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EXISTENCE OF SOLUTIONS FOR NONLOCAL BOUNDARY VALUE PROBLEM WITH SINGULARITY IN PHASE VARIABLES

Y. TIAN and W. GE

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Abstract. In this paper, we prove existence results for singular problem

$$\begin{cases} x^{(n)}(t) + f(t, x(t), \dots, x^{(n-2)}(t)) = 0, \quad 0 < t < 1, \\ x^{(i)}(0) = 0, \quad 0 \le i \le n-2, \quad x^{(n-2)}(1) = \int_0^1 x^{(n-2)}(s) dg(s). \end{cases}$$

Here the positive Carathédory function f may be singular at the zero value of all its phase variables. Proofs are based on the Leray-Schauder degree and Vitali's convergence theorem.

1. Introduction

Let $J = [0,1]$, $\mathbb{R}_- = (-\infty, 0)$, $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$.

We investigate the existence of solutions for singular boundary value problem

$$x^{(n)}(t) + f\big(t, x(t), \dots, x^{(n-2)}(t)\big) = 0, \quad 0 < t < 1, \tag{1.1}$$

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$$x^{(i)}(0) = 0, \quad 0 \le i \le n-2, \quad x^{(n-2)}(1) = \int_0^1 x^{(n-2)}(s)dg(s), \tag{1.2}$$

where $n \ge 2$, the integral is in the sense of Riemann-Stieltjes and nonlinear term f satisfies local Carathédory conditions on $J \times D\big(f \in \mathrm{Car}(J \times D)\big)$ with

$$D = \underbrace{\mathbb{R}_+ \times \cdots \times \mathbb{R}_+}_{n-2}.$$

The function f in (1.1) may be singular at the zero value of all its phase variables.

Definition 1.1. A function $x \in AC^{n-1}(J)$ (i.e. x has absolutely continuous the $(n-1)^{st}$ derivative on J) is said to be a solution of boundary value problem (1.1)–(1.2), if $x^{(i)}(t) > 0$ on $(0,1]$ for $0 \le i \le n-2$, x satisfies the boundary condition (1.2) and (1.1) holds a.e. on J.

The purpose of this paper is to give conditions which guarantee the existence of a positive solution to BVP (1.1), (1.2).

This paper is mainly motivated by the works [8]–[9], [13], where the existence of two-point higher order BVPs with singularities in phase variables was studied. In [3], Agarwal et al. consider the existence of solutions for Lidstone boundary value problem as follows

$$\begin{cases} (-1)^n x^{(2n)}(t) = f\big(t, x(t), \ldots, x^{(2n-2)}(t)\big), & t \in (0,T), \\ x^{(2j)}(0) = x^{(2j)}(T) = 0, & 0 \le j \le n-1, \end{cases} \tag{1.3}$$

where $f \in \mathrm{Car}(J \times D)$, and satisfying for a.e. $t \in J$ and for each $(x_0, \ldots, x_{2n-2}) \in D$,

$$f(t, x_0, \ldots, x_{2n-2}) \le \phi(t) + \sum_{j=0}^{2n-2} q_j(t)\omega_j(|x_j|) + \sum_{j=0}^{2n-2} h_j(t)|x_j|,$$

where $\phi, h_j \in L_1(J)$ and $q_j \in L_\infty(J)$ are nonnegative, $\omega_j \colon \mathbb{R}^+ \to \mathbb{R}^+$ are non-increasing, and

$$S = \sum_{i=0}^{n-1} \frac{T^{2(n-i)-3}}{6^{n-i-1}} \int_0^T t(T-t)h_{2i}(t)dt + \sum_{i=0}^{n-2} \frac{T^{2(n-i-2)}}{6^{n-i-2}} \int_0^T t(T-t)h_{2i+1}(t)dt < 1$$

and

$$\int_0^T \omega_j(s)ds < \infty, \quad \omega_j(uv) \le \Lambda\omega_j(u)\omega_j(v),$$

for $0 \le j \le 2n-2$ and $u, v \in \mathbb{R}_+$ with a positive constant Λ.

Another motivation for this paper is the work [8] and [9], where the nonlocal boundary value problem was considered. But nonlinear term f in all these papers have not singularity. For example, in [9] the existence of a solution of the following boundary value problem

$$\begin{cases} x^{(n)}(t) = f\big(t, x(t), x'(t), \dots, x^{(n-1)}(t)\big), \quad t \in (0,1), \\ x^{(i)}(0) = 0 \quad \text{for } i = 0, 1, \dots, k-1, \\ x^{(j)}(1) = \displaystyle\int_0^1 x^{(j)}(s) dG_{n-j}(s) \quad \text{for } j = k, \dots, n-1 \end{cases} \tag{1.4}$$

was studied, where $f\colon [0,1] \times (\mathbb{R}^m)^n \to \mathbb{R}^m$ is a Carathéodory function, f has not singularity in phase variables, $k \in \{1, \dots, n-1\}$, the function G_i $(i = k, \dots, n-k)$ takes value in linear space of all $m \times m$ square matrices. The method used in [9] is Leray-Schauder degree theory.

