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# SOLUTIONS OF NONLINEAR SINGULAR BOUNDARY VALUE PROBLEMS

**H. SEBAI**

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**Abstract.** We study the existence of solutions to a class of problems

$$u'' + f(t,u) = 0, \qquad u(0) = u(1) = 0,$$

where $f(t,\cdot)$ is allowed to be singular at $t = 0$, $t = 1$.

## 1. Introduction

Consider the singular boundary value problem (BVP)

$$u'' + f(t,u) = 0, \tag{1.1}$$

$$u(0) = u(1) = 0, \tag{1.2}$$

where $f : (0,1) \times \mathbb{R}^k \to \mathbb{R}^k$, $k \in \mathbb{N}$, is singular both at the end points $t = 0$, $t = 1$ and $f$ will be either a Carathéodory function, or a continuous function. A model example for a continuous function $f$ is

$$f(t,u) = \frac{g(u)}{t^\gamma (1-t)^\gamma},$$

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where $g$ is continuous. In the case $\gamma < 2$ the Dirichlet problem (1.1)–(1.2) was investigated by several authors ([1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]). By a solution $u$ of (1.1)–(1.2) in the continuous case we mean a function $u \in C([0,1],\mathbb{R}^k) \cap C^2((0,1),\mathbb{R}^k)$ satisfying (1.1) everywhere and (1.2). In the Carathéodory case, this means a function $u \in C([0,1],\mathbb{R}^k)$ with absolutely continuous derivative $u$ which satisfies (1.1) almost everywhere and (1.2). In the case $\gamma \geq 2$ the standard method of finding a fixed point to an appropriate integral operator (see below) will not be acting in the space $C$ of all continuous functions $u : [0,1] \to \mathbb{R}^k$.

## 2. The general framework

The present work shows that having $\gamma \geq 2$, we shall obtain the existence of solutions for the problem (1.1)–(1.2). The results here are different. Our method of proof is based on the definition of the following subspace

$$X_\alpha := \left\{ u \in C([0,1],\mathbb{R}^k) : \sup_{t\in(0,1)} \frac{|u(t)|}{(t(1-t))^\alpha} < \infty \right\},$$

$\alpha \in (0,1)$ with the norm

$$||u||_\alpha = \sup_{t\in(0,1)} \frac{|u(t)|}{t^\alpha(1-t)^\alpha},$$

where $C([0,1],\mathbb{R}^k)$ is the space of all continuous functions from $[0,1]$ into $\mathbb{R}^k$, and $|\cdot|$ means the Euclidean norm in $\mathbb{R}^k$.

Functions from $X_\alpha$ vanish immediately at the ends of the interval $[0,1]$ and the family of $X_\alpha$ is extending (increasing) when $\alpha$ decreases to 0. For $\alpha \geq 1$, the space is not sufficiently large. It is obvious that the convergence in $X_\alpha$ is the uniform convergence after multiplication by the function

$$t \to (t(1-t))^{-\alpha}$$

and that the compactness criterion in $X_\alpha$ is the classical Ascoli-Arzèla's theorem after the same operation. The first result Theorem 2.1 shows conditions which guarantee that the Hammerstein operator connected with the problem maps $X_\alpha$ into itself and is completely continuous. After that the existence of a solution can be obtained by using either the Schauder fixed point Theorem 3.1 or the Leray-Schauder continuation Theorem 4.1 is devoted similar results for the Carathéodory case. The last result Theorem 6.2 concerns positive solutions. Now (1.1)–(1.2) has a solution $u = u(t) \in C^2(0,1)$ if and only if $u \in C[0,1]$ solves the operator equation

$$u(t) = \int_0^1 G(t,s) f(s,u(s)) ds. \tag{2.1}$$

Define the operator $u \mapsto Tu$ by

$$Tu(t) := \int_0^1 G(t,s) f(s, u(s)) ds, \tag{2.2}$$

where

$$G(t,s) := \begin{cases} s(1-t) & \text{for } 0 \le s < t, \\ t(1-s) & \text{for } t \le s \le 1, \end{cases}$$

is the Green function corresponding to the linear differential operator $-u''$ with the boundary value $u(0) = u(1) = 0$. Notice that

$$G(t,s) \le s(1-s) \quad \text{for all } s, t \in [0,1]. \tag{2.3}$$

**Remark 2.1.** By definition of the Banach space $X_\alpha$, we can see that for any $u \in X_\alpha$ there is $M > 0$ such that for all $t \in (0,1)$, one has

$$|u(t)| \le M t^\alpha (1-t)^\alpha.$$

**Theorem 2.1.** *Let $f : (0,1) \times \mathbb{R}^k \to \mathbb{R}^k$ be a continuous function. Assume that $f$ satisfies the following condition*
*$(H)$: there exist*

$$0 < \delta \le \frac{1}{2}, \quad c > 0, \quad \gamma \ge 2, \quad p > \gamma - 1$$

