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PRODUCTS OF STRONG ŚWIĄTKOWSKI FUNCTIONS

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Abstract. In this paper we characterize the products of four or more strong Świątkowski functions.

1. Preliminaries

The letters $\mathbb{R}$ and $\mathbb{N}$ denote the real line and the set of positive integers, respectively. For all $a, b \in \mathbb{R}$, we define $\mathrm{I}(a, b] \stackrel{\mathrm{df}}{=} (a, b]$, if $a < b$, and $\mathrm{I}(a, b] \stackrel{\mathrm{df}}{=} [b, a)$ otherwise. The symbol $\mathrm{I}(a, b)$ is defined analogously.

For each $A \subset \mathbb{R}$ we use the symbols $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{bd} A$ and $\operatorname{card} A$ to denote the interior, the closure, the boundary, and the cardinality of A, respectively. We say that a set $A \subset \mathbb{R}$ is *simply open* [1], if it can be written as the union of an open set and a nowhere dense set.

Let $f : I \to \mathbb{R}$, where I is a nondegenerate interval. The symbol $\mathcal{C}(f)$ stands for the set of all points of continuity of f. We say that f is a *Darboux function*, if it maps connected sets onto connected sets. We say that f is *quasi-continuous* in the sense of Kempisty [4], if for all $x \in I$ and open sets $U \ni x$ and $V \ni f(x)$, the set $\operatorname{int}\big(U \cap f^{-1}(V)\big)$ is nonempty. We say that

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f is *cliquish* [10], if the set of points of continuity of f is dense in I. We say that f is a *strong Świątkowski function* [6] ($f \in \acute{\mathcal{S}}_s$), if whenever $\alpha, \beta \in I$ and $y \in \mathrm{I}\big(f(\alpha), f(\beta)\big)$, there is an $x_0 \in \mathrm{I}(\alpha, \beta) \cap \mathcal{C}(f)$ such that $f(x_0) = y$. One can easily see that strong Świątkowski functions are both Darboux and quasi-continuous. The symbol $[f = a]$ stands for the set $\{x \in I : f(x) = a\}$. Similarly we define the symbols $[f < a]$, $[f > a]$, etc. The symbol $\mathcal{M}$ denotes the class of all real functions f defined on a nondegenerate interval such that f has a zero in each subinterval in which it changes sign.

2. Introduction

In 1996 Maliszewski proved the following theorem [7].

Theorem 2.1. *For each function f the following conditions are equivalent*:

i) *f is a finite product of Darboux quasi-continuous functions,*
ii) *$f \in \mathcal{M}$, f is cliquish, and the set $[f = 0]$ is simply open,*
iii) *there are Darboux quasi-continuous functions g and h such that $f = gh$.*

He showed also that there is a bounded Darboux quasi-continuous function which cannot be written as the finite product of strong Świątkowski functions [7, Proposition III.4.1]. Moreover he remarked that the sign function can be written as the product of three strong Świątkowski functions but it cannot be written as the product of two strong Świątkowski functions [7, Propositions III.4.2 and III.4.3].

In 2003 I showed that there is a function which can be written as the product of four strong Świątkowski functions, and which cannot be written as the product of three strong Świątkowski functions [9]. In this paper I characterize products of four or more strong Świątkowski functions. However, the following problem is still open.

Problem 1. *Characterize the products of two strong Świątkowski functions and the products of three strong Świątkowski functions.*

3. Auxiliary lemmas

Lemmas 3.1 and 3.2 can be easily proved using [5, Theorem 12].

Lemma 3.1. *Let $I \subset \mathbb{R}$ be an interval, $g\colon I \to \mathbb{R}$ and $x \in I$. If $g{\restriction}I \cap (-\infty, x) \in \acute{\mathcal{S}}_s$, $g{\restriction}I \cap (x, \infty) \in \acute{\mathcal{S}}_s$ and $x \in \mathcal{C}(g)$, then $g \in \acute{\mathcal{S}}_s$.*

Lemma 3.2. *Let $I \subset \mathbb{R}$ be an interval, $g\colon I \to \mathbb{R}$ and $x \in I$. If $g{\restriction}I \cap (-\infty, x] \in \acute{\mathcal{S}}_s$, $g{\restriction}I \cap (x, \infty) \in \acute{\mathcal{S}}_s$ and $g(x) \in g\big[[x,t] \cap \mathcal{C}(g)\big]$ for each $t \in (x, \sup I)$, then $g \in \acute{\mathcal{S}}_s$.*

The next lemma is interesting in itself.

