Journal of Applied Analysis
Vol. 11, No. 2 (2005), pp. 261–282

EULER-POINCARÉ FORMALISM OF COUPLED KDV TYPE SYSTEMS AND DIFFEOMORPHISM GROUP ON S^1

P. GUHA

Received September 1, 2003 and, in revised form, March 21, 2004

Abstract. This paper describes a wide class of coupled KdV equations. The first set of equations directly follow from the geodesic flows on the Bott-Virasoro group with a complex field. But the set of 2-component systems of nonlinear evolution equations, which includes dispersive water waves, Ito's equation, many other known and unknown equations, follow from the geodesic flows of the right invariant L^2 metric on the semidirect product group $\widehat{\mathrm{Diff}(S^1) \ltimes C^\infty}(S^1)$, where $\mathrm{Diff}(S^1)$ is the group of orientation preserving diffeomorphisms on a circle. We compute the Lie-Poisson brackets of the Antonowicz-Fordy system, and the mode expansion of these beackets yield the twisted Heisenberg-Virasoro algebra. We also give an outline to study geodesic flows of a H^1 metric on $\widehat{\mathrm{Diff}(S^1) \ltimes C^\infty}(S^1)$.

2000 *Mathematics Subject Classification.* 53A07, 53B50.
Key words and phrases. Diffeomorphism, geodesic flows, coupled KdV equations, Bott-Virasoro group, twisted Heisenberg-Virasoro algebra.

ISSN 1425-6908

Contents

1. Introduction

It is known that the periodic Korteweg-de Vries (KdV) equation can be interpreted as a geodesic flow of the right invariant metric on the Bott-Virasoro group, which at the identity is given by the L^2-inner product [25, 26, 27, 30].

Recently Misiolek [23] showed that an analogous correspondence can be established for the Camassa-Holm equation [7]. It gives rise to a geodesic flow of a certain right invariant Sobolev metric H^1 on the Bott-Virasoro group. In fact another well known equation, the Hunter-Saxton (or Harry-Dym) equation, also follows from geodesic flow [12, 16] on the Bott-Virasoro group.

Thus we see the KdV and the Camassa-Holm equations arise in a unified geometric construction, both are integrable systems which describe geodesic flows on the Bott-Virasoro group. Earlier it was known that both the KdV and the Camassa-Holm are obtained from different regularisations of the Euler equation for a one dimensional compressible fluid. The Euler equation [5], of course, describes geodesic motion on the group of orientation preserving diffeomorphisms of the circle $\mathrm{Diff}(S^1)$ with respect to L^2 metric [9].

In this paper we study integrable systems associated with the semi-direct product group [cf. 8] $\mathrm{Diff}(S^1) \ltimes C^\infty(S^1)$. The Lie algebra of $\mathrm{Diff}(S^1) \ltimes C^\infty(S^1)$ is known in physical literature [4, 13, 15, 21]. It has a three dimensional extension (explained in the next section) $\mathrm{Vect}(S^1) \ltimes C^\infty(S^1) \oplus \mathbb{R}^3$, where $\mathrm{Vect}(S^1) = Lie(\mathrm{Diff}(S^1))$ is the Lie algebra of smooth vector fields on S^1. The Lie algebra $\mathrm{Vect}(S^1)$ admits a nontrivial one-dimensional central extension defined by the Gelfand-Fuchs 2-cocycle

$$\omega(f_1, f_2) = \int_{S^1} f_1'(x) f_2''(x)\, dx,$$

where $f_i(x)\dfrac{d}{dx}$ is the Lie algebra of $\mathrm{Vect}(S^1)$. The classical Virasoro algebra is the central extension of $\mathrm{Vect}(S^1)$. Then a typical element of this algebra would be

$$(f\frac{d}{dx}, u(x), \alpha) \quad \text{where } f\frac{d}{dx} \in \mathrm{Vect}(S^1),\ u(x) \in C^\infty(S^1),\ \alpha \in \mathbb{R}^3.$$

It was shown by Ovsienko and Roger [25] that the cocycles define the universal central extension of the Lie algebra of $\mathrm{Vect}(S^1) \ltimes C^\infty(S^1)$. This means $H^2(\mathrm{Vect}(S^1) \ltimes C^\infty(S^1)) = \mathbb{R}^3$. The $\mathrm{Diff}(\widehat{S^1) \ltimes C}^\infty(S^1)$ is the nontrivial extension of $\mathrm{Diff}(S^1) \ltimes C^\infty(S^1)$.

In this paper we investigate geodesic flows [5, 22] on the $\mathrm{Diff}(\widehat{S^1) \ltimes C}^\infty(S^1)$, which at the identity is given by the L^2 inner product. These are all completely integrable coupled nonlinear third order partial differential equations.

1.1. Brief history and formulation of coupled KdV type systems.

Since 80's, the coupled KdV systems are considered to be important mathematical models. These set of equatios are used in various physical phenomena. In 1981, Fuchssteiner [10] made a detailed study of four coupled KdV equation and formulated the bihamiltonian structure of them. One of them turned out to be a complex version of the KdV. Immediately after that Hirota and Satsuma[1] introduced a coupled KdV equations

$$\begin{aligned} p_t + p_{xxx} + pp_x - qq_x &= 0\,, \\ q_t + q_{xxx} + pq_x + p_x q &= 0\,. \end{aligned} \tag{1}$$

Researchers have studied the bihamiltonian structures and Lax pairs of these class of systems. So it became necessary to have a Lie algebraic framework to study such systems. Given an isospectral flow of an appropriate eigenvalue problem, it is known that the strong symmetry constructed via

[1] The history and detail structure of these equations can be found in Ablowitz and Clarkson [1] and J. P. Wang's review [29].

the spectral gradient approach is a hereditary operator provided spectral gradient functions are dense.

In late eighties Antonowicz and Fordy [3] investigated second order energy dependent spectral parameter and found their isospectral flows have multi-Hamiltonian structure. This approach gave a very simple and elegant construction of the associated Hamiltonian operators. This method can be applied to Kuperschmidt's [19] nonstandard Lax operators and also to super Lax equations.

Let us apply the Antonowicz-Fordy scheme to Kuperschmidt's nonstandard Lax operators

$$\mathbf{L}\phi = (\varepsilon\partial^2 - r\partial + q)\phi = 0, \tag{2}$$

where ε, r and q are now polynomials in λ and construct the associated Hamiltonian operators. Equations (1) follow directly from the compatibility condition of (2) and

$$\phi_t = \mathbf{P}\phi \equiv \frac{1}{2}(P\partial + Q)\phi, \tag{3}$$

where P and Q are functions of u_i and the spectral parameter λ.