*such that for all $t \in (0,\delta) \cup (1-\delta, 1)$, and $|u| \le \delta$,*

$$|f(t,u)| \le \frac{c|u|^p}{t^\gamma (1-t)^\gamma}. \tag{2.4}$$

*Then if $\alpha \in (0,1)$ satisfies the inequality*

$$\alpha p + 1 > \gamma, \tag{2.5}$$

*the operator $T$ maps $X_\alpha$ into itself and is completely continuous.*

**Proof.** Let $u \in X_\alpha$. By Remark 2.1, there is $M > 0$ such that for any $t \in (0,1)$ we have

$$|u(t)| \le M t^\alpha (1-t)^\alpha.$$

This implies that there exists $0 < \delta_1 < \delta$ such that for $t \in (0,\delta_1) \cup (1-\delta_1, 1)$, one obtains $|u(t)| \le \delta$. Hence, by (2.2) and (2.3)

$$\begin{aligned} |Tu(t)| &= \left| \int_0^1 G(t,s) f(s,u(s)) \right| ds \\ &\le \left( \int_0^{\delta_1} + \int_{\delta_1}^{1-\delta_1} + \int_{1-\delta_1}^1 \right) s(1-s) |f(s,u(s))| ds \\ &\le c||u||_\alpha^p \left( \int_0^{\delta_1} + \int_{1-\delta_1}^1 \right) (s(1-s))^{\alpha p + 1 - \gamma} ds \end{aligned}$$

$$+\int_{\delta_1}^{1-\delta_1} s(1-s)|f(s,u(s))|ds.$$

The first summand is finite by (2.5). Since the interval $[\delta_1, 1-\delta_1]$ is closed and the function $u : [\delta_1, 1-\delta_1] \to \mathbb{R}^k$ is continuous, hence the subset $u([\delta_1, 1-\delta_1])$ is compact in $\mathbb{R}^k$.

Put

$$B := [\delta_1, 1-\delta_1] \times u([\delta_1, 1-\delta_1]).$$

Then $f_{|B}$ is a bounded function on the set $B$, i.e., there exists $M_1 > 0$ such that

$$|f(t,x)| \leq M_1, \tag{2.6}$$

for all $(t,x) \in B$, and in consequence $\int_{\delta_1}^{1-\delta_1} s(1-s)|f(s,u(s))|ds$ exists.Thus $T$ is well defined. Now we shall prove that $T$ maps $X_\alpha$ into itself. First we verify, using the Lebesgue dominated convergence theorem, that:

$$Tu \in C[0,1] \tag{2.7}$$

and

$$\sup_{t\in(0,1)} \frac{|Tu(t)|}{t^\alpha(1-t)^\alpha} < \infty. \tag{2.8}$$

In fact, let $\lim_{n\to\infty} t_n = t$. One has

$$\begin{aligned}|Tu(t_n) - Tu(t)| &\leq \left(\int_0^{\delta_1} + \int_{\delta_1}^{1-\delta_1} + \int_{1-\delta_1}^{1}\right) |G(t_n,s) - G(t,s)||f(s,u(s))|ds \\ &\leq \left(\int_0^{\delta_1} + \int_{1-\delta_1}^{1}\right) \varphi_n(s)(s(1-s))^{\alpha p-\gamma}ds + \int_{\delta_1}^{1-\delta_1} \psi_n(s)ds.\end{aligned}$$

where

$$\begin{aligned}\varphi_n(s) =& c||u||_\alpha^p |G(t_n,s) - G(t,s)| \\ \psi_n(s) =& M|G(t_n,s) - G(t,s)|,\end{aligned}$$

where

$$M = \sup\{|f(t,u)| : (t,u) \in B\} < \infty,$$

one obtains

$$\lim_{n\to\infty} \psi_n(s) = 0$$

(since $G$ is continuous), and

$$|\psi_n(s)| \leq 2s(1-s) =: g(s).$$

We see that the function $g$ is integrable. Therefore when $n \to \infty$

$$\int_{\delta_1}^{1-\delta_1} |G(t_n,s) - G(t,s)||f(s,u(s))|ds \to 0 \tag{2.9}$$

due to Lebesgue dominated convergence theorem. Now let

$$\varphi_n(s) = c||u||_\alpha^p |G(t_n, s) - G(t, s)|(s(1-s))^{\alpha p - \gamma}.$$

By the continuity of the function $G$, we have

$$\lim_{n\to\infty} \varphi_n(s) = 0,$$

and

$$\varphi_n(s) \le c(s(1-s))^{\alpha p - \gamma + 1}$$

which is an integrable function. Hence

$$\lim_{n\to\infty} \left( \int_0^{\delta_1} + \int_{1-\delta_1}^1 \right) \varphi_n(s) ds = 0. \tag{2.10}$$

Using (2.9)–(2.10), we have

$$\lim_{n\to\infty} Tu(t_n) = Tu(t).$$

Hence $Tu \in C[0,1]$. Notice that the map

$$t \mapsto \frac{1-t}{(t(1-t))^\alpha}$$

is decreasing, and the map

$$t \mapsto \frac{t}{(t(1-t))^\alpha}$$

is increasing for any $t \in (0,1)$, so one obtains

$$\frac{1-t}{(t(1-t))^\alpha} \le \frac{1-s}{(s(1-s))^\alpha} \quad \text{for } s \le t, \tag{2.11}$$

and

$$\frac{s}{(s(1-s))^\alpha} \le \frac{t}{(t(1-t))^\alpha} \quad \text{for } s \le t. \tag{2.12}$$