Besides, there are many papers studied singular boundary value problems. For example second order singular boundary value problems was investigated in Agarwal [2], Liu Bing [10], Zhang Zhongxin [13] and the references therein. The existence of positive solutions for higher order singular boundary value problem was considered in [1]. Generality speaking, nonlinear term $f(t, x_0, x_1, \dots, x_q)$ satisfies the following conditions:

(1) $f(t, x_0, x_1, \dots, x_q)$ is non-increasing in x_i for each fixed $(t, x_0, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_q)$, $0 \le i \le q$;

(2) $\lim_{x_i \to \infty} f(t, x_0, x_1, \dots, x_q) = 0$ uniformly on compact subsets of $(0,1) \times (0,\infty)^{n-2}$, $1 \le i \le n-1$.

By using Leray-Schauder degree theory we get a new result on the existence of solution to boundary value problem (1.1)–(1.2). Meanwhile we remove the restraint (1) and (2) on nonlinear term f. The approaches to estimate a *priori* bound of the solutions to boundary value problem (1.1)–(1.2) are different from the corresponding ones of the past work [8, 9]. At last we give an example to illustrate our results.

From now on, $\|x\| = \max\{|x(t)|\colon t \in J\}$, $\|x\|_{L^1} = \int_0^1 |x(t)| dt$ and $\|x\|_\infty = \operatorname{ess\,max}\{|x(t)|\colon 0 \le t \le 1\}$ stand for the norm in $C^0(J)$, $L_1(J)$, and $L_\infty(J)$, respectively. For any measurable set $\mathcal{M} \subset \mathbb{R}$, $\mu(\mathcal{M})$ denotes the Lebesgue measure of $\mathcal{M}$.

The following assumptions imposed upon the function in (1.1) will be used in the paper:

(H_1) $f \in \mathrm{Car}(J \times D)$ and there exists nonnegative functions $\phi, q_i \in L_1(J)$, $\phi(t) \not\equiv 0$, $h_i \in C(J \times \mathbb{R})$ and non-increasing nonnegative function $\omega_i \in L_1(\mathbb{R}_+)$, $0 \le i \le n-2$ such that for $(t, x) \in J \times D$,

$$f(t, x_0, \dots, x_{n-2}) = \phi(t) + \sum_{i=0}^{n-2} q_i(t)\omega_i(|x_i|) + \sum_{i=0}^{n-2} h_i(t, x_i)$$

and h_i satisfies

$$\lim_{|x_i|\to\infty}\sup_{t\in[0,1]}\frac{h_i(t,x_i)}{|x_i|}=\alpha_i\ge 0,\quad \alpha_i \text{ are any constants in } (0,1),\quad 0\le i\le n-2, \tag{1.5}$$

ω_i satisfies

$$\omega_i(xy)\le \Lambda\omega_i(x)\omega_i(y)\quad \text{for } x,y\in(0,\infty),\quad \Lambda>0 \text{ is a positive constant}, \tag{1.6}$$

$$\int_0^1 \omega_i\left(\int_0^t (t-s)^{n-3-i}s(1-s)ds\right)dt<\infty,\ 0\le i\le n-3,\quad \int_0^1 \omega_{n-2}(s(1-s))ds<\infty; \tag{1.7}$$

(H_2) g is Lebesgue measurable, increasing on J and satisfies $g(0)=0$, $g(1)<1$.

The paper is organized as follows. Section 2 presents priori bound of solutions for BVP (1.1)–(1.2). Besides, we prove that some sets of functions containing solutions of our auxiliary regular BVPs are uniformly absolutely continuous on J. Section 3 we prove the existence of solution for boundary value problem (1.1)–(1.2). Proof is based on the Arzelà-Ascoli theorem and the Vitali's convergence theorem, see, e.g. [5], [6], [11]. Section 4 present an example to illustrate our main result.

2. Auxiliary results

Lemma 2.1. *Let $\phi\in L_1(J)$ be nonnegative and $\phi(t)\not\equiv 0$. Suppose $x\in AC^{n-1}(J)$ satisfy* (1.2) *and*

$$\phi(t)\le -x^{(n)}(t),\quad t\in J. \tag{2.1}$$

Then we have on J for $0\le i\le n-1$

$$x^{(i)}(t)\ge\frac{\|x^{(n-2)}\|}{(n-3-i)!}\int_0^t (t-s)^{n-3-i}s(1-s)ds,\quad 0\le i\le n-3,$$
$$x^{(n-2)}(t)\ge\|x^{(n-2)}\|t(1-t).$$