Taking some special values of ε_i, they derive a tri-Hamiltonian dispersive water waves hierarchy. The first nontrivial member of this hierarchy is

$$\begin{aligned} u_{0t} &= \frac{1}{4}u_{1xxx} + \frac{1}{2}u_1u_{0x} + u_0u_{1x}\ , \\ u_{1t} &= u_{0x} + \frac{3}{2}u_1u_{1x}\ . \end{aligned} \tag{4}$$

An invertible change of variables

$$\begin{aligned} q &= u_0 + \frac{1}{4}{u_1}^2 - \frac{1}{2}u_{1x}\ , \\ r &= u_1\ . \end{aligned} \tag{5}$$

transforms equation (4) into a standard dispersive water waves equation

$$\begin{aligned} q_t &= \frac{1}{2}(q_x + 2qr)_x\ , \\ r_t &= \frac{1}{2}(r_x + 2q + r^2)_x\ . \end{aligned} \tag{6}$$

The hierarchy has a tri-Hamiltonian structure and is the first among a new kind of integrable system which have come to be known in the literature as non-standard integrable systems.

The Antonowicz-Fordy scheme can be used to generate several interesting coupled integrable systems. Recently, Alber et. al. showed that in case of certain potentials, a limiting procedure can be applied to generate solutions, which results in solutions with peaks [2 and references therein].

Finally, it must be worth to note that the the another class of coupled systems can be obtained from the L^2 geodesic flows on the superconformal group. The geodesic flow on superconformal group yields super KdV (sKdV) equation

$$\begin{aligned} u_t &= 6uu_x - u_{xxx} + 3\phi\phi_x \,, \\ \phi_t &= 3u_x\phi + 6u\phi_x - 4\phi_{xxx} \,, \end{aligned} \tag{7}$$

where u, x and t are even (commutating), while ϕ is odd (anticommutating).

We also compute the Lie-Poisson structure of the Antonowicz-Fordy system. It turns out that the mode expansion of these brackets yield the classical analogue of twisted Heisenberg-Virasoro algebra [4, 6]. This algebra has an infinite-dimensional Heisenberg subalgebra and a Virasoro subalgebra [17, 18].

The twisted Heisenberg-Virasoro algebra has been studied by Arbarello et. al. in [4]. They established a connection between the second cohomology of certain moduli space of curves and the second cohonology of this Lie algebra of differential operators of orcer at most one.

1.2. Motivation and result.

In this paper we study various coupled KdV type systems. We show that one class of systems can be derived straight from the geodesic flows on the Bott-Viarsoro group. These are known as Hirota-Satsuma systems.

We demontrate that almost all the coupled KdV type systems derived from the Antonowicz-Fordy scheme can be obtained from the geodesic flows on the $\mathrm{Diff}(S^1)\widehat{) \ltimes C}^\infty(S^1)$.

There are several equations arose from their scheme can be manifested as geodesics flows on $\mathrm{Diff}(S^1)\widehat{) \ltimes C}^\infty(S^1)$. Thus we unify all these coupled KdV systems geometrically. In our earlier papers [11, 13] we have already shown that the Ito and various dispersive water waves equations follow from the geodesic flows on $\mathrm{Diff}(S^1)\widehat{) \ltimes C}^\infty(S^1)$. We have also studied [14] the bihamiltonian structures of these flows.

Let us state the results of our paper.

Theorem 1. *Let $t \mapsto \hat{\mathcal{C}}$ be a curve in the $\mathrm{Diff}(S^1)\widehat{) \ltimes C}^\infty(S^1)$. Let $\hat{\mathcal{C}} = (e, e, 0)$ be the initial point, directing to the vector*

$$\hat{\mathcal{C}}(0) = (u(x)\frac{d}{dx}, v(x), c),$$

where $c \in \mathbb{R}^3$. Then $\hat{\mathcal{C}}(t)$ is a geodesic of the L^2 metric

(A) *if and only if* $(u(x,t)\frac{d}{dx}, v(x,t), c)$ *satisfies the dispersive water waves type equation when the motion is restricted to hyperplane* $c = (-1, 0, \frac{1}{2})$

$$\begin{aligned} u_t &= v_{xxx} + 3(uv)_x + uv_x \\ v_t &= u_x + 4vv_x \end{aligned} \tag{8}$$

(B) *if and only if* $(u(x,t)\frac{d}{dx}, v(x,t), c)$ *satisfies the Ito equation when the motion is restricted to hyperplane* $c = (-1, 0, 0)$

$$\begin{aligned} u_t &= u_{xxx} + 6uu_x + 2vv_x \\ v_t &= 2(uv)_x \end{aligned} \tag{9}$$

(C) *if and only if* $(u(x,t)\frac{d}{dx}, v(x,t), c)$ *satisfies the Kaup-Boussinesq system when the motion is restricted to hyperplane* $c = (-\frac{1}{4}, 0, \frac{1}{2})$

$$\begin{aligned} u_t &= uu_x + v_x \\ v_t &= \frac{1}{4}u_{xxx} + (uv)_x \end{aligned} \tag{10}$$

(D) *if and only if* $(u(x,t)\frac{d}{dx}, v(x,t), c)$ *satisfies the Broer-Kaup system when the motion is restricted to hyperplane* $c = (0, -1, -1)$

$$\begin{aligned} u_t &= -u_{xx} + 2(uv)_x + uu_x \\ v_t &= v_{xx} + 2vv_x - 2u_x \end{aligned} \tag{11}$$

Theorem 2. *The Fourier expansion of the Lie-Poisson brackets associated with the Antonowicz-Fordy Hamiltonian structure*

$$\mathcal{O} = \begin{pmatrix} -c_1D^3 + 2uD + u_x & vD + c_2D^2 \\ v_x + vD - c_2D^2 & 2c_3D \end{pmatrix}$$

yield the classical analogue of twisted Heisenberg-Virasoro algebra.

We also derive the Hirota-Satsuma, the Nutku-Oguz and the Hénon-Heiles systems as an Euler-Poincaré flows on the Bott-Virasoro group.

This paper is organized as follows: In Section 2 we discuss the geodesic flows on $\widehat{\mathrm{Diff}(S^1) \ltimes C^\infty}(S^1)$ with respect to L^2 metric. This construction yields several examples of coupled KdV equations considered by Antonowicz and Fordy. In Section 3 we present lots of examples. Section 4 is devoted to the Lie-Poisson structures of the Antonowicz-Fordy system. In Section 5 we discuss another class of coupled KdV equations follow from the geodesic flows on the Bott-Virasoro group. Section 6 is devoted geodesic flows with respect to H^1 metric.

2. Coupled KdV type equations and $\mathbf{L^2}$ metric on Bott-Virasoro group

Let $\mathrm{Diff}(S^1)$ be the group of orientation preserving diffeomorphisms of a circle. It is known that the group $\mathrm{Diff}(S^1)$ as well as its Lie algebra of vector fields on S^1, $T_{id}\,\mathrm{Diff}(S^1) = \mathrm{Vect}(S^1)$, have non-trivial one-dimensional central extensions, the Bott-Virasoro group $\widehat{\mathrm{Diff}}(S^1)$ and the Virasoro algebra Vir respectively [16, 17].