Let

$$H(t,s) := \frac{G(t,s)}{(t(1-t))^\alpha}. \tag{2.13}$$

By (2.11)–(2.12) we have

$$H(t,s) \le (s(1-s))^{1-\alpha} \quad \text{for all } s,t \in [0,1]. \tag{2.14}$$

Using (2.11), (2.12) and (2.6), one has

$$\begin{aligned} \sup_{t\in(0,1)} \frac{|Tu(t)|}{(t(1-t))^\alpha}| \le & c||u||_\alpha^p \left( \int_0^{\delta_1} + \int_{1-\delta_1}^1 \right) (s(1-s))^{\alpha p + 1 - \alpha - \gamma} ds \\ & + M_1 \int_{\delta_1}^{1-\delta_1} (s(1-s))^{1-\alpha} ds < \infty. \end{aligned}$$

This shows that for all $u \in X_\alpha$, $Tu \in X_\alpha$, so that $T : X_\alpha \to X_\alpha$. We shall show that $T$ is completely continuous. First we verify the continuity of $T$. In fact, let $\lim_{n\to\infty} u_n = u$ in $X_\alpha$. We prove that $Tu_n \to Tu$ in $X_\alpha$. The sequence $u_n$ as convergent is bounded, so there exists $M > 0$ such that $||u_n||_\alpha \le M$ for all $n \in \mathbb{N}$ and $||u||_\alpha \le M$. Therefore there is $\delta_1 \in (0, \delta)$ such that for $t \in (0, \delta_1) \cup (1 - \delta_1, 1)$, one has $|u_n(t)| \le \delta$ for $n \in \mathbb{N}$, and $|u(t)| \le \delta$. We have

$$\begin{aligned} ||Tu_n - Tu||_\alpha &= \sup_{t\in(0,1)} \left| \frac{Tu_n(t) - Tu(t)}{(t(1-t))^\alpha} \right| \\ &\le \left( \int_0^{\delta_1} + \int_{\delta_1}^{1-\delta_1} + \int_{1-\delta_1}^{1} \right) (s(1-s))^{1-\alpha} |f(s, u_n(s)) - f(s, u(s))| ds. \end{aligned}$$

Put

$$\psi_n(s) := (s(1-s))^{1-\alpha} |f(s, u_n(s)) - f(s, u(s))|.$$

One obtains

$$|\psi_n(s)| \le 2(s(1-s))^{1-\alpha} |h_M(s)|,$$

where

$$h_M(s) = \begin{cases} cM^p (s(1-s))^{\alpha p - \gamma}, & s \in (0, \delta_1) \cup (1 - \delta_1, 1) \\ M_1, & s \in [\delta_1, 1 - \delta_1]. \end{cases}$$

Then by the Lebesgue dominated convergence theorem $Tu_n \to Tu$, as $n \to \infty$, in $X_\alpha$.

We can see that the image $T(D)$ of any bounded set $D \subset X_\alpha$ is relatively compact in $X_\alpha$, i.e., the family $\{Fu : \ u \in D\}$, where

$$Fu(t) := \frac{Tu(t)}{(t(1-t))^\alpha},$$

is uniformly bounded by (2.8) and equicontinuous since the function $H$ has a continuous extension on the product $[0, 1] \times [0, 1]$ , so that $H$ is uniformly continuous, i.e., for a given $\varepsilon > 0$ there is $\eta > 0$, such that for any $s \in [0, 1]$ and $t_1, t_2 \in [0, 1]$ if $|t_1 - t_2| < \eta$ then

$$|H(t_1, s) - H(t_2, s)| \le \max \left( \frac{\varepsilon}{3cR^p \mu}, \frac{\varepsilon}{3M_0} \right),$$

where

$$\begin{aligned} M_0 &:= \max\{|f(t, u(t))| : \ t \in [\delta_1, 1 - \delta_1], \ u \in D\}, \\ \mu &:= \int_0^1 (s(1-s))^{\alpha p - \gamma} ds, \end{aligned}$$

and $R$ is a radius of the ball containing the set $D$. One has

$$|Fu(t_1) - Fu(t_2)| = \left| \int_0^1 (H(t_1, s) - H(t_2, s)) f(s, u(s)) ds \right|$$

$$
\begin{aligned}
&\leq \left(\int_0^{\delta_1}+\int_{\delta_1}^{1-\delta_1}+\int_{1-\delta_1}^{1}\right)|H(t_1,s)-H(t_2,s)||f(s,u(s))|ds\\
&\leq c||u||_\alpha^p\left(\int_0^{\delta_1}+\int_{1-\delta_1}^{1}\right)|H(t_1,s)-H(t_2,s)|(s(1-s))^{\alpha p-\gamma}ds\\
&+\int_{\delta_1}^{1-\delta_1}|H(t_1,s)-H(t_2,s)||f(s,u(s))|ds =: J_1+J_2.
\end{aligned}
$$

We have

$$
\begin{aligned}
J_1 \leq & c||u||_\alpha^p\left(\int_0^{\delta_1}+\int_{1-\delta_1}^{1}\right)\frac{\varepsilon}{3cR^p\mu}(s(1-s))^{\alpha p-\gamma}ds\\
\leq & \frac{\varepsilon}{3\mu}\left(\int_0^{\delta_1}+\int_{1-\delta_1}^{1}\right)(s(1-s))^{\alpha p-\gamma}ds \leq 2\varepsilon/3 \text{ if } |t_1-t_2|<\eta,
\end{aligned}
$$

and

$$
J_2 \leq \int_{\delta_1}^{1-\delta_1}|H(t_1,s)-H(t_2,s)||f(s,u(s))|ds \leq \frac{\varepsilon}{3} \text{ if } |t_1-t_2|<\eta.
$$

