1) = 1/n_0 > \alpha(0) = \alpha(1) = 0$, then there exists $[a, b] = \{a, a+1, \ldots, b\} \subset N$ such that

$$\beta(i) < \alpha(i) \text{ on } [a, b], \quad \beta(a-1) \geq \alpha(a-1), \quad \beta(b+1) \geq \alpha(b+1).$$

Thus for $i \in [a, b]$, we have from (3.20) and $M > \sup_{i \in N^+} \alpha(i)$ that

$$\begin{aligned} -\Delta(\phi(\Delta\beta(i-1))) &= q(i)g(\beta(i))(1 + \frac{h(M)}{g(M)}) \\ &\geq q(i)g(\alpha(i))(1 + \frac{h(\alpha(i))}{g(\alpha(i))}) \\ &\geq q(i)f(i, \alpha(i)) \geq -\Delta(\phi(\Delta\alpha(i-1))). \end{aligned}$$

Since $\beta(a-1) \geq \alpha(a-1)$, $\beta(b+1) \geq \alpha(b+1)$, it follows from Lemma 2.4 that $\beta(i) \geq \alpha(i)$ for $i \in [a-1, b+1] = \{a-1, a, \ldots, b+1\} \subset N^+$, a contradiction.

Next we show

$$\beta(i) \leq M, \quad i \in N^+. \tag{3.23}$$

Since $\Delta(\phi(\Delta\beta(i-1))) \leq 0$ on $N$ implies $\Delta^2\beta(i-1) \leq 0$ on $N$, then $\beta(i) \geq 1/n_0$ on $N^+$ and there exists $i_0 \in N$ with $\Delta\beta(i) \geq 0$ on $[0, i_0) = \{0, 1, \ldots, i_0 - 1\}$ and $\Delta\beta(i) \leq 0$ on $[i_0, T+1) = \{i_0, i_0+1, \ldots, T\}$, and $\beta(i_0) = ||\beta||$.

Also notice that for $z \in N$, we have

$$-\Delta(\phi(\Delta\beta(z-1))) = g(\beta(z))\left(1 + \frac{h(M)}{g(M)}\right)q(z). \tag{3.24}$$

We sum the equation (3.24) from $j+1$ $(0 \leq j < i_0)$ to $i_0$ to obtain

$$\phi(\Delta\beta(j)) = \phi(\Delta\beta(i_0)) + \left(1 + \frac{h(M)}{g(M)}\right) \sum_{z=j+1}^{i_0} g(\beta(z))q(z).$$

Since $\Delta\beta(i_0) \leq 0$, and $\beta(z) \geq \beta(j+1)$ when $j+1 \leq z \leq i_0$, we have

$$\phi[\Delta\beta(j)] \leq g(\beta(j+1))\left(1 + \frac{h(M)}{g(M)}\right) \sum_{z=j+1}^{i_0} q(z), \quad j < i_0,$$

i.e.,

$$\frac{\Delta\beta(j)}{\phi^{-1}(g(\beta(j+1)))} \leq \phi^{-1}\left(1+\frac{h(M)}{g(M)}\right)\phi^{-1}(\sum_{z=j+1}^{i_0} q(z)),\ j < i_0. \tag{3.25}$$

Since $g(\beta(j+1)) \leq g(u) \leq g(\beta(j))$ for $\beta(j) \leq u \leq \beta(j+1)$ when $j < i_0$, we have

$$\int_{\beta(j)}^{\beta(j+1)} \frac{du}{\phi^{-1}(g(u))} \leq \frac{\Delta\beta(j)}{\phi^{-1}(g(\beta(j+1)))}, \quad j < i_0. \tag{3.26}$$

It follows from (3.25) and (3.26) that

$$\int_{\beta(j)}^{\beta(j+1)} \frac{du}{\phi^{-1}(g(u))} \leq \phi^{-1}\left(1+\frac{h(M)}{g(M)}\right)\phi^{-1}(\sum_{z=j+1}^{i_0} q(z)), \quad j < i_0,$$

and then we sum the above from 0 to $i_0 - 1$ to obtain

$$\begin{aligned}\int_{1/n_0}^{\beta(i_0)} \frac{du}{\phi^{-1}(g(u))} &\leq \phi^{-1}\left(1+\frac{h(M)}{g(M)}\right)\sum_{j=0}^{i_0-1}\phi^{-1}(\sum_{z=j+1}^{i_0} q(z)) \\ &= \phi^{-1}\left(1+\frac{h(M)}{g(M)}\right)\sum_{j=1}^{i_0}\phi^{-1}(\sum_{z=j}^{i_0} q(z)).\end{aligned} \tag{3.27}$$

Similarly, we sum the equation (3.24) from $i_0$ to $j$ ($i_0 \leq j < T+1$) to obtain

$$-\phi(\Delta\beta(j)) = -\phi(\Delta\beta(i_0-1)) + \left(1+\frac{h(M)}{g(M)}\right)\sum_{z=i_0}^{j} g(\beta(z))q(z),\ s \geq t_0.$$

Since $\Delta\beta(i_0 - 1) \geq 0$, we have

$$\frac{-\Delta\beta(j)}{\phi^{-1}(g(\beta(j)))} \leq \phi^{-1}\left(1+\frac{h(M)}{g(M)}\right)\phi^{-1}(\sum_{z=i_0}^{j} q(z)), \quad j \geq i_0.$$