Proof. By (1.2) we have

$$x^{(i)}(t)=\int_0^t x^{(i+1)}(s)ds,\quad i=0,\dots,n-3. \tag{2.2}$$

By (2.1), we have $x^{(n-2)}(t)$ is concave on J. So

$$\min_{t\in[0,1]} x^{(n-2)}(t) = \min\{x^{(n-2)}(0), x^{(n-2)}(1)\}.$$

We claim $x^{(n-2)}(1) \geq 0$. If not,

$$\begin{aligned} x^{(n-2)}(1) = \int_0^1 x^{(n-2)}(s)dg(s) \geq \min_{t\in[0,1]} x^{(n-2)}(s) \int_0^1 dg(s) \\ =x^{(n-2)}(1)g(1) > x^{(n-2)}(1), \end{aligned}$$

a contradiction. Thus we obtain $x^{(n-2)}(t) \geq 0$ for $t \in J$. So

$$x^{(n-2)}(t) \geq \|x^{(n-2)}\| t(1-t). \tag{2.3}$$

By (2.1)–(2.2) we have

$$x^{(i)}(t) \geq \frac{\|x^{(n-2)}\|}{(n-3-i)!} \int_0^t (t-s)^{n-3-i} s(1-s)ds.$$

□

Lemma 2.2. *Let $\phi \in L_1(J)$ be nonnegative and $\phi(t) \not\equiv 0$. Then there exists a positive constant $c = c(\phi)$ such that for each function $x \in AC^{n-1}(J)$ satisfying* (1.2) *and*

$$\phi(t) \leq -x^{(n)}(t), \quad \text{for a.e. } t \in J,$$

the estimate $\|x^{(n-2)}\| \geq c$ holds.

Proof. By $-x^{(n)}(t) \geq \phi(t) \geq 0$, we know $x^{(n-2)}(t)$ is concave on J. If $x^{(n-2)}(t) \equiv 0$, $t \in J$, then $x^{(n)}(t) \equiv 0$, $t \in J$, which contradicts that $-x^{(n)}(t) \geq \phi(t)$ and $\phi(t)$ be nonnegative and $\phi(t) \not\equiv 0$. □

Remark 2.1. It follows from Lemma 2.1 and Lemma 2.2 that for any solution of BVP (1.1)–(1.2)

$$\begin{aligned} |x^{(i)}(t)| \geq& \frac{c}{(n-3-i)!} \int_0^t (t-s)^{n-3-i} s(1-s)ds, \quad i = 0, \dots, n-3, \\ x^{(n-2)} \geq& ct(1-t), \end{aligned}$$

where $c = c(\phi)$.

For each $m \in \mathbb{N}$, define $\mathcal{X}_m$, and $f_m \in \mathrm{Car}(J \times \mathbb{R}^n)$ by the formulas

$$\mathcal{X}_m(u) = \begin{cases} u, & \text{for } u \geq \dfrac{1}{m}, \\ \dfrac{1}{m}, & \text{for } u < \dfrac{1}{m}, \end{cases}$$

and

$$f_m(t, x_0, x_1, \dots, x_{n-2}) = \phi(t) + \sum_{i=0}^{n-2} q_i(t)\omega_i\big(\mathcal{X}_m(x_i)\big) + \sum_{i=0}^{n-2} h_i(t, x_i) \tag{2.4}$$

for $(t, x_0, \dots, x_{n-2}) \in J \times \mathbb{R}^{n-1}$. Hence

$$0 < \phi(t) \le f_m(t, x_0, \dots, x_{n-2}) \le \phi(t) + \sum_{i=0}^{n-2} q_i(t)\omega_i(|x_i|) + \sum_{i=0}^{n-2} h_i(t, x_i) \tag{2.5}$$

for a.e. $t \in J$ and each $(x_0, \dots, x_{n-2}) \in \mathbb{R}_0^{n-1}$.

Consider auxiliary regular differential equation

$$x^{(n)}(t) + f_m\big(t, x(t), \dots, x^{(n-2)}(t)\big) = 0 \tag{2.6}$$

and

$$x^{(n)}(t) + \lambda f_m\big(t, x(t), \dots, x^{(n-2)}(t)\big) = 0, \quad \lambda \in [0,1] \tag{2.7}$$

depending on the parameters $m \in \mathbb{N}$.