Therefore

$$
|Fu(t_1)-Fu(t_2)| \leq \varepsilon \quad \text{if } |t_1-t_2|<\eta.
$$

This means that the subset $F(D)$ consists of equibounded and equicontinuous functions. Using the Arzèla-Ascoli theorem we can conclude that $T$ is completely continuous. □

## 3. Application of Schauder theorem

The following theorem gives a solution to the problem under assumption of the sublinearity of $f$. Unfortunately, $f$ loses his strong singularity at $t=0$, $t=1$ for large $|u|$.

**Theorem 3.1.** *Assume*
*(i) there exist*

$$
0<\delta\leq\frac{1}{2},\quad c>0,\quad \gamma\geq 2,\quad p>\gamma-1
$$

*such that for any* $t\in(0,\delta)\cup(1-\delta,1)$, $u\in\mathbb{R}^k$

$$
|f(t,u)| \leq \frac{c|u|^p}{(t(1-t))^\gamma}, \tag{3.1}
$$

*(ii) there exist*

$$
E>0,\quad c_1>0,\quad \rho\in(0,1),\quad \nu\leq\frac{\gamma}{p-\rho+1}
$$

*such that for all* $t \in (0,1)$, *and* $|u| \geq E$, *one has*

$$|f(t,u)| \leq \frac{c_1|u|^\rho}{(t(1-t))^\nu}. \tag{3.2}$$

*Let* $\alpha \in (0,1)$ *satisfy the condition* (2.5). *Set*

$$\omega := \alpha p + 1 - \alpha - \gamma, \quad \xi := \alpha\rho + 1 - \alpha - \nu,$$

$$\lambda_2 := \int_\delta^{1-\delta} (s(1-s))^\xi ds \quad \text{and} \quad \lambda_1 := \int_\delta^{1-\delta} (s(1-s))^{1-\alpha} ds.$$

*Let*

$$M := \sup\{|f(t,u)| : |u| \leq E; \delta \leq t \leq 1-\delta\}.$$

*If there is a positive number* $R$ *such that*

$$R \ \geq \ cR^p \left(\int_0^\delta + \int_{1-\delta}^1\right) (s(1\,-\,s))^\omega ds \,+\, \max(M\lambda_1, c_1\lambda_2 R^\rho). \tag{3.3}$$

*Then the problem* (1.1)–(1.2) *has a solution in* $X_\alpha$ *with the norm* $||u||_\alpha \leq R$.

**Proof.** Let

$$\overline{B}(0,R) := \{u \in X_\alpha : ||u||_\alpha \leq R\}$$

be a closed ball in $X_\alpha$ centered at 0 with radius defined in (3.3). We shall prove that $T$ maps this ball into itself, i.e., $T(\overline{B}(0,R)) \subset \overline{B}(0,R)$. In fact, let $u \in \overline{B}(0,R)$, one obtains

$$\begin{aligned}\sup_{t\in(0,1)} \left|\frac{Tu(t)}{(t(1-t))^\alpha}\right| &\leq \left(\int_0^\delta + \int_\delta^{1-\delta} + \int_{1-\delta}^1\right) (s(1-s))^{1-\alpha}|f(s,u(s))|ds \\ &\leq cR^p \left(\int_0^\delta + \int_{1-\delta}^1\right) (s(1-s))^\omega ds \\ &+ \int_\delta^{1-\delta} (s(1-s))^{1-\alpha} \max(M, c_1||u||_\alpha^\rho (s(1-s))^{\alpha\rho-\nu}) ds.\end{aligned}$$

So for $||u||_\alpha \leq R$, using (3.3), one has $||Tu||_\alpha \leq R$. Hence $T$ has a fixed point due to Schauder Fixed Point Theorem. □

**Remark 3.1.** First assumption of the above theorem is slightly stronger than condition $(H)$ which guarantees the complete continuity of $T$.

**Example 3.1.**

$$f(t,u)=\begin{cases} \dfrac{u^4}{(t(1-t))^3} & \text{for } |u|\le 1 \\ \dfrac{|u|^{1/2}}{(t(1-t))^{1/2}} & \text{for } |u|\ge 4 \\ [\dfrac{2}{3(t(1-t))^{1/2}}-\dfrac{1}{3(t(1-t))^3}]|u|+\dfrac{4}{3(t(1-t))^3}-\dfrac{2}{3(t(1-t))^{1/2}} & \\ \qquad\qquad \text{for } |u|\in(1,4). & \end{cases}$$