The Lie algebra $\mathrm{Vect}(S^1)$ is the algebra of smooth vector fields on S^1. This satisfies the commutation relations

$$[f\frac{d}{dx}, g\frac{d}{dx}] := (f(x)g'(x) - f'(x)g(x))\frac{d}{dx}. \tag{12}$$

One parameter family of $\mathrm{Vect}(S^1)$ acts on the space of smooth functions $C^\infty(S^1)$ by

$$L^{(\mu)}_{f(x)\frac{d}{dx}} a(x) = f(x)a'(x) - \mu f'(x)a(x), \tag{13}$$

where

$$L^{(\mu)}_{f(x)\frac{d}{dx}} = f(x)\frac{d}{dx} - \mu f'(x)$$

is the derivative with respect to the vector field $f(x)\dfrac{d}{dx}$.

The Lie algebra of $\mathrm{Diff}(S^1) \ltimes C^\infty(S^1)$ is the semidirect product Lie algebra

$$\mathcal{G} = \mathrm{Vect}(S^1) \ltimes C^\infty(S^1).$$

An element of $\mathcal{G}$ is a pair $(f(x)\dfrac{d}{dx}, a(x))$, where $f(x)\dfrac{d}{dx} \in \mathrm{Vect}(S^1)$ and $a(x) \in C^\infty(S^1)$.

It is known that this algebra has a three dimensional central extension given by the non-trivial cocycles

$$\omega_1((f\frac{d}{dx}, a), (g\frac{d}{dx}, b)) = \int_{S^1} f'(x)g''(x)dx \tag{14}$$

$$\omega_2((f\frac{d}{dx}, a), (g\frac{d}{dx}, b)) = \int_{S^1} f''(x)b(x) - g''a(x))dx \tag{15}$$

$$\omega_3((f\frac{d}{dx}, a), (g\frac{d}{dx}, b)) = 2\int_{S^1} a(x)b'(x)dx. \tag{16}$$

The first cocycle ω_1 is the well known Gelfand-Fuchs cocycle. The Virasoro algebra is the unique non-trivial central extension of $\mathrm{Vect}(S^1)$ via this ω_1 cocycle. Hence we define the Virasoro algebra

$$Vir = \mathrm{Vect}(S^1) \oplus \mathbb{R}.$$

The space $C^\infty(S^1) \oplus \mathbb{R}$ is identified with a part of the dual space to the Virasoro algebra. It is called the *regular part*, and the pairing between this space and the Virasoro algebra is given by:

$$\langle (u(x), a), (f(x)\frac{d}{dx}, \alpha) \rangle = \int_{S^1} u(x)f(x)dx + a\alpha.$$

Similarly we consider an extension of $\mathcal{G}$. This extended algebra is given by

$$\hat{\mathcal{G}} = \mathrm{Vect}^s(S^1) \ltimes C^\infty(S^1) \oplus \mathbb{R}^3. \tag{17}$$

Definition 1. The commutation relation in $\hat{\mathcal{G}}$ is given by

$$[(f\frac{d}{dx}, a, \alpha), (g\frac{d}{dx}, b, \beta)] := ((fg' - f'g)\frac{d}{dx}, fb' - ga', \omega) \tag{18}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$, $\omega = (\omega_1, \omega_2, \omega_3)$ are the two cocycles.

The dual space of smooth functions $C^\infty(S^1)$ is the space of distributions (generalized functions) on S^1. Of particular interest are the orbits in $\hat{\mathcal{G}}^*_{reg}$. In the case of current group, Gelfand, Vershik and Graev have constructed some of the corresponding representations.

Definition 2. The regular part of the dual space $\hat{\mathcal{G}}^*$ to the Lie algebra $\hat{\mathcal{G}}$ as follows: consider

$$\hat{\mathcal{G}}^*_{reg} = C^\infty(S^1) \oplus C^\infty(S^1) \oplus \mathbb{R}^3$$

and fix the pairing between this space and $\hat{\mathcal{G}}$, $\langle \cdot, \cdot \rangle \colon \hat{\mathcal{G}}^*_{reg} \otimes \hat{\mathcal{G}} \to \mathbb{R}$:

$$\langle \hat{u}, \hat{f} \rangle = \int_{S^1} f(x)u(x)dx + \int_{S^1} a(x)v(x)dx + \alpha\gamma, \tag{19}$$

where $\hat{u} = (u(x), v, \gamma)$, $\hat{f} = (f\frac{d}{dx}, a, \alpha)$.

Extend this to a right invariant metric on the semi-direct product group $\mathrm{Diff}\widehat{(S^1) \ltimes C}^\infty(S^1)$ by setting

$$\langle \hat{u}, \hat{f} \rangle_{\hat{\xi}} = \langle d_{\hat{\xi}} R_{\hat{\xi}^{-1}} \hat{u}, d_{\hat{\xi}} R_{\hat{\xi}^{-1}} \hat{f} \rangle_{L^2} \tag{20}$$

for any $\hat{\xi} \in \hat{\mathcal{G}}$ and $\hat{u}, \hat{f} \in T_{\hat{\xi}}\widehat{\mathcal{G}}$, where

$$R_{\hat{\xi}} \colon \hat{\mathcal{G}} \longrightarrow \hat{\mathcal{G}}$$

is the right translation by $\hat{\xi}$.

We shall show that the Antonowicz-Fordy equation is precisely the Euler-Poincaré equation on the dual space of $\hat{\mathcal{G}}$ associated with the L^2 inner product.

Given any three elements

$$\hat{f} = (f\frac{d}{dx}, a, \alpha), \quad \hat{g} = (g\frac{d}{dx}, b, \beta), \ \hat{u} = (u\frac{d}{dx}, v, c), \quad \text{in } \hat{\mathcal{G}}.$$

Lemma 1.

$$ad^*_{\hat{f}}\hat{u} = \begin{pmatrix} 2f'(x)u(x) + f(x)u'(x) + a'v(x) - c_1 f''' + c_2 a'' \\ f'v(x) + f(x)v'(x) - c_2 f''(x) + 2c_3 a'(x) \\ 0 \end{pmatrix}$$

Proof. This follows from

$$\begin{aligned}
&\langle ad^*_{\hat{f}}\hat{u}, \hat{g}\rangle_{L^2} = \langle \hat{u}, [\hat{f}, \hat{g}]\rangle_{L^2} \\
&= \langle (u(x)\frac{d}{dx}, v(x), c), [(fg' - f'g)\frac{d}{dx}, fb' - ga', \omega)\langle_{L^2} \\
&= -\int_{S^1}(fg' - f'g)u(x)dx - \int_{S^1}(fb' - ga')vdx - c_1\int_{S^1} f'(x)g''(x)dx \\
&- c_2\int_{S^1}(f''(x)b(x) - g''(x)a(x))dx - 2c_3\int_{S^1} a(x)b'(x)dx.
\end{aligned}$$

Since f, g, u are periodic functions, hence integrating by parts we obtain

$$\begin{aligned}
R.H.S. = &\langle (2f'(x)u(x) + f(x)u'(x) + a'(x)v(x) - c_1 f'''(x) \\
&+ c_2 a''(x), f'(x)v(x) + f(x)v'(x) - c_2 f'' b(x) + 2c_3 a'(x), 0)\rangle
\end{aligned}$$

□

The Hamiltonian structure associated with the coadjoint action is given by

$$\mathcal{O} = \begin{pmatrix} -c_1 D^3 + 2uD + u_x & vD + c_2 D^2 \\ v_x + vD - c_2 D^2 & 2c_3 D \end{pmatrix}. \tag{21}$$

This is most general Hamiltonian structure for the Antonowicz-Fordy system. So all other Hamiltonian structures follow from this.