Lemma 2.3. *Let $h\colon [0,1] \to \mathbb{R}_+$ be continuous. Suppose $x(t)$ is a solution of the following boundary value problem*

$$\begin{cases} x^{(n)}(t) + h(t) = 0, \quad t \in (0,1), \\ x^{(i)}(0) = 0, \quad i = 0, \dots, n-2, \qquad x^{(n-2)}(1) = \displaystyle\int_0^1 x^{(n-2)}(s)dg(s). \end{cases}$$

Then $x(t)$ can be uniquely expressed as

$$x(t) = \frac{At^{n-1}}{(n-1)!} - \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} h(s)ds$$

where

$$A = \frac{1}{1 - \displaystyle\int_0^1 s dg(s)} \left(\int_0^1 (1-s)h(s)ds - \int_0^1 \left(\int_0^r (r-s)h(s)ds \right) dg(r) \right).$$

Proof. *Sufficiency.* First integrating both sides of equation $x^{(n)}(t) + h(t) = 0$ on $[0,t]$, we have

$$x^{(n-1)}(t) = x^{(n-1)}(0) - \int_0^t h(s)ds.$$

Integrating again the above equation on $[0,t]$ and using the second boundary condition we get

$$x^{(n-2)}(t) = x^{(n-1)}(0)t - \int_0^t (t-s)h(s)ds.$$

It follows that

$$\int_0^1 x^{(n-2)}(s)dg(s) = x^{(n-1)}(0)\int_0^1 sdg(s) - \int_0^1 \left(\int_0^r (r-s)h(s)ds\right) dg(r).$$

Noticing boundary condition

$$x^{(n-2)}(1) = \int_0^1 x^{(n-2)}(s)dg(s),$$

we obtain the following equality

$$\begin{aligned} x^{(n-1)}(0) - \int_0^1 (1-s)h(s)ds =& x^{(n-1)}(0)\int_0^1 sdg(s) \\ &- \int_0^1 \left(\int_0^r (r-s)h(s)ds\right) dg(r), \end{aligned}$$

holds, which means

$$\begin{aligned} x^{(n-1)}(0) =& \frac{1}{1-\int_0^1 sdg(s)} \left(\int_0^1 (1-s)h(s)ds - \int_0^1 \left(\int_0^r (r-s)h(s)ds\right) dg(r)\right) \\ =& A. \end{aligned}$$

So

$$x^{(n-2)}(t) = At - \int_0^t (t-s)h(s)ds.$$

Integrating the above equation on $[0,t]$ for $n-2$ times, we get

$$x(t) = \frac{At^{n-1}}{(n-1)!} - \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}h(s)ds$$

holds.

Necessity. From the expression of x, it is easy to obtain x is a solution of boundary value problem in Lemma 2.3. □

Lemma 2.4. *Let $m \in \mathbb{N}$. If there exists a positive constant K such that*

$$\|x^{(j)}\| \le K, \quad 0 \le j \le n-1 \tag{2.8}$$

for any solution x of BVP (2.7), (1.2) *with $\lambda \in [0,1]$, then BVP* (2.6), (1.2) *has a solution x satisfying* (2.8).

Proof. By Lemma 2.3 we know that solving (2.7), (1.2) is equivalent to find $x \in C^{n-1}(J)$ satisfying

$$x(t) = \lambda \frac{At^{n-1}}{(n-1)!} - \lambda \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f_m(s, x(s), \ldots, x^{(n-1)}(s)) ds, \tag{2.9}$$

where A is defined in Lemma 2.3. It is easy to see that

$$S \colon C^{n-1}(J) \to C^{n-1}(J),$$
$$(Sx)(t) = \frac{At^{n-1}}{(n-1)!} - \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f_m(s, x(s), \ldots, x^{(n-1)}(s)) ds$$

is a completely continuous operator. Since we can rewrite (2.9) as

$$x = \lambda Sx, \quad \lambda \in [0,1]. \tag{2.10}$$

By our assumption, (2.10) holds for any solution of (2.7), there exists a solution x of the operator equation $x = Sx$ by [6]. Of course, x is a solution of BVP (2.6), (1.2) satisfying (2.8). □

For convenience we denote

$$\Gamma := \int_0^1 \left(\phi(s) + \Lambda \sum_{i=0}^{n-3} q_i(s) \omega_i \left(\frac{c}{(n-3-i)!} \right) \omega_i \left(\int_0^s (s-\theta)^{n-3-i} \theta (1-\theta) d\theta \right) + q_{n-2}(s) \Lambda \omega_{n-2}(c) \omega_{n-2}\big(s(1-s)\big) \right) ds.$$

Lemma 2.5. *Let assumptions* (H_1)–(H_2) *be satisfied. Furthermore, the following inequality is satisfied*

$$(H_3) \qquad \sum_{i=0}^{n-2} \frac{\alpha_i}{(n-i-1)!} < 1.$$

Then there exists a positive constant P *such that* $\|x^{(j)}\| \leq P$, $0 \leq j \leq n-1$ *for any solution* x *of BVP* (2.7), (1.2) *with* $m \in \mathbb{N}$.

Proof. Let x be a solution of BVP (2.7), (1.2) for some $m \in \mathbb{N}$.

In what follows we will prove $\|x^{(i)}\| \leq P$, $0 \leq j \leq n-1$. The proof of this lemma is divided into three steps.