Let $\delta = 1/2$ and $p = 4$, $\gamma = 3$ when $t$ is near the end points $t = 0$, $t = 1$ of the interval $[0, 1]$, and $u \in \mathbb{R}$ with $|u|$-small, and

$$\rho = \nu = \frac{1}{2}$$

when $t$ is everywhere on $(0, 1)$ with $|u| \ge 4 =: E$ and let $\alpha \in (1/2, 5/7)$, then by assumption, one has

$$\frac{1}{2} = \nu \le \frac{\gamma}{p-\rho+1} = \frac{2}{3}.$$

Let now

$$\alpha = \frac{5}{7}, \quad p = 4, \quad \gamma = 3, \quad \rho = \nu = \frac{1}{2}, \quad \delta = \frac{1}{2}, \quad \text{and} \quad c = c_1 = 1,$$

then for these values the condition $\alpha p + 1 > \gamma$ is satisfied and

$$\omega = 1-\alpha+\alpha p-\gamma = \frac{1}{7}, \quad \xi = 1-\alpha+\alpha\rho-\nu = \frac{1}{7},$$
$$\lambda_1 = \int_\delta^{1-\delta}(s(1-s))^{1-\alpha}ds = 0, \quad \lambda_2 = \int_\delta^{1-\delta}(s(1-s))^{\xi}ds = 0,$$
$$\left(\int_0^\delta + \int_{1-\delta}^1\right)(s(1-s))^{\omega}ds = \int_0^1 (s(1-s))^{\frac{1}{7}}ds = \frac{1}{294}.$$

It is easy to verify that there exists $R > 0$ which satisfies

$$\frac{1}{294}R^4 \le R.$$

**Remark 3.2.** The integral is computed by MAPLE 6.

## 4. Application of the topological degree

We shall use the following theorem

**Theorem 4.1** ([6, Lemma 2.5.1])**.** *Let $\Omega$ be a bounded open set in a real Banach space $E$, $0 \in \Omega$ and $A : \overline{\Omega} \to E$ be completely continuous. Suppose $Au \neq \mu u$, for all $u \in \partial\Omega$, $\mu \geq 1$. Then the operator $A$ has a fixed point in $\Omega$.*

**Theorem 4.2.** *Suppose condition $(H)$ holds and $\alpha$ satisfies* (2.5)*. Assume that there exists $M > 0$ such that for any $|u| > M(t(1-t))^\alpha$, one has*

$$(f(t,u),u) \leq 0 \quad \text{for all } t \in (0,1). \tag{4.1}$$

*Then the Dirichlet problem* (1.1)–(1.2) *has a solution in $X_\alpha$.*

**Proof.** First let us assume that inequality (4.1) is sharp. Let $B(0, M_0)$ be a ball in $X_\alpha$ centered at 0 with radius $M_0$, where $M_0 = M + 1$. We shall prove that the BVP

$$u'' = -\lambda f(t,u), \quad u(0) = u(1) = 0, \quad \text{for } \lambda \in (0,1], \tag{4.2}$$

has no solutions on $\partial B(0, M_0)$. Suppose on the contrary that there exist $\varphi$ and $\lambda > 0$ satisfying (4.2) such that $||\varphi||_\alpha = M_0$. Put

$$\psi(t) := \frac{\varphi(t)}{(t(1-t))^\alpha}.$$

Since $\varphi$ satisfies (4.2), then it is of the form

$$\varphi(t) = \lambda \int_0^1 G(t,s) f(s, \varphi(s)) ds.$$

We can see that

$$\varphi'(t) = -\lambda \int_0^t s f(s,\varphi(s)) ds + \lambda \int_t^1 (1-s) f(s,\varphi(s)) ds$$

So by condition $(H)$, one has $\varphi'$ is a bounded function and then, due to the l'Hospital theorem

$$\lim_{t \to 0,1} \psi(t) \to 0.$$

Since the function $\psi$ is continuous, hence there is $t_0 \in (0,1)$ such that

$$\left| \frac{\varphi(t_0)}{(t_0(1-t_0))^\alpha} \right| = M_0.$$

Then $|\psi(t_0)| = M_0$, and one has

$$0 = \frac{d}{dt} |\psi(t)|^2_{t=t_0} = \frac{d}{dt} (\psi(t), \psi(t))_{t=t_0} = 2(\psi'(t_0), \psi(t_0)).$$

Using (4.1)

$$0 \geq \frac{d^2}{dt^2}|\psi(t)|^2_{t=t_0} = 2(\psi''(t_0), \psi(t_0)) > 0 \tag{4.3}$$

a contradiction. Any solution of (4.2) is a zero of the operator $(1/\lambda)I - T$. Hence, by Theorem 4.1 with $A = T$, $\mu = 1/\lambda$, the equation $u - Tu = 0$ has a solution $u \in \Omega$.

Now, pass to the general case: inequality (4.1) is as in the statement of the theorem. Perturbing the right-hand side of the differential equation by $-(1/n)u$, where $n \in \mathbb{N}$, we have a solution $u_n$ by the first part of the proof. It is easily seen that the sequence $(u_n)_n$ satisfies the assumptions of the Arzéla-Ascoli theorem. Thus it has a uniformly convergent subsequence $u_{n_m} \to u$. The limit is a solution of the main problem. □

**Example 4.1.** Let

$$f(t,u) = \frac{-u^{2n+1}}{(t(1-t))^\gamma} + h(t),$$

where $h \in X_\alpha$, $t \in (0,1)$, $u \in \mathbb{R}^+$, $2n+1 > \gamma - 1$, and $1 > \alpha > (\gamma-1)/(2n+1)$. If $h \in X_\alpha$ we have

$$|h(t)| \leq M(t(1-t))^\alpha,$$

and for $|u| \geq M(t(1-t))^\alpha$, one has

$$uf(t,u) \leq 0.$$

It is obvious, since

$$|uh(t)| \leq \frac{|u|^{2n+2}}{(t(1-t))^\gamma}$$

for such $u$.