The Euler-Poincaré equation is the Hamiltonian flow on the coadjoint orbit in $\hat{\mathcal{G}}^*$, generated by the Hamiltonian

$$H(\hat{u}) \equiv H(u, v) = \langle (u(x), v(x)), (u(x), v(x))\rangle, \tag{22}$$

given by

$$\frac{d\hat{u}}{dt} = ad^*_{\hat{u}}\hat{u}. \tag{23}$$

Let V be a vector space and assume that the Lie group G acts on the left by linear maps on V, thus G acts on the left on its dual space V^* [5, 22].

Proposition 1. *Let $G \ltimes V$ be a semidirect product space (possibly infinite dimensional), equipped with a metric $\langle \cdot, \cdot \rangle$ which is right translation. A curve $t \to c(t)$ in $G \ltimes V$ is a geodesic of this metric if and only if $\hat{u}(t) = d_{c(t)} R_{c(t)^{-1}} \dot{c}(t)$ satisfies the Euler-Poincaré equation.*

If we assume

$$\frac{\delta H}{\delta u} = 2v, \qquad \frac{\delta H}{\delta v} = u,$$

then we prove the first part of the theorem.

3. Examples of Euler-Poincaré flows on semidirect product group

In this section we construct several examples of Euler-Poincaré flows on the semidirect product group $\widehat{\mathrm{Diff}(S^1) \ltimes C^\infty}(S^1)$.

3.1. Dispersive water waves equation.

We begin with a prototypical example, the dispersive water waves equation

$$\begin{aligned} w_{0t} &= w_{1xxx} + 3(w_1 w_0)_x + w_0 w_{1x} \ , \\ w_{1t} &= w_{0x} + 4 w_1 w_{1x} \ . \end{aligned} \tag{24}$$

We show that this is a geodesic flow on the extension of the Bott-Virasoro group, and this flow

$$\begin{pmatrix} w_{0t} \\ w_{1t} \end{pmatrix} = \begin{pmatrix} D^3 + D w_0 + w_0 D & w_1 D \\ D w_1 & D \end{pmatrix} \begin{pmatrix} \dfrac{\delta H}{w_0} \\ \dfrac{\delta H}{w_1} \end{pmatrix} \tag{25}$$

is connected to a hyperplane in the coadjoint orbit of the Bott-Virasoro group.

It is possible to define a Miura map for this system

$$\begin{aligned} u &= w_1 \ , \\ v &= w_0 + \frac{3}{4} w_1^2 \ . \end{aligned} \tag{26}$$

The Miura map transformed this equation to

$$\begin{aligned} v_t &= u_{xxx} + u v_x + 2(uv)_x - \frac{3}{2} u^2 u_x, \\ u_t &= v_x \ . \end{aligned} \tag{27}$$

It is easy to see that the Hamiltonian structure also transformed to

$$\mathcal{O} = \begin{pmatrix} \frac{1}{2}D^3 + D(v-2u^2) + (v-2u^2)D + 2v_x & uD \\ Du & D \end{pmatrix}$$

with the Hamiltonian functionals satisfy

$$\begin{aligned} \frac{\delta H_1}{\delta v} &= 2u \ , \\ \frac{\delta H_1}{\delta u} &= v - 2u^2 \ . \end{aligned} \tag{28}$$

3.2. The Ito equation.

Let us choose the *hyperplane* in the dual space. The coadjoint action leaves the parameter space invariant. Let us consider a hyperplane $c_1 = -1$, $c_2 = c_3 = 0$.

Corollary 1.

$$ad^*_{\hat{f}}\hat{u} = \begin{pmatrix} 2f'(x)u(x) + f(x)u'(x) + a'v(x) + f''' \\ f'v(x) + f(x)v'(x) \\ 0 \end{pmatrix} .$$

The Hamiltonian structure of the well known Ito system

$$\begin{aligned} u_t =& u_{xxx} + 6uu_x + 2vv_x \\ v_t =& 2(uv)_x \end{aligned}$$

is given by

$$\mathcal{O}_{\mathrm{Ito}} = \begin{pmatrix} D^3 + 4uD + 2u_x & 2vD \\ 2v_x + 2vD & 0 \end{pmatrix},$$

where

$$\frac{\delta H}{\delta u} = u, \quad \frac{\delta H}{\delta v} = v.$$

This system is connected to a hyperplane $c_1 = -1$, $c_2 = c_3 = 0$.

3.3. Modified dispersive water wave equation.

When we restrict to a hyperplane $c_1 = 0$, $c_2 = 1$, $c_3 = 0$, we obtain the modified dispersive water wave equation

$$\begin{aligned} u_t &= 6uu_x + 2vv_x + v_{xx} \ , \\ v_t &= 2(vu)_x - u_{xx}. \end{aligned} \tag{29}$$

Thus the Hamiltonian structure of the modified dispersive water wave is

$$\mathcal{O}_2 = \begin{pmatrix} 4uD + 2u_x & 2vD \\ 2v_x + 2vD + D^2 & 0 \end{pmatrix}.$$

3.4. The Kaup-Boussinesq system.

The Kaup-Boussinesq equation

$$u_t = uu_x + v_x, \qquad v_t = (uv)_x + \frac{1}{4}u_{xxx}$$

is another system apart from KdV which is often model for the shallow water undular bose. The KB system has a natural two wave structure, which enables one to capture the effects of interaction of unmodular bores or rarefaction waves araising in the decay of an jump discontinuity. This equation is also related to a hyperplane $c_1 = 1/4$ and $c_3 = 1/2$ in the coadjoint orbit of the extension of the Bott-Virasoro group. Its Hamiltonian structure is

$$\mathcal{O}_2 = \begin{pmatrix} 2vD + v_x + \frac{1}{4}D^3 & uD \\ Du & D \end{pmatrix},$$

with

$$\frac{\delta H}{\delta v} = u \quad \text{and} \quad \frac{\delta H}{\delta u} = v.$$