Step 1. It follows from boundary condition that

$$x^{(i)}(t) = \int_0^t \frac{(t-\theta)^{n-i-2}}{(n-i-2)!} x^{(n-1)}(\theta) d\theta, \quad t \in J, \quad 0 \leq i \leq n-2. \tag{2.11}$$

Thus we have

$$\|x^{(i)}\| \leq \frac{1}{(n-i-1)!} \|x^{(n-1)}\|, \quad 0 \leq i \leq n-2. \tag{2.12}$$

Step 2. Prove there exists a positive constant P such that

$$\|x^{(n-1)}\| \le P.$$

We claim there exists $\xi \in [0,1]$ such that $x^{(n-1)}(\xi) = 0$.

Otherwise, if $x^{(n-1)}(t) \ge 0$, $t \in [0,1]$, then

$$x^{(n-2)}(1) = \max_{t\in J} x^{(n-2)}(t).$$

But

$$x^{(n-2)}(1) = \int_0^1 x^{(n-2)}(s)dg(s) \le \max_{t\in J} x^{(n-2)}(t)g(1) < x^{(n-2)}(1),$$

a contradiction;
if $x^{(n-1)}(t) \le 0$, $t \in [0,1]$, then $x^{(n-2)}(t) \le 0$ for $t \in J$. But

$$x^{(n-2)}(1) = \int_0^1 x^{(n-2)}(s)dg(s) \ge \min_{t\in J} x^{(n-2)}(t)g(1) > x^{(n-2)}(1),$$

a contradiction.

Noticing $x^{(n-1)}(t)$ is decreasing on $[0,1]$, one has

$$x^{(n-1)}(t) > 0 \text{ for } t \in [0,\xi), \quad x^{(n-1)}(t) < 0 \text{ for } t \in (\xi,1]. \tag{2.13}$$

Let sufficiently small $\varepsilon > 0$ be such that

$$\sum_{i=0}^{n-2} \frac{\alpha_i + \varepsilon}{(n-i-1)!} < 1. \tag{2.14}$$

Then for this $\varepsilon > 0$, there is $\delta > 0$ so that

$$|h_i(t,x_i)| < (\alpha_i + \varepsilon)|x_i| \quad \begin{array}{l}\text{uniformly for } t \in [0,1],\\ \text{and } |x_i| > \delta,\ i = 0,\dots,n-2.\end{array} \tag{2.15}$$

Let, for $i = 0,\dots,n-2$,

$$\begin{aligned} \Delta_{1,i} &= \{t\colon t \in [0,1],\ |x_i(t)| \le \delta\},\\ \Delta_{2,i} &= \{t\colon t \in [0,1],\ |x_i(t)| > \delta\},\\ h_{\delta,i} &= \max_{t\in[0,1],|x_i|\le\delta} h_i(t,x_i). \end{aligned}$$

On the one hand, integrating both sides of (2.7) from t to ξ, $(t \in [0,\xi])$, using (2.5), Remark 2.1, (2.11) and (2.15) we have

$$\begin{aligned} x^{(n-1)}(t) =& \lambda \int_t^\xi f_m\big(s, x(s), \dots, x^{(n-2)}(s)\big)ds\\ \le& \lambda \int_t^\xi \left(\phi(s) + \sum_{i=0}^{n-2} q_i(s)\omega_i\big(|x^{(i)}(s)|\big)\right) ds \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=0}^{n-2}\int_{[t,\xi]\cap\Delta_{1,i}} h_i(s,x^{(i)}(s))ds+\sum_{i=0}^{n-2}\int_{[t,\xi]\cap\Delta_{2,i}} h_i(s,x^{(i)}(s))ds\\
\le&\int_t^\xi\Bigg[\phi(s)+\sum_{i=0}^{n-3}q_i(s)\Lambda\omega_i\left(\frac{c}{(n-3-i)!}\right)\\
&\times\omega_i\left(\int_0^s(s-\theta)^{n-3-i}\theta(1-\theta)d\theta\right)\\
&+q_{n-2}(s)\Lambda\omega_{n-2}(c)\omega_{n-2}\big(s(1-s)\big)\Bigg]\\
&+\sum_{i=0}^{n-2}\int_{[t,\xi]\cap\Delta_{1,i}} h_i\big(s,x^{(i)}(s)\big)ds+\sum_{i=0}^{n-2}\int_{[t,\xi]\cap\Delta_{2,i}} h_i\big(s,x^{(i)}(s)\big)ds,
\end{aligned}
$$

thus we have for $t\in[0,\xi]$, noticing (2.12)

$$
\begin{aligned}
x^{(n-1)}(t)&\le\Gamma+\sum_{i=0}^{n-2}h_{\delta,i}+\int_t^\xi\sum_{i=0}^{n-2}(\alpha_i+\varepsilon)|x^{(i)}(s)|ds\\
&\le\Gamma+\sum_{i=0}^{n-2}h_{\delta,i}+\sum_{i=0}^{n-2}\frac{\alpha_i+\varepsilon}{(n-i-1)!}\|x^{(n-1)}\|,
\end{aligned}
$$

i.e.