## 5. The Carathéodory conditions

Consider the Dirichlet problem (1.1)–(1.2) with $f : (0,1) \times \mathbb{R}^k \to \mathbb{R}^k$ satisfying the Carathéodory conditions, i.e., $f(\cdot,u) : t \mapsto f(t,u)$ is measurable on $(0,1)$ for each $u \in \mathbb{R}^k$ and $f(t,\cdot) : u \mapsto f(t,u)$ is continuous on $\mathbb{R}^k$ for almost all $t \in (0,1)$. Let

$$L_{\{1-\alpha\}} := \left\{ h : \int_0^1 (s(1-s))^{1-\alpha}|h(s)|ds < \infty \right\}, \quad \alpha \in (0,1) \tag{5.1}$$

be the $L^1$-space for the measure $\mu$ on $[0,1]$ defined by the formula

$$\mu(A) = \int_A (s(1-s))^{1-\alpha} ds.$$

**Theorem 5.1.** *Suppose that $f$ satisfies the Carathéodory conditions, and for any $M > 0$ there is $h_M \in L_{\{1-\alpha\}}$, such that for any $|u| \leq M(t(1-t))^{\alpha}$ we obtain*

$$|f(t,u)| \leq h_M(t)$$

*for a. e. $t \in (0,1)$. Then the operator $T$ is completely continuous from $X_\alpha$ into $X_\alpha$, where $T$ is of the form (2.2).*

**Proof.** By assumption and by the fact that

$$t(1-t) \leq (t(1-t))^{1-\alpha},$$

for $\alpha \in (0,1)$, and for any $t \in [0,1]$, one has

$$|G(t,s)f(s,\varphi(s))| \leq s(1-s)|h_M(s)|$$

for $||\varphi||_\alpha \leq M$. This implies that

$$\int_0^1 s(1-s)|h_M(s)|ds < \infty,$$

and in consequence (2.2) exists. We prove that the operator $T$ maps $X_\alpha$ into itself. Let

$$Fu(t) := \frac{Tu(t)}{(t(1-t))^{\alpha}}.$$

So the operator $F$ is of the form:

$$Fu(t) := \frac{Tu(t)}{(t(1-t))^{\alpha}} = \int_0^1 H(t,s)f(s,u(s))ds.$$

Let $\varphi \in X_\alpha$ such that $||\varphi||_\alpha \leq M$, so there exists $h_M \in L_{\{1-\alpha\}}$, and

$$\begin{aligned}|F\varphi(t) - F\varphi(t_0)| &\leq \int_0^1 |H(t,s) - H(t_0,s)||f(s,\varphi(s))|ds \\ &\leq \int_0^1 |H(t,s) - H(t_0,s)||h_M(s)|ds.\end{aligned}$$

Let

$$\psi_t(s) := |H(t,s) - H(t_0,s)||h_M(s)|.$$

Since the function $H$ is uniformly continuous on the product $[0,1] \times [0,1]$, then

$$\lim_{t \to t_0} |H(t,s) - H(t_0,s)||h_M(s)| = 0$$

uniformly with respect to $t_0 \in [0,1]$, and

$$|\psi_t(s)| \leq 2(s(1-s))^{1-\alpha}|h_M(s)| =: g(s).$$

Hence, using the Lebesgue dominated convergence theorem again, one has

$$\lim_{t \to t_0} \int_0^1 |H(t,s) - H(t_0,s)||h_M(s)|ds = 0.$$

So

$$\lim_{t \to t_0} |F\varphi(t) - F\varphi(t_0)| = 0 \tag{5.2}$$

for any $t_0 \in (0, 1)$. This means that the function $F\varphi$ is continuous on $[0, 1]$, so that $F\varphi$ is bounded and in consequence $T\varphi \in X_\alpha$ for any $\varphi \in X_\alpha$. Now we shall prove that the subset $T(\overline{B}(0, M))$ is relatively compact in $X_\alpha$, i.e., the subset $F(\overline{B}(0, M))$ consists of equibounded and equicontinuous functions. By (5.2), for any $\varepsilon > 0$ there is $\delta > 0$ such that if $|t - t_0| < \delta$ implies

$$\int_0^1 |H(t, s) - H(t_0, s)||h_M(s)|ds \leq \varepsilon,$$

then for any function $||\varphi||_\alpha \leq M$ and $|t - t_0| < \delta$

$$|F\varphi(t) - F\varphi(t_0)| \leq \int_0^1 |H(t, s) - H(t_0, s)||h_M(s)|ds \leq \varepsilon,$$

i.e., the family $\{F\varphi : ||\varphi||_\alpha \leq M\}$ is equicontinuous. Moreover for $\varphi \in X_\alpha$ and $||\varphi||_\alpha \leq M$ there exists $h_M \in L_{\{1-\alpha\}}$ such that

$$\sup_{t \in (0,1)} |F\varphi(t)| \leq \sup_{t \in (0,1)} \int_0^1 H(t, s)|f(s, \varphi(s)|ds$$
$$\leq \int_0^1 (s(1 - s))^{1-\alpha}|h_M(s)|ds =: N < +\infty.$$