Lemma 3.3. *Let $I \subset \mathbb{R}$ be an interval, $g\colon I \to \mathbb{R}$ and $h\colon \mathbb{R} \to \mathbb{R}$. If $g, h \in \acute{\mathcal{S}}_s$, then $h \circ g \in \acute{\mathcal{S}}_s$.*

Proof. Let $\alpha, \beta \in I$, $\alpha < \beta$ and $z \in \mathrm{I}\big((h \circ g)(\alpha), (h \circ g)(\beta)\big)$. Since $h \in \acute{\mathcal{S}}_s$, there is a $y \in \mathrm{I}\big(g(\alpha), g(\beta)\big) \cap \mathcal{C}(h)$ such that $h(y) = z$. Since $y \in \mathrm{I}\big(g(\alpha), g(\beta)\big)$ and $g \in \acute{\mathcal{S}}_s$, there is an $x \in (\alpha, \beta) \cap \mathcal{C}(g)$ with $g(x) = y$. Clearly $x \in \mathcal{C}(h \circ g)$. $\square$

Lemma 3.4. *Assume $F \subset C$ are closed and $\mathcal{J}$ is a family of components of $\mathbb{R} \setminus C$ such that $C \subset \operatorname{cl} \bigcup \mathcal{J}$. There is a family $\mathcal{J}' \subset \mathcal{J}$ such that*

i) for each $J \in \mathcal{J}$, if $F \cap \operatorname{bd} J \neq \emptyset$, then $J \in \mathcal{J}'$,
ii) for each $c \in F$, if c is a right-hand (left-hand) limit point of C, then c is a right-hand (respectively left-hand) limit point of the union $\bigcup \mathcal{J}'$,
iii) $\operatorname{cl} \bigcup \mathcal{J}' \subset F \cup \bigcup_{J \in \mathcal{J}'} \operatorname{cl} J$.

Proof. Let $\mathcal{P}$ be the family of all components of $\mathbb{R} \setminus F$ and $P \in \mathcal{P}$. One can easily see that there is a family $\mathcal{J}_P \subset \mathcal{J}$ such that $\bigcup \mathcal{J}_P \subset P$ and the following conditions hold:

$$\text{if } \cap C \neq \emptyset, \text{ then } \mathcal{J}_P \neq \emptyset, \tag{1}$$

$$\text{for each } J \in \mathcal{J}, \text{ if } J \subset P \text{ and } \operatorname{bd} P \cap \operatorname{bd} J \neq \emptyset, \text{ then } J \in \mathcal{J}_P, \tag{2}$$

$$\text{if } \inf P \in \operatorname{cl}(P \cap C), \text{ then } \inf P \in \operatorname{cl} \bigcup \mathcal{J}_P, \tag{3}$$

$$\text{if } \sup P \in \operatorname{cl}(P \cap C), \text{ then } \sup P \in \operatorname{cl} \bigcup \mathcal{J}_P, \tag{4}$$

$$\operatorname{cl} \bigcup \mathcal{J}_P \subset \operatorname{bd} P \cup \bigcup_{J \in \mathcal{J}_P} \operatorname{cl} J. \tag{5}$$

Define $\mathcal{J}' = \bigcup_{P \in \mathcal{P}} \mathcal{J}_P$. Clearly $\mathcal{J}' \subset \mathcal{J}$. We will show that $\mathcal{J}'$ satisfies the conditions i)–iii) of the lemma.

Assume that $F \cap \operatorname{bd} J \neq \emptyset$ for some $J \in \mathcal{J}$. Since $F \subset C$, there is a $P \in \mathcal{P}$ with $J \subset P$. Then by (2), $J \in \mathcal{J}_P \subset \mathcal{J}'$. This proves condition i).

To prove condition ii) assume that $c \in F$ is a right-hand limit point of C. We consider two cases.