$$
x^{(n-1)}(0)\le\Gamma+\sum_{i=0}^{n-2}h_{\delta,i}+\sum_{i=0}^{n-2}\frac{\alpha_i+\varepsilon}{(n-i-1)!}\|x^{(n-1)}\|. \tag{2.16}
$$

On the other hand, integrating both sides of (2.7) from ξ to t, ($t\in[\xi,1]$), using (2.5), Remark 2.1 , (2.11), (2.15), we have

$$
\begin{aligned}
|x^{(n-1)}(t)|=&\lambda\int_\xi^t f_m\big(s,x(s),\dots,x^{(n-2)}(s)\big)ds\\
\le&\lambda\int_\xi^t\left(\phi(s)+\sum_{i=0}^{n-2}q_i(s)\omega_i(|x^{(i)}(s)|)\right)ds\\
&+\sum_{i=0}^{n-2}\int_{[\xi,t]\cap\Delta_{1,i}} h_i\big(s,|x^{(i)}(s)|\big)+\sum_{i=0}^{n-2}\int_{[\xi,t]\cap\Delta_{2,i}} h_i\big(s,|x^{(i)}(s)|\big)ds\\
\le&\lambda\int_\xi^t\Bigg[\phi(s)+\sum_{i=0}^{n-2}q_i(s)\Lambda\omega_i\left(\frac{c}{(n-3-i)!}\right)\\
&\times\omega_i\left(\int_0^s(s-\theta)^{n-3-i}\theta(1-\theta)d\theta\right)
\end{aligned}
$$

$$+ q_{n-2}(s)\Lambda\omega_{n-2}(c)\omega_{n-2}\big(s(1-s)\big)\Bigg] ds \tag{2.17}$$

$$+ \sum_{i=0}^{n-2} \int_{[\xi,t]\cap\Delta_{1,i}} h_i\big(s, x^{(i)}(s)\big) + \sum_{i=0}^{n-2} \int_{[\xi,t]\cap\Delta_{2,i}} h_i\big(s, x^{(i)}(s)\big) ds,$$

thus for $t \in [\xi, 1]$, noticing (2.12) we have

$$\begin{aligned} |x^{(n-1)}(s)| &\le \Gamma + \sum_{i=0}^{n-2} h_{\delta,i} + \sum_{i=0}^{n-2} \int_\xi^t (\alpha_i + \varepsilon)|x^{(i)}(s)| ds \\ &\le \Gamma + \sum_{i=0}^{n-2} h_{\delta,i} + \sum_{i=0}^{n-2} \frac{\alpha_i + \varepsilon}{(n-i-1)!} \|x^{(n-1)}\|, \end{aligned}$$

i.e.

$$|x^{(n-1}(1)| \le \Gamma + \sum_{i=0}^{n-2} h_{\delta,i} + \sum_{i=0}^{n-2} \frac{\alpha_i + \varepsilon}{(n-i-1)!} \|x^{(n-1)}\|. \tag{2.18}$$

By $x^{(n-1)}$ is decreasing on J, it follows from (2.16) (2.18) that

$$\|x^{(n-1)}\| = \max\{x^{(n-1)}(0), |x^{(n-1)}|\} \le \Gamma + \sum_{i=0}^{n-2} h_{\delta,i} + \sum_{i=0}^{n-2} \frac{\alpha_i + \varepsilon}{(n-i-1)!} \|x^{(n-1)}\|$$

we have

$$\|x^{(n-1)}\| \le \frac{\Gamma + \sum_{i=0}^{n-2} h_{\delta,i}}{1 - \sum_{i=0}^{n-2} \frac{\alpha_i + \varepsilon}{(n-i-1)!}} := P.$$

By (H_1), (H_3) we have $P < \infty$.

Step 3. Prove $\|x^{(i)}\| \le P$ for $i = 0, 1, \dots, n-1$.

By Step 1 the result is clear. This completes the proof. □

Lemma 2.6. *Let assumptions (H_1), (H_3) be satisfied. Let $\{x_m\}$ be a sequence of solutions to BVP (2.7), (1.2) for each $m \in \mathbb{N}$. Then the sequence*

$$\{f_m\big(t, x_m(t), \dots, x_m^{(n-2)}(t)\big)\} \subset L_1(J)$$

is uniformly absolutely continuous on J, that is for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_{\mathcal{M}} f_m\big(t, x_m(t), \dots, x_m^{(n-2)}(t)\big) dt < \varepsilon$$

for any measurable set $\mathcal{M} \subset J$, $\mu(\mathcal{M}) < \delta$.