Then the family $\{F\varphi : ||\varphi||_\alpha \leq M\}$ is equibounded. By Arzéla-Ascoli theorem the operator $T$ is compact in $X_\alpha$. Now we shall prove that $T$ is continuous. In fact, let $\varphi_n$ be a sequence of elements in $X_\alpha$, converging to some function $\varphi$ of $X_\alpha$, i.e.,

$$||\varphi_n - \varphi||_\alpha \to 0,$$

when $n \to \infty$. There is $M > 0$ such that $||\varphi_n||_\alpha \leq M$ for all $n \in \mathbb{N}$ and $||\varphi||_\alpha \leq M$. By assumption on $f$, one has

$$\lim_{n \to \infty} f(t, \varphi_n(t)) = f(t, \varphi(t))$$

for almost all $t$. Let $\varepsilon > 0$. Since the integral

$$\int_0^1 (s(1 - s))^{1-\alpha}|h_M(s)|ds$$

exists then there is $\delta > 0$ such that for $J \subset I$, where $I = [0, 1]$, and $\mu(J) < \delta$

$$\int_J (s(1 - s))^{1-\alpha}|h_M(s)|ds \leq \frac{\varepsilon}{4\mu(I)}.$$

Using the Egoroff theorem there is $J_1 \subset I$ such that for $\mu(J_1) \leq \delta$

$$\lim_{n \to \infty} f(t, \varphi_n(t)) = f(t, \varphi(t))$$

uniformly on $I - J_1$, so there exists $n_0 \in \mathbb{N}$, and for $n \geq n_0$

$$(s(1-s))|f(s,\varphi_n(s)) - f(s,\varphi(s))| \leq \frac{\varepsilon}{2\mu(I)}$$

for $s \in I - J_1$. Therefore for all $n \geq n_0$

$$\begin{aligned}\sup_{t\in(0,1)} |F\varphi_n(t) - F\varphi(t)| \leq & \left(\int_{I-J_1} + \int_{J_1}\right) H(t,s)|f(s,\varphi_n(s)) - f(s,\varphi(s))|ds \\ \leq & \frac{\varepsilon}{2\mu(I)} \int_{I-J_1} \mu ds + 2\int_{J_1} (s(1-s))^{1-\alpha}|h_M(s)|ds \\ \leq & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Then

$$||T\varphi_n - T\varphi||_\alpha \leq \varepsilon.$$

This means that the operator $T$ is continuous. □

Now to prove that the problem (1.1)–(1.2) has a solution, we can repeat Theorem 4.2 and the application of the Theorem 3.1 for Carathéodory functions.

## 6. Positive solutions

Now we look for a positive solution to problem (1.1)–(1.2) in dimension $k = 1$, for simplicity. Function $f : (0,1) \times \mathbb{R}_+ \to \mathbb{R}_+$ ($\mathbb{R}_+ = [0,\infty)$) is supposed to be continuous or to satisfy the Carathéodory conditions. Let $E$ be a real Banach space and $P$ denote a cone in $E$, i.e. $P \subset E$ is a nonempty closed convex set such that

$$\begin{aligned} u \in & P, \quad \lambda \geq 0 \quad \Rightarrow \quad \lambda u \in P; \\ u \in & P \cap (-P) \quad \Rightarrow \quad u = 0.\end{aligned}$$

This cone defines a partial order in $E$:

$$u \leq v \quad \Leftrightarrow \quad v - u \in P$$

and one can set

$$[u,v] := \{w \in E : \quad u \leq w \leq v\}.$$

Any operator defined on a subset of $E$ is called increasing if

$$u \leq v \quad \Rightarrow \quad Tu \leq Tv.$$

If $u \leq v$ and $u \neq v$, we write $u < v$.

We shall use the following theorem on fixed points for increasing operators.

**Theorem 6.1** ([6, Theorem 2.1.3])**.** *Let $E$ be a real Banach space, let $u_0, v_0 \in E$, $u_0 < v_0$ and $T : [u_0, v_0] \to E$ be an increasing operator such that $u_0 \leq Tu_0$, $Tv_0 \leq v_0$. Suppose that $T([u_0, v_0])$ is a relatively compact subset of $E$. Then $T$ has at least one fixed point in $[u_0, v_0]$.*

Now let $P_\alpha := \{u \in X_\alpha : \ u(t) \geq 0, \ t \in [0,1]\}$ be a cone in the real Banach space $X_\alpha$. Similarly as in Section 2, we can prove that under condition $(H)$, (Theorem 2.1), the operator $T$ maps $X_\alpha$ into itself, and $T : X_\alpha \to X_\alpha$ is completely continuous.