3.5. The Broer-Kaup system.

The Broer-Kaup system

$$u_t = -u_{xx} + 2(uv)_x + uu_x, \qquad v_t = v_{xx} + 2vv_x - 2u_x$$

is a geodesic flow associated to the hyperplane $c_2 = -1$ and $c_3 = -1$. Hence the Hamiltonian structure is

$$\mathcal{O}_{BK} = \begin{pmatrix} uD_x + D_x u & -D_x^2 + vD_x \\ D_x^2 + D_x v & -2D_x \end{pmatrix}, \qquad \text{with } H = \int_{S^1} uv.$$

3.6. The Wadati-Konno-Ichikawa system.

In late seventies, Wadati et. al. [28] proposed two highly nonlinear equations

$$u_t = D_x^2\left(\frac{u}{\sqrt{1+uv}}\right), \qquad v_t = -D_x^2\left(\frac{v}{\sqrt{1+uv}}\right).$$

The Hamiltonian structure of this pair is associated to the hyperplane $c_2 = \kappa$, where κ is very large. Then the Hamiltonian structure becomes

$$\begin{pmatrix} uD_x + D_x u & -\kappa D_x^2 + vD_x \\ \kappa D_x^2 + D_x v & 0 \end{pmatrix} \underset{\kappa \to 0}{\longrightarrow} \sim \begin{pmatrix} 0 & -D_x^2 \\ D_x^2 & 0 \end{pmatrix} \equiv \mathcal{O}_{WKI}.$$

If we use $H = 2\sqrt{1+uv}$, then we obtain WKI system. Thus the above Hamiltonian structure $\mathcal{O}_{WKI}$ can be obtained from frozen bracket [14, 16] at $(u(x), v(x), c) \equiv (0, 0, c)$, where $c = (0, 1, 0)$.

4. Twisted Heisenberg-Virasoro algebra and Lie-Poisson structure

The Hamiltonian operator associated with the Antonowicz-Fordy system give rise to Kac-Moody algebras. There is an explicit algorithim for the construction of Kac-Moody algebras from the Hamiltonian operator which is essentially based on Fourier analysis.

4.1. Lie-Poisson brackets.

It is now customary to define a Lie-Poisson bracket as

$$\{f, g\} = \int \nabla f \; \mathcal{O}_{AF} \nabla g \; dx, \tag{30}$$

where ∇f denotes the gradient of the functional f with respect to (u, v), and $\mathcal{O}_{AF}$ is the Hamiltonian operator of the Antonowicz-Fordy system

$$\mathcal{O}_{AF} = \begin{pmatrix} -c_1 D^3 + 2uD + u_x & vD + c_2 D^2 \\ v_x + vD - c_2 D^2 & 2c_3 D \end{pmatrix}.$$

Let us calculate the Lie-Poisson brackets of $u(x)$ and $v(x)$.

Proposition 2.

$$\{u(x), u(x')\} = -c_1 \delta'''(x-x') + 2u\delta'(x-x') + u'\delta(x-x'), \tag{31}$$

$$\{u(x), v(x')\} = c_2 \delta''(x-x') + v\delta'(x-x'), \tag{32}$$

$$\{v(x), v(x')\} = 2c_3 \delta'(x-x'). \tag{33}$$

Proof. It follows directly from the formula, for example,

$$\{u(x), u(x')\}$$

$$= \int_{S^1} dx_1 \begin{pmatrix} \dfrac{\delta u(x)}{\delta u(x_1)} \\ \dfrac{\delta u(x)}{\delta v(x_1)} \end{pmatrix}^T \begin{pmatrix} -c_1 D^3 + 2uD + u_x & vD + c_2 D^2 \\ v_x + vD - c_2 D^2 & 2c_3 D \end{pmatrix} \begin{pmatrix} \dfrac{\delta u(x')}{\delta u(x_1)} \\ \dfrac{\delta v(x')}{\delta v(x_1)} \end{pmatrix}$$

$$= \int_S^1 dx_1 \begin{pmatrix} \delta(x - x_1) \\ 0 \end{pmatrix}^T \mathcal{O}_{AF} \begin{pmatrix} \delta(x' - x_1) \\ 0 \end{pmatrix}$$

etc. $\square$

4.2. Fourier expansion and twisted Heisenberg-Virasoro algebra. Let us perform the Fourier expansion of $u(x)$ and $v(x)$, given by

$$u(x) = \sum_{p=1}^{\infty} L_p e^{ipx} + \alpha, \tag{34}$$

$$v(x) = \sum_{p=1}^{\infty} S_p e^{ipx} + \beta. \tag{35}$$

Hence we obtain the twisted Heisenberg-Virasoro algebra corresponding to the above Lie-Poisson brackets

Proposition 3.

$$\begin{aligned} i\{L_n, L_m\} =&(n-m)L_{n+m} + (c_1 n^3 - \ \alpha n)\delta_{n+m,0}, \\ i\{L_n, S_m\} =& - mS_{n+m} - (n\beta + ic_2\ n^2)\delta_{n+m,0}, \\ i\{S_n, S_m\} =& 2c_3 n\delta_{n+m,0}. \end{aligned}$$

Proof. By direct computation. □

This Lie algebra is the classical analogue of the twisted Heisenberg-Virasoro algebra. It has an infinite-dimensional Heisenberg subalgebra and a Virasoro subalgebra intertwined with the cocycle (15). The infinite-dimensional Heisenberg algebra has the basis $\{S_j,\ c\ |j \in \mathbb{Z}\}$ and the Lie bracket is given by (38), and c_3 is a central term. By the direct exponentiation we construct the Heisenberg group:

$$\{\exp(\kappa\ c_3)\ \exp(\sum_{j\leq 0} a_j S_j)\ \exp(\sum_{j\leq 0} a_j S_j)\},$$

where $a_k \in \mathbb{R}$ with finitely many non-zero a_k.

The center of the twisted Heisenberg-Virasoro algebra is four dimensional and is spanned by $\{S_0, c_1, c_2, c_3\}$.

5. The Euler-Poincaré framework of the Hirota-Satsuma type equation

Another class of coupled KdV equations can be derived as Euler-Poincaré flows on Bott-Virasoro group using complex fields. This is different from the Antonowicz and Fordy classes. In this section we consider such classes of coupled KdV systems.

5.1. Diffeomorphism and Virasoro algebra.

Let us consider the Lie algebra of vector fields on S^1, $\mathrm{Vect}(S^1)$. The dual of this algebra is identified with space of quadratic differential forms $u(x)dx^{\otimes 2}$ by the following pairing,

$$\langle u(x), f(x)\rangle = \int_0^{2\pi} u(x)\ f(x)dx$$

where

$$f(x)\frac{d}{dx} \in \mathrm{Vect}(S^1).$$

The Virasoro algebra Vir has a unique non-trivial central extension by means of $\mathbb{R}$

$$0 \longrightarrow \mathbb{R} \longrightarrow Vir \longrightarrow \mathrm{Vect}(S^1)$$

described by the Gelfand-Fuchs cocycle

$$\omega_1(f,g) = \frac{1}{2}\int_{S^1} f'g''dx.$$

The elements of Vir can be identified with the pairs (2π periodic function, real number). The commutator in Vir takes the form

$$[(f(x)\frac{d}{dx}, a), (g(x)\frac{d}{dx}, b)] = ((fg' - gf')\frac{d}{dx}, \int_{S^1} f'g'').$$

The dual space Vir^* can be identified to the set $\{(\mu, udx^2)|\mu \in \mathbb{R}\}$.