If there is a $P \in \mathcal{P}$ with $c = \inf P$, then by (3),

$$c \in \operatorname{cl} \bigcup \mathcal{J}_P \subset \operatorname{cl}\big((c, \infty) \cap \bigcup \mathcal{J}'\big).$$

In the opposite case fix a $d > c$. Since $C \subset \operatorname{cl} \bigcup \mathcal{J}$, we obtain $(c, d) \cap \bigcup \mathcal{J} \neq \emptyset$. By our assumption, there is a $J \in \mathcal{J}$ such that $J \subset (c, d)$ and $(\sup J, d) \cap F \neq \emptyset$. Choose $P \in \mathcal{P}$ with $J \subset P$. Clearly $P \subset (c, d)$.

If $P \cap C = \emptyset$, then $P = J \in \mathcal{J}$, and by (2), $P \in \mathcal{J}_P \subset \mathcal{J}'$. So, $(c, d) \cap \bigcup \mathcal{J}' \neq \emptyset$.

If $P \cap C \neq \emptyset$, then by (1), $\mathcal{J}_P \neq \emptyset$. Since $\bigcup \mathcal{J}_P \subset P$, we have $(c, d) \cap \bigcup \mathcal{J}' \neq \emptyset$. This completes the proof of ii).

Finally we will show iii). Note that by (5),

$$\begin{aligned} \operatorname{cl} \bigcup \mathcal{J}' &= \operatorname{cl} \bigcup_{P \in \mathcal{P}} \bigcup \mathcal{J}_P \subset \operatorname{cl} \bigcup_{P \in \mathcal{P}} \operatorname{bd} P \cup \bigcup_{P \in \mathcal{P}} \operatorname{cl} \bigcup \mathcal{J}_P \\ &\subset \operatorname{cl} F \cup \bigcup_{P \in \mathcal{P}} \Big(\bigcup_{J \in \mathcal{J}_P} \operatorname{cl} J \cup \operatorname{bd} P \Big) = F \cup \bigcup_{J \in \mathcal{J}'} \operatorname{cl} J. \end{aligned}$$

This completes the proof of the lemma. □

Lemma 3.5. *Let $I = (a, b)$ and $y_1, y_2 \in [0, 1]$. There is a strong Świątkowski function $\psi\colon \operatorname{cl} I \to [0, 1]$ such that $\psi[I] = \psi[I \cap \mathcal{C}(\psi)] = (0, 1]$, $\psi(a) = y_1$, $\psi(b) = y_2$, and $\operatorname{bd} I \subset \mathcal{C}(\psi)$.*

Proof. Define the function $\bar{\psi}\colon \mathbb{R} \to (0, 1]$ by

$$\bar{\psi}(x) = \begin{cases} \min\{1, \sin x^{-1} + |x| + 1\} & \text{if } x \neq 0, \\ 2^{-1} & \text{if } x = 0. \end{cases}$$

Then clearly $\bar{\psi} \in \acute{\mathcal{S}}_s$. Choose elements $a < x_1 < x_2 < x_3 < b$ and define the function $\psi\colon \operatorname{cl} I \to [0, 1]$ by the formula:

$$\psi(x) = \begin{cases} \bar{\psi}(x - x_2) & \text{if } x \in [x_1, x_3], \\ y_1 & \text{if } x = a, \\ y_2 & \text{if } x = b, \\ \text{linear} & \text{in intervals } [a, x_1] \text{ and } [x_3, b]. \end{cases}$$

One can easily show that the function ψ has all required properties. □

Lemmas 3.6–3.11 and Proposition 4.1 will be used in the proof of the main result of this paper. In all of them we assume the following.

$$E \text{ is a compact interval,} \tag{6}$$

$$f\colon E \to \mathbb{R} \text{ is cliquish,} \tag{7}$$

$$f \in \mathcal{M}, \tag{8}$$

$$\text{the set } [f=0] \text{ is simply open.} \tag{9}$$

Moreover we define

$$Y_f \stackrel{\text{df}}{=} \operatorname{cl}[f<0] \cap \operatorname{cl}[f>0]. \tag{10}$$

One can easily see that by (8), $Y_f \subset \operatorname{bd}[f=0]$. But $[f=0]$ is simply open, so Y_f is nowhere dense.