Proof. With respect to (2.5) and properties of measurable sets, it is sufficient to verify that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any at most countable set $\{(a_j, b_j)\}_{j\in J}$ of mutually disjoint intervals $\{(a_j, b_j)\}_{j\in J}$ with $\sum_{j\in J}(b_j - a_j) < \delta$, we have for each $m \in \mathbb{N}$,

$$\sum_{j\in J}\int_{a_j}^{b_j}\left[\phi(t) + \sum_{i=0}^{n-2} q_i(t)\omega_i(|x_m^{(i)}|) + \sum_{i=0}^{n-2} h_i(t, x_m^{(i)})\right] dt < \varepsilon. \tag{2.19}$$

By (2.5) we have

$$\phi(t) \le f_m\big(t, x(t), \dots, x^{(n-2)}(t)\big), \quad t \in J.$$

Thus the conditions in Lemma 2.1 and Lemma 2.2 are satisfied. There exists $c = c(\phi)$ such that

$$x_m^{(i)}(t) \ge \frac{c}{(n-i-3)!}\int_0^t (t-s)^{n-3-i} s(1-s)ds, \quad i = 0, \dots, n-3, \quad t \in J, \tag{2.20}$$

$$x_m^{(n-2)}(t) \ge ct(1-t), \quad t \in [0,1]. \tag{2.21}$$

In addition by Lemma 2.5 one has

$$\|x_m^{(i)}\| \le P, \quad i = 0, \dots, n-2. \tag{2.22}$$

It follows from (2.20)–(2.22) that

$$\begin{aligned}
&\sum_{j\in J}\int_{a_j}^{b_j}\left[\phi(t) + \sum_{i=0}^{n-2} q_i(t)\omega_i(|x_m^{(i)}|) + \sum_{i=0}^{n-2} h_i(t, x_m^{(i)})\right] dt \\
&\le \sum_{j\in J}\int_{a_j}^{b_j}\Bigg[\phi(t) + \Lambda\sum_{i=0}^{n-3} q_i(t)\omega_i\left(\frac{c}{(n-3-i)!}\right)\omega_i\left(\int_0^t (t-s)^{n-3-i}s(1-s)ds\right) \\
&\quad + \Lambda\sum_{i=0}^{n-2} q_i(t)\omega_i(c)\omega_i(t(1-t)) + \sum_{i=0}^{n-2}\max_{(t,x_i)\in[0,1]\times[0,P]} h_i(t, x_i)\Bigg] dt.
\end{aligned}$$

By (H_1), we have $\phi, q_i, h_i, \in L_1(J)$ and (1.6) hold. Consequently, for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any at most countable set $\{(a_j, b_j)\}_{j\in J}$ of mutually disjoint intervals $(a_j, b_j) \subset J$ with $\sum_{j\in J}(b_j - a_j) < \delta$, (2.19) holds. This completes the proof. □

3. Existence results

Theorem 3.1. *Suppose the assumptions (H_1), (H_2) and (H_3) be satisfied. Then there exists a solution x such that $x \in AC^{(n-1)}(J)$, $x^{(i)}(t) \ge 0$, $t \in J$ for BVP* (1.1), (1.2).

Proof. For each $m \in \mathbb{N}$, there exists a solution x_m of BVP (2.6), (1.2) by Lemma 2.4, Lemma 2.5. Consider the solution sequence $\{x_m\}$. Lemma 2.5 shows that $\{x_m\}$ is bounded in $C^{n-1}(J)$. Lemma 2.1 and Lemma 2.2 means that

$$x_m^{(i)}(t) \geq c \int_0^t (t-s)^{n-3-i} s(1-s)ds, \quad 0 \leq i \leq n-3, \tag{3.1}$$

$$x_m^{(n-2)}(t) \geq ct(1-t), \quad t \in J. \tag{3.2}$$

The Arzelà-Ascoli theorem guarantees the existence of a subsequence $\{x_{m_k}\}$ converging in $C^{n-2}(J)$. Lemma 2.2 means that c is independent m, so (3.1) and (3.2) gives

$$x^{(i)}(t) \geq c \int_0^t (t-s)^{n-3-i} s(1-s)ds, \quad i = 0, \ldots, n-3,$$

$$x^{(n-2)}(t) \geq ct(1-t), \quad t \in J.$$

Moreover,

$$\begin{aligned} \lim_{m\to\infty} x_m^{(i)}(0) &= x^{(i)}(0) = 0, \quad i = 0, \ldots, n-2, \\ \lim_{m\to\infty} x_m^{(n-2)}(1) &= x^{(n-2)}(1). \end{aligned} \tag{3.3}$$

By Riemann-Stieltjes dominated convergence theorem we have

$$\lim_{m\to\infty} \int_0^1 x_m^{(n-2)}(s)dg(s) = \int_0^1 x^{(n-2)}(s)dg(s). \tag{3.4}$$

(3.3), (3.4) means x satisfies the boundary condition (1.2).