A pairing between a point

$$(\lambda, f(x)\frac{d}{dx}) \in Vir$$

and a point $(\mu, udx^{\otimes 2})$ is given by

$$\lambda\mu + \int_{S^1} f(x)u(x)\ dx.$$

The next theorem follows from the work of Lazutkin and Pankratova [20].

Theorem 3. *The space Vir^* can be identified with the Vir-module of Hill's operators $\{\mu\frac{d^2}{dx^2} + u(x)\}$ acting on distributions of weight $-\frac{1}{2}$ as $\widehat{\mathrm{Diff}}_+(S^1)$ modules.*

This is verifiable by the direct computaion of the action of Vir^* on its dual.

5.2. Coadjoint action and Hamiltonian structure of Hirota-Satsuma system.

It is known from the work of Kirillov that the dual of Virasoro algebra is a $\mathrm{Diff}_+(S^1)$ module.

Lemma 2.
$$Ad^*(\lambda,\phi)(\mu,u) = (\mu, Ad^*(\phi)u + \lambda\Lambda(\phi))$$
where $\Lambda(\phi) = \mathcal{S}(\phi)\circ\phi^{-1}$, $\mathcal{S}(\phi)$ *is called Schwarzian derivative of an analytic function* ϕ *and it is defined by*
$$\frac{1}{2}\left(\frac{\phi'''}{\phi'} - \frac{3}{2}\left(\frac{\phi''}{\phi'}\right)^2\right).$$

The infinitesimal version of this action is given by

Lemma 3.
$$ad^*_{(\lambda, f(x)\frac{d}{dx})}(\mu,u) = (\mu, \frac{1}{2}\mu f''' + 2f'u + fu') \equiv (\mu,\tilde{u}). \tag{36}$$

Proof. It follows from the definition
$$\begin{aligned}\langle ad^*_{(\lambda,f)}(\mu,u),(\nu,g)\rangle =&\langle(\mu,u), ad_{(\lambda,\ f)}(\nu,g)\rangle\\ =&\langle(\mu,u),(\frac{1}{2}\int_{S^1} f'g''dx, [f\frac{d}{dx}, g\frac{d}{dx}])\rangle.\end{aligned}$$
□

Alternatively, we know that the Hill's operator maps
$$\Delta\colon \mathcal{F}_{1/2} \longrightarrow \mathcal{F}_{-3/2}, \tag{37}$$
where $\mathcal{F}_\lambda$ can be interpreted as a tensor-densities on S^1 of degree $-\lambda$.

The action of $\mathrm{Vect}(S^1)$ on the space of Hill's operator Δ is defined by the commutation with the Lie derivative
$$\mathcal{L}^\lambda_{f(x)\frac{d}{dx}} = f(x)\frac{d}{dx} - \lambda f'(x),$$
given by
$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta] := \mathcal{L}^{-3/2}_{f(x)\frac{d}{dx}} \circ \Delta - \Delta \circ \mathcal{L}^{1/2}_{f(x)\frac{d}{dx}}. \tag{38}$$
The above equation yields the coadjoint action of $\mathrm{Vect}(S^1)$.

Hence the Hamiltonian operator of the KdV equation equation is given by
$$\mathcal{O}_{KdV} \equiv ad^*_u = (\frac{1}{2}\partial_x^3 + 2u\partial_x + u_x). \tag{39}$$

In the Hirota-Satsuma case the unknown variable is

$$u(x,t) = p(x,t) + iq(x,t),$$

where $p(x,t)$ is the real part of $u(x,t)$, $q(x,t)$ is the imaginary part of $u(x,t)$. Then the Hirota-Satsuma type equation follows from this scheme.

Proposition 4. *The Hamiltonian structure of the Hirota-Satsuma system can be written as*

$$\mathcal{O}_{HS} = \begin{pmatrix} \frac{1}{2}\partial^3 + 2p\partial + p_x & 0 \\ 0 & \frac{1}{2}\partial^3 + 2q\partial_x + q_x \end{pmatrix} \tag{40}$$

5.3. Example: the Nutku-Oguz system.

There are several systems closely related to this system. A few years ago Y. Nutku and O. Oguz [24] proposed a new class of coupled KdV type system

$$\begin{aligned} q_t &= q_{xxx} + 2aqq_x + pp_x + (qp)_x \\ p_t &= p_{xxx} + 2bpp_x + qq_x + (qp)_x, \end{aligned} \tag{41}$$

where $a+b=1$. If we change the variables to $u = q+p$ and $v = q+p$, then the above set is boiled down to

$$\begin{aligned} u_t &= u_{xxx} + uu_x + vv_x \\ v_t &= v_{xxx} + \lambda vv_x + (uv)_x, \end{aligned} \tag{42}$$

where λ is a parameter which is assumed as real. If $\lambda = 0$, the ystem is a completely coupled KdV system discussed by Fuchssteiner. This system can be easily incorporated in our programme.

The symmetrically coupled system

$$\begin{aligned} u_t &= u_{xxx} + v_{xxx} + 6uu_x + 4uv_x + 2u_xv \\ v_t &= u_{xxx} + v_{xxx} + 6vv_x + 4vu_x + 2v_xu \end{aligned} \tag{43}$$

is also a geodesic flow on the space of the Bott-Virasoro group. This can be easily checked if one replaces

$$\lambda = u+v \quad \text{and} \quad H = \frac{1}{2}(u+v)^2 = \frac{1}{2}\lambda^2.$$

6. $\mathbf{H^1}$ metric and integrable equation

Let us introduce H^1 norm on the algebra $\hat{\mathcal{G}}$

$$\langle \hat{f}, \hat{g} \rangle_{H^1} = \int_{S^1} f(x)g(x)dx + \int_{S^1} a(x)b(x)dx \int_{S^1} \partial_x f(x)\partial_x g(x)dx$$
$$+ \int_{S^1} \partial_x a(x)\partial_x b(x)dx + \alpha\beta,$$

where $\hat{g}$ and $\hat{f}$ are defined as above.