**Theorem 6.2.** *Let $f$ be a non-negative continuous function satisfying the condition $(H)$, and there exist $r_0 > 0$, $c_0 > 0$, $\beta > -2$ such that for any $t \in (0,1)$, one has:*

$$f(t, r_0 t(1-t)) \geq c_0 (t(1-t))^\beta, \tag{6.1}$$

$$\lambda c_0 \geq r_0, \tag{6.2}$$

*where*

$$\lambda = \int_0^1 (s(1-s))^{1+\beta} ds. \tag{6.3}$$

*Suppose that there exists $\alpha \leq 1/2$ satisfying the condition*

$$\alpha p + 1 > \gamma$$

*and*

$$\lim_{u \to \infty} \sup_{t \in (0,1)} \frac{f(t,u)}{u} (t(1-t))^\alpha = 0, \tag{6.4}$$

$$f(t, \cdot) \text{ is non-decreasing on } \mathbb{R}^+ \text{ for } 0 < t < 1. \tag{6.5}$$

*Then the operator $T$ has a fixed point in $P_\alpha$.*

**Proof.** Define the operator $u \mapsto Tu$ as in (2.2). Put

$$u_0(t) = r_0 t(1-t). \tag{6.6}$$

Since $t(1-t) \leq (t(1-t))^\alpha$ then $u_0(t) \leq r_0(t(1-t))^\alpha$ and using (6.1), one has

$$\begin{aligned} Tu_0(t) =& \int_0^1 G(t,s) f(s, u_0(s)) ds \\ \geq& c_0 r_0 \left( (1-t) \int_0^t s^{1+\beta} (1-s)^\beta ds + t \int_t^1 s^\beta (1-s)^{\beta+1} ds \right) \\ \geq& c_0 r_0 t(1-t) \int_0^1 (s(1-s))^{1+\beta} ds. \end{aligned}$$

By (6.2)–(6.3), one has

$$c_0 r_0 t(1-t)\int_0^1 (s(1-s))^{1+\beta} ds \geq r_0 t(1-t) = u_0(t).$$

Therefore $Tu_0 \geq u_0$. Using (6.4) one obtains: there exists $R > r_0$ such that

$$\frac{f(t,R)}{R} \leq \frac{1}{(t(1-t))^{\alpha}}, \tag{6.7}$$

for any $t \in (0,1)$. Let $v_0(t) = R(t(1-t))^{1-\alpha}$. We observe that $v_0 \in X_\alpha$, (since $\alpha \leq 1/2$), $v_0(t) < R$, for all $t \in [0,1]$, and $\alpha \in (0,1)$, $u_0(t) < v_0(t)$, for any $t \in [0,1]$. By (6.7), (2.11) and (2.12) we have

$$\begin{aligned} Tv_0(t) = &\int_0^1 G(t,s) f(s, v_0(s)) ds \leq \int_0^1 G(t,s) f(s,R) ds \\ \leq &R\int_0^t (1-t)\frac{sds}{(s(1-s))^{\alpha}} + R\int_t^1 t\frac{(1-s)ds}{(s(1-s))^{\alpha}} \\ \leq &R\int_0^t (1-t)\frac{tds}{(t(1-t))^{\alpha}} + R\int_t^1 t\frac{(1-t)ds}{(t(1-t))^{\alpha}} \\ = &R(t(1-t))^{1-\alpha}\int_0^1 ds = v_0(t). \end{aligned}$$

So for $0 \leq t \leq 1$ we have

$$Tv_0(t) \leq v_0(t).$$

We can apply Theorem 6.1. Therefore the operator $T$ has one positive solution. □

**Example 6.1.** Let $p > \gamma - 1$. The following function satisfies all assumptions of the last theorem.

$$f(t,u) := \begin{cases} \dfrac{u^p}{t^\gamma (1-t)^\gamma} & \text{for } 0 \leq u \leq t(1-t), \\ t^{p-\gamma}(1-t)^{p-\gamma} & \text{for } u > t(1-t). \end{cases}$$

For $0 \leq u \leq v \leq t(1-t)$,

$$f(t,u) - f(t,v) = \frac{u^p - v^p}{t^\gamma (1-t)^\gamma} \leq 0,$$

(since $p > \gamma - 1 \geq 1$). For $u \leq t(1-t)$, and $v > t(1-t)$, one has

$$\begin{aligned} f(t,u) - f(t,v) = &\frac{u^p}{t^\gamma(1-t)^\gamma} - t^{p-\gamma}(1-t)^{p-\gamma} \\ \leq &t^{p-\gamma}(1-t)^{p-\gamma} - t^{p-\gamma}(1-t)^{p-\gamma} = 0. \end{aligned}$$

Now for $t(1-t) < u \leq v$, we have

$$f(t,u) - f(t,v) = 0.$$

Hence the function $f(t,\cdot)$ is increasing for any $t \in (0,1)$. We can see that

$$\lim_{u\to\infty} \frac{f(t,u)}{u} t^{\alpha}(1-t)^{\alpha} = 0.$$

If $u > r_0 t(1-t)$ and $r_0 \geq 1$, then by (6.5) we note that

$$f(t,u) \geq c_0 t^{\beta}(1-t)^{\beta}$$

and from the assumption, one has

$$f(t,u) = t^{p-\gamma}(1-t)^{p-\gamma}.$$

So the inequality

$$t^{p-\gamma}(1-t)^{p-\gamma} \geq c_0 t^{\beta}(1-t)^{\beta}$$

must be satisfied and, in consequence, $p-\gamma \leq \beta$.

**Remark 6.1.** All examples in this paper are not natural and complicated but they demonstrate the fact that such examples exist.

**Remark 6.2.** We have tried to apply the Krasnoselskii Fixed Point Theorem for cone-expansion maps but it seems that it is impossible in our situation.

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Habib Sebai
Faculty of Mathematics
University of Łódź
Banacha 22
90-238 Łódź, Poland
e-mail: habibsebai@op.pl