From $f_{m_k} \in \mathrm{Car}(J \times \mathbb{R}^n)$, and their construction, it follows that there exists $\mathcal{M} \in J$, $\mu(\mathcal{M}) = 0$ such that $f_{m_k}(t, \cdot, \ldots, \cdot)$ are continuous on $\mathbb{R}^{n-1}$ for each $t \in J \setminus \mathcal{M}$, which implies that

$$\lim_{k\to\infty} f_{m_k}\big(t, x_{m_k}(t), \ldots, x_{m_k}^{(n-2)}(t)\big) = f\big(t, x(t), \ldots, x^{(n-2)}(t)\big)$$

for $t \in J \setminus \mathcal{M} \cup \{0\}$.

By Lemma 2.6 $\{f_{m_k}\big(t, x_{m_k}(t), \ldots, x_{m_k}^{(n-2)}(t)\big)\}$ is uniformly absolutely continuous on J. Hence $f \in L_1(J)$ and

$$\lim_{k\to\infty} \int_0^t f_{m_k}\big(s, x_{m_k}(s), \ldots, x_{m_k}^{(n-2)}(s)\big)ds = \int_0^t f\big(s, x(s), \ldots, x^{(n-2)}(s)\big)ds$$

for $t \in J$ by the Vitali's convergence theorem. Since $\{x_{m_k}^{(n-1)}\}$ is bounded, we can assume that it is convergent, say

$$\lim_{k\to\infty} x_{m_k}^{(n-1)}(0) = C.$$

Taking limits as $k \to \infty$ in the following equalities

$$x_{m_k}^{(n-2)}(t) = x_{m_k}^{(n-1)}(0)t - \int_0^t \int_0^s f_{m_k}(r, x_{m_k}(r), \dots, x_{m_k}^{(n-2)}(r))drds, \quad t \in J$$

we get

$$x^{(n-2)}(t) = Ct - \int_0^t \int_0^s f(r, x(r), \dots, x^{(n-2)}(r))drds, \quad t \in J.$$

Then $x \in AC^{n-1}(J)$ and

$$x^{(n)}(t) + f(t, x(t), \dots, x^{(n-1)}(t)) = 0, \quad \text{for a.e. } t \in J.$$

Therefore, x is a solution of BVP (1.1), (1.2). □

4. Example

Let us consider the following third-order boundary value problem

$$\begin{cases} x^{(3)} + \phi(t) + \dfrac{q_0(t)}{x_0^{1/4}} + \dfrac{q_1(t)}{x_1^{1/2}} + a_0(t)\sin(|x_0|) + a_1(t)|x_1| = 0, \\ (0) = x'(0) = 0, \qquad\qquad x'(1) = \int_0^1 x'(s)dg(s) \end{cases} \tag{4.1}$$

with $\phi, q_i \in L_1(J)$, $a_i \in C(J)$ be positive for $i = 0, 1$, g is Lebesgue measurable, increasing on J and satisfies $g(0) = 0$, $g(1) < 1$. Corresponding to BVP (1.1)–(1.2), we have

$$\begin{aligned} &f(t, x_0, x_1) = \phi(t) + \frac{q_0(t)}{x_0^{1/4}} + \frac{q_1(t)}{x_1^{1/2}} + a_0(t)\sin(|x_0|) + a_1(t)|x_1|, \\ &\omega_0(|x_0|) = \frac{1}{x_0^{1/4}}, \qquad\qquad \omega_1(x_1) = \frac{1}{x_0^{1/2}}, \\ &h_0(t, x_0) = a_0(t)\sin(|x_0|), \qquad h_1(t, x_1) = a_1(t)|x_1|. \end{aligned}$$

Assume

$$\sup_{t \in [0,1]} a_1(t) < 1 \tag{4.2}$$

holds. Then (4.1) has at least one positive solution $x \in C^2(J)$, $x'' \in AC(J)$.

To see that (4.1) has a positive solution $x \in C^2(J)$, $x'' \in AC(J)$, we notice

$$\begin{aligned} \int_0^1 \omega_0\left(\int_0^t s(1-s)ds\right)dt = \sqrt[4]{6}\int_0^1 \frac{1}{\sqrt[4]{3t^2 - 2t^3}} < \sqrt[4]{6}\int_0^1 \frac{1}{\sqrt{t}}dt = 2\sqrt[4]{6} < \infty, \\ \int_0^1 \omega_1(s(1-s))ds = \int_0^1 \frac{1}{\sqrt{s(1-s)}}ds < \infty, \end{aligned}$$

so (H_1) is satisfied. (H_2) is clear to be satisfied. It is easy to see (H_3) are true since (4.2) hold. Theorem 3.1 now guarantees that BVP (4.1) has a solution $x \in AC^{n-1}(J)$, $x^{(i)}(t) \geq 0$, $i = 0, 1$.

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Yu Tian
Department of Applied Mathematics
Beijing Institute of Technology
Beijing 100081
P. R. China
e-mail: tianyu2992@163.com

Weigao Ge
Department of Applied Mathematics
Beijing Institute of Technology
Beijing 100081
P. R. China