Now we compute:

Lemma 4. *The coadjoint operator for H^1 norm is given by*

$$ad^*_{\hat{f}}\hat{u}$$
$$= \begin{pmatrix} 2f'(x)(1-\partial_x^2)u(x)+f(x)(1-\partial_x^2)u'(x)+a'(1-\partial_x^2)v(x)-c_1 f'''+c_2 a'' \\ f'(1-\partial_x^2)v(x)+f(x)(1-\partial_x^2)v'(x)-c_2 f'' b(x)+2c_3 a'(x) \\ 0 \end{pmatrix}.$$

Proof. From the definition it follows that

$$\langle ad^*_{\hat{f}}\hat{u}, \hat{g}\rangle_{H^1}$$
$$= -\int_{S^1}(fg'-f'g)u(x)dx - \int_{S^1}(fb'-ga')vdx - c_1\int_{S^1} f'(x)g''(x)dx$$
$$- c_2\int_{S^1}(f''(x)b(x)-g''(x)a(x))dx - 2c_3\int_{S^1} a(x)b'(x)dx$$
$$- \int_{S^1}\partial_x(fg'-f'g)u(x)dx - \int_{S^1}\partial_x(fb'-ga')vdx.$$

In the preceding section we have already computed the first five terms. After computing the last two terms by integrating by parts and using the fact that the functions $f(x), g(x), u(x)$ and $a(x), b(x), v(x)$ are periodic, this expression can be expressed as above.

Let us compute now the left hand side:

$$L.H.S. = \int_{S^1}(ad^*_{\hat{f}}\hat{u})\hat{g}dx + \int_{S^1}(ad^*_{\hat{f}}\hat{u})'\hat{g}'dx$$
$$= \int_{S^1}[(1-\partial^2)ad^*_{\hat{f}}\hat{u}]\hat{g}dx.$$

Thus by equating the R.H.S. and L.H.S. we obtain the above formula. □

Corollary 2.

$$ad^*_{\hat f}\hat u = \begin{pmatrix} 2f'(x)(1-\partial_x^2)u(x) + f(x)(1-\partial_x^2)u'(x) + a'(1-\partial_x^2)v(x) + f''' \\ f'(1-\partial_x^2)v(x) + f(x)(1-\partial_x^2)v'(x) \\ 0 \end{pmatrix}.$$

Hence the Hamiltonian operator is

$$\begin{pmatrix} \frac{1}{2}D^3 + D\tilde w_0 + \tilde w_0 D & \tilde w_1 D \\ D\tilde w_1 & D \end{pmatrix}, \tag{44}$$

where $\tilde w_i = (1-\partial_x^2)w_i$.

Thus we prove:

Theorem 4. *Let* $\widehat{\mathrm{Diff}}^s(S^1)$ *be the group of orientation preserving Sobolev* H^s *diffeomorphisms of a circle. Let* $t \mapsto \hat c$ *be a curve in the* $\mathrm{Diff}^s\widehat{(S^1) \ltimes C}^\infty(S^1)$*. Let* $\hat c = (e, e, 0)$ *be the initial point, directing to the vector*

$$\hat c(0) = (w_0(x)\frac{d}{dx}, w_1(x), \gamma_0),$$

where $\gamma_0 \in \mathbb{R}^3$*. Then* $\hat c(t)$ *is a geodesic of the* H^1 *metric if and only if* $(w_0(x,t)\frac{d}{dx}, w_1(x,t), \gamma)$ *satisfies*

$$\begin{aligned} w_{0t} - w_{0xxt} &= w_{1xxx} + 3(w_0 w_1)_x + w_0 w_{1x} - 2(w_{0xx}w_1)_x - 2w_{0xx}w_{1x} - w_{1xx}w_{0x} \\ w_{1t} - w_{1xxt} &= w_{0x} + 4w_1 w_{1x} - 2(w_1 w_{1xx})_x. \end{aligned} \tag{45}$$

Thus one can obtain the H^1 flows of other integrable systems following this prescription.

7. Conclusion and outlook

There are various ways to derive coupled KdV systems. Firstly, one class of systems can be derived as geodesic flows with respect to the L^2 metrics on the Bott-Virasoro groups using complexified field. These systems are called Hirota-Satsuma type systems.

One can also generate coupled KdV from the geodesic flows on the Superconformal group, super analogue of geodesic flows on $\mathrm{Diff}\widehat{(S^1) \ltimes C}^\infty(S^1)$. These are known as super KdV systems. These are coupled integrable systems.

Finally, we have seen that the most interesting class of coupled KdV systems can be derived as geodesic flows with respect to the L^2 metrics on $\mathrm{Diff}\widehat{(S^1) \ltimes C}^\infty(S^1)$. These are known as Antonowicz-Fordy systems. We have seen that the mode expansion of the Lie-Poisson brackets of the AF system is related to the twisted Heisenberg-Virasoro algebra. This algebra has an infinite-dimensional Heisenberg subalgebra and a Virasoro subalgebra. These subalgebras do not form a semidirect product, but instead, the natural action of the Virasoro subalgebra on a Heisenberg subalgebra is twisted with a two cocycle. It would be rather interesting to study such algebra from the super analogue of our construction. Finally, we have also briefly discussed the method to obtain the H^1 analogue of these flows.