So we have

$$\int_{\beta(j+1)}^{\beta(j)} \frac{du}{\phi^{-1}(g(u))} \leq \frac{-\Delta\beta(j)}{\phi^{-1}(g(\beta(j)))} \leq \phi^{-1}\left(1+\frac{h(M)}{g(M)}\right)\phi^{-1}(\sum_{z=i_0}^{j} q(z)),\ j \geq i_0,$$

and then we sum the above from $i_0$ to $T$ to obtain

$$\int_{1/n_0}^{\beta(i_0)} \frac{du}{\phi^{-1}(g(u))} \leq \phi^{-1}\left(1+\frac{h(M)}{g(M)}\right)\sum_{j=i_0}^{T}\phi^{-1}(\sum_{z=i_0}^{j} q(z)). \tag{3.28}$$

Now (3.27) and (3.28) imply

$$\int_{\varepsilon}^{\beta(i_0)} \frac{du}{\phi^{-1}(g(u))} \leq \int_{1/n_0}^{\beta(i_0)} \frac{du}{\phi^{-1}(g(u))} \leq b_0\phi^{-1}\left(1+\frac{h(M)}{g(M)}\right).$$

This together with (3.19) implies $||\beta|| = \beta(i_0) \leq M$.

Observe that

$$f(i,\beta(i)) \leq g(\beta(i))\left(1+\frac{h(\beta(i))}{g(\beta(i))}\right)$$
$$\leq g(\beta(i))\left(1+\frac{h(M)}{g(M)}\right), \quad i \in N.$$

Thus we have $\beta(i) \geq 1/n_0$ and $\beta(i) \geq \alpha(i)$ for $i \in N^+$ with

$$-\Delta(\phi(\Delta(\beta(i-1))) = q(i)g(\beta(i))(1+\frac{h(M)}{g(M)}) \geq q(i)f(i,\beta(i)),\ i \in N,$$

so that $\beta(i)$ satisfies (3.5). The result follows from Theorem 3.1. □

Combining Theorem 3.4 with the comments before Theorem 3.3 yields the following theorem.

**Theorem 3.4.** *Let* $n_0 \in \{1,2,\dots\}$ *be fixed and suppose* (3.2), (3.3), (3.15) *and* (3.17) *hold. In addition assume there is a constant* $M > 0$ *with* (3.18) *holding. Then* (3.1) *has a solution* $y \in C(N^+,\mathbb{R})$ *with* $y(i) > 0$ *for* $i \in N$.

**Proof.** This follows immediately from Theorem 3.4 once we show there exists $\alpha \in C(N^+,\mathbb{R})$ such that (3.4) hold, and

$$M > \alpha(i) \text{ for each } i \in N^+. \tag{3.29}$$

Let $\alpha(i) = kv(i)$, $i \in N^+$, where $v$ is defined by (3.16), and

$$0 < k < \min\left\{[k_0]^{1/(p-1)}, \frac{1}{n_0||v||}, \frac{M}{||v||}\right\}.$$

Thus, $\alpha(i) \leq 1/n_0$, $-\Delta(\phi(\Delta\alpha(i-1))) = k^{p-1} \leq k_0$, $\alpha(0) = \alpha(T+1) = 0$, $\alpha > 0$ for $i \in N$ with (3.4) holding, since

$$q(i)f(i,\alpha(i)) \geq k_0 \geq -\Delta(\phi(\Delta\alpha(i-1))), \quad i \in N.$$

Then $\alpha \in C(N^+,\mathbb{R})$ and (3.4), and (3.29) hold. □

Next we present an example which illustrates how easily the theory is applied in practice.

**Example 3.1.** The boundary value problem

$$\begin{cases} \Delta(\phi(\Delta y(i-1))) + \sigma([y(i)]^{-\alpha} + [y(i)]^{\beta} + \sin\dfrac{8\pi i}{T}), \quad i \in N \\ y(0) = y(T+1) = 0 \end{cases} \tag{3.30}$$

with $\alpha > 0$, $\beta \geq 0$ and $\sigma > 0$ has a solution $y \in C(N^+, \mathbb{R})$ with $y(i) > 0$ for $i \in N$, if

$$\sigma < \left[\frac{p-1}{b_1(\alpha+p-1)}\right]^{p-1} \sup_{c\in(0,\infty)} \frac{c^{\alpha+p-1}}{1+c^\alpha+c^{\alpha+\beta}}; \tag{3.31}$$

here

$$b_1 = \max_{i\in N}\left(\sum_{j=1}^{i}(i-j+1)^{1/(p-1)}, \sum_{j=i}^{T}(j-i+1)^{1/(p-1)}\right).$$

To see this we will apply Theorem 3.5 with

$$q(i) = \sigma, \quad g(u) = u^{-\alpha}, \quad h(u) = u^\beta + 1.$$

Clearly (3.2), (3.3), (3.15) and (3.17) hold. Also notice that (3.31) implies that there exists $M > 0$ such that

$$\sigma < \left[\frac{p-1}{b_1(\alpha+p-1)}\right]^{p-1} \frac{M^{\alpha+p-1}}{1+M^\alpha+M^{\alpha+\beta}};$$

and so (3.18) holds.

Thus all the conditions of Theorem 3.5 are satisfied so existence is guaranteed.

**Remark 3.1.** If $\beta < p-1$ then (3.31) is automatically satisfied.

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