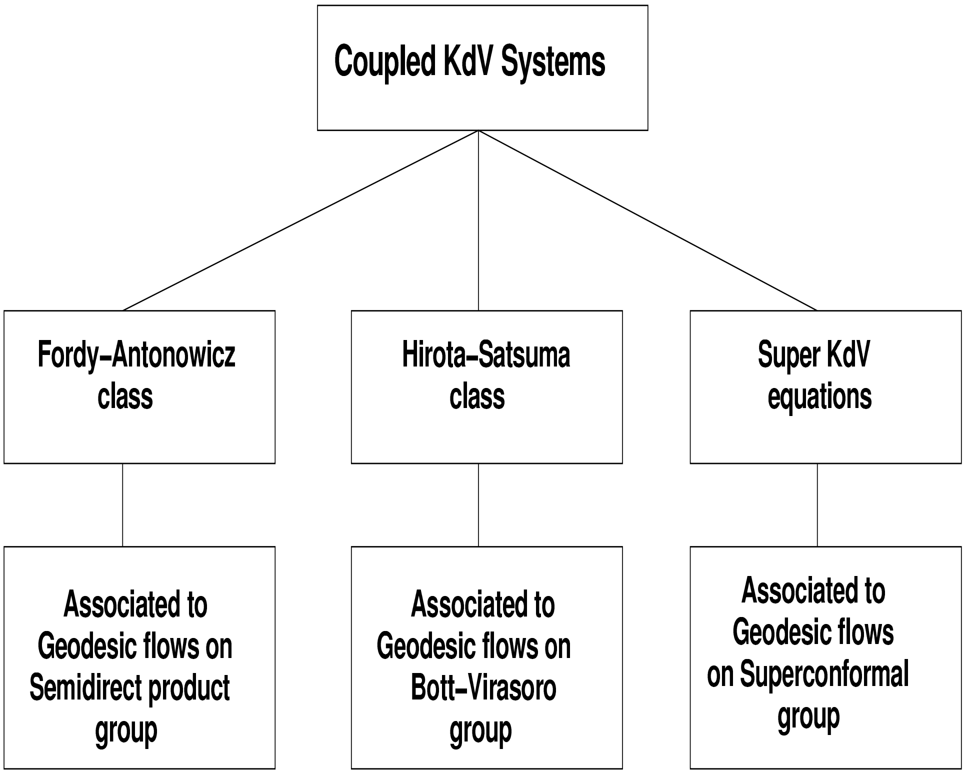

Figure 1. Various methods of constructing coupled KdV systems

Acknowledgement. The author is grateful to Professors Jerry Marsden and Valentin Ovsienko for several stimulating discussions. He is also grateful to Professor Thanasis Fokas for various remarks. He like to thanks Dr. Ain-Ul Huda for Tex assistence. The author expresses grateful thanks to the faculty and staff of Department of mathematics for gracious hospitality at the University of Colorado at Colorado Springs, where this work was done. Finally he is grateful to anonymous referee for valuable suggestions that improve the quality of the manuscript.

References

[1] Ablowitz, M., Clarkson, P. A., *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, London Math. Soc. Lecture Note Ser. **149**, Cambridge University Press, Cambridge, 1991.

[2] Alber, M., Camassa, R., Fedorov, Yu., Holm, D., Marsden, J. E., *The complex geometry of weak piecewise smooth solutions of integrable nonlinear PDE's of shallow water and Dym type*, Comm. Math. Phys. **221**(1) (2001), 197–227.

[3] Antonowicz, M., Fordy, A., *Coupled KdV equation with multi-Hamiltonian structures*, Physica D **28** (1987), 345–357.

[4] Arbarello, E., De Concini, C., Kac, V. G., Procesi, C., *Moduli space of curves and representation theory*, Comm. Math. Phys. **117** (1988), 1–36.

[5] Arnold, V. I., *Mathematical Methods of Classical Mechanics*, Grad. Texts in Math. **60**, 2nd ed., Springer-Verlag, New York, 1989.

[6] Billig, Y., *Representations of the twisted Heisenberg-Virasoro algebra at level zero*, Canad. Math. Bull. **46**(4) (2003), 529–537.

[7] Camassa, R., Holm, D., *A completely integrable dispersiveshallow water equation with peaked solutions*, Phys. Rev. Lett. **71** (1993), 1661–1664.

[8] Cendra, H., Holm, D., Marsden, J., Ratiu, T., *Lagrangian reduction, the Euler-Poincaré equations, and semidirect products*, Trans. Amer. Math. Soc. **186** (1998), 1–25.

[9] Ebin, D., Marsden, J., *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. of Math. (2) **92** (1970), 102–163.

[10] Fuchssteiner, B., *The Lie algebra structure of nonlinear evolution equations admitting infinite-dimensional abelian symmetry groups*, Progr. Theoret. Phys. **65**(3) (1981), 861–876.

[11] Guha, P., *Ito equation as a geodesic flow on* $(\mathrm{Diff}^s(S^1) \odot C^\infty(S^1))$, Arch. Math. (Brno) **36**(4) (2000), 305–312.

[12] Guha, P., *Diffeomorphism with some Sobolev metric, geodesic flow and integrable systems*, J. Dynam. Control Systems **8**(4) (2002), 529–545.

[13] Guha, P., *Integrable geodesic flows on the (super)extension of Bott-Virasoro group*, Lett. Math. Phys. **52** (2000), 311–328.

[14] Guha, P., *Geodesic flows, bi-Hamiltonian structure and coupled KdV type systems*, J. Math. Anal. Appl. **310**(1) (2005), 45–56.

[15] Harnad, J., Kupershmidt, B. A., *Symplectic geometries on* T^*G*, Hamiltonian group actions and integrable systems*, J. Geom. Phys. **16** (1995), 168–206.

[16] Khesin, B., Misiolek, G., *Euler equations on homogeneous spaces and Virasoro orbits*, Adv. Math. **176**(1) (2003), 116–144.

[17] Kirillov, A. A., *Infinite dimensional Lie groups: their orbits, invariants and representations. The geometry of moments*, Lecture Notes in Math. **970** (1982), 101–123.

[18] Kirillov, A. A., *Merits and demerits of the orbit method*, Bull. Amer. Math. Soc. (N.S.) **36**(4) (1999), 433–488.

[19] Kupershmidt, B. A., *Mathematics of dispersive water waves*, Comm. Math. Phys. **99**(1) (1985), 51–73.

[20] Lazutkin, V. F., Penkratova, T. F., *Normal forms and versal deformations for Hill's equation* (in Russian), Funktsional. Anal. i Prilozhen. **9**(4) (1975), 41–48.

[21] Marcel, P., Ovsienko, V., Roger, C., *Extension of the Virasoro and Neveu-Schwartz algebras and generalized Sturm-Liouville operators*, Lett. Math. Phys. **40** (1997), 31-39.

[22] Marsden, J. E., Ratiu, T. S., *Introduction to Mechanics and Symmetry. A Basic Exposition of Classical Mechanical Systems*, Texts Appl Math. **17**, 2nd ed., Springer-Verlag, New York, 1999.

[23] Misiolek, G., *A shallow water equation as a geodesic flow on the Bott-Virasoro group*, J. Geom. Phys. **24** (1998), 203–208.

[24] Nutku, Y., Oguz, O., *Bi-Hamiltonian structure of a pair of coupled KdV equations*, Nuovo Cimento B (11) **105**(12) (1990), 1381–1383.

[25] Ovsienko, V. Yu., Khesin, B. A., *KdV super equation as an Euler equation*, Funct. Anal. Appl. **21** (1987), 329–331.

[26] Segal, G., *Unitary representations of some infinite dimensional groups*, Comm. Math. Phys. **80** (1981), 301–342.

[27] Segal, G., *The geometry of the KdV equation. Topological methods in quantum field theory* (Trieste, 1990), Internat. J. Modern Phys. A **6**(16) (1991), 2859–2869.

[28] Wadati, M., Konno, K., Ichikawa,I., *A generalization of inverse scattering method*, J. Phys. Soc. Japan **46**(6) (1979), 1965–1966.

[29] Wang, J. P., *A list of* $1+1$ *dimensional integrable equations and their properties*, Recent advances in integrable systems (Kowloon, 2000), J. Nonlinear Math. Phys. **9** (2002), suppl. 1, 213–233.

[30] Witten, E., *Coadjoint orbits of the Virasoro group*, Comm. Math. Phys. **114**(1) (1988), 1–53.

Partha Guha
S.N. Bose National Centre for Basic Sciences
JD Block, Sector-3, Salt Lake
Calcutta-700098, INDIA
e-mail: partha@bose.res.in
and
Department of Mathematics
University of Colorado at Colorado Springs
1420 Austin Bluffs Parkway
Colorado Springs, Colorado 80933-7150
USA