Lemma 3.6. *Let $I \subset E$ be an open interval such that $I \cap \operatorname{int}[f=0] = \emptyset$ and $f=0$ on the set $K \stackrel{\text{df}}{=} (Y_f \cap I) \cup \operatorname{bd} I$. Then there is a strong Świątkowski function $\varphi\colon \operatorname{cl} I \to [-1,1]$ with $\operatorname{sgn} \circ \varphi = \operatorname{sgn} \circ f \restriction \operatorname{cl} I$ such that $\operatorname{bd} I \subset \mathcal{C}(\varphi)$.*

Proof. Put $G_1 = I \cap \operatorname{int}[f<0]$ and $G_2 = I \cap \operatorname{int}[f>0]$. Let $\mathcal{J}$ be the family of all components of $G_1 \cup G_2$.

For each $J \in \mathcal{J}$, use Lemma 3.5 to construct a strong Świątkowski function $\psi_J\colon \operatorname{cl} J \to [0,1]$ such that

$$\psi_J[J] = \psi_J[J \cap \mathcal{C}(\psi_J)] = (0,1],$$
$$\psi_J(x) = \min\bigl\{\operatorname{dist}(K,x)/2, |\operatorname{sgn} f(x)|\bigr\} \quad \text{if } x \in \operatorname{bd} J,$$

and

$$\operatorname{bd} J \subset \mathcal{C}(\psi_J), \tag{11}$$

and put

$$c_J = \min\bigl\{1, \max\{\operatorname{dist}(K, \inf J), \operatorname{dist}(K, \sup J), |J|\}\bigr\},$$

where $|J|$ denotes the length of the interval J.

Define the function $\varphi\colon \operatorname{cl} I \to [-1,1]$ by

$$\varphi(x) = \begin{cases} c_J \psi_J(x) \operatorname{sgn} f(x) & \text{if } x \in \operatorname{cl} J,\ J \in \mathcal{J}, \\ \min\bigl\{1, \operatorname{dist}(K,x)/2\bigr\} \operatorname{sgn} f(x) & \text{otherwise.} \end{cases}$$

(Notice that if $J_1, J_2 \in \mathcal{J}$ and $\operatorname{cl} J_1 \cap \operatorname{cl} J_2 = \{x\}$, then $f(x) = 0$. So, φ is well defined.)

Clearly K is closed. So, $\operatorname{sgn} \circ \varphi = \operatorname{sgn} \circ f \restriction \operatorname{cl} I$. We will show that $K \subset \mathcal{C}(\varphi)$.

Let $x_0 \in K \setminus \{\inf I\}$. If there is a $J \in \mathcal{J}$ such that $x_0 = \sup J$, then by (11), φ is continuous from the left at x_0.

In the opposite case first let $(x_m) \subset [\inf I, x_0) \cap \bigcup \mathcal{J}$ be a sequence convergent to x_0. For each $m \in \mathbb{N}$ there is a $J_m = (a_m, b_m) \in \mathcal{J}$ such that $x_m \in J_m$. Observe that

$$\lim_{m\to\infty} (a_m - b_m) = 0 \tag{12}$$

and $\operatorname{dist}(K, b_m) \le x_0 - b_m < x_0 - x_m$, whence

$$\lim_{m\to\infty} \operatorname{dist}(K, b_m) = 0. \tag{13}$$

By (12) and (13), we obtain

$$\lim_{m\to\infty} \operatorname{dist}(K, a_m) = 0. \tag{14}$$

Since

$$\begin{aligned}0 &\le |\varphi(x_m)| = |c_{J_m}\psi_{J_m}(x_m)| \le c_{J_m}\\ &= \min\bigl\{1, \max\{\operatorname{dist}(K, a_m), \operatorname{dist}(K, b_m), b_m - a_m\}\bigr\},\end{aligned}$$

by (12), (13) and (14), we obtain $\lim_{m\to\infty} \varphi(x_m) = 0 = \varphi(x_0)$. It proves that x_0 is a left-hand side point of continuity of the restriction $\varphi\restriction \bigcup \mathcal{J} \cup \{x_0\}$.

Now put $G_3 = \operatorname{cl} I \setminus (G_1 \cup G_2)$. Since $G_3 = \operatorname{cl}[f = 0] = \operatorname{bd}[f = 0]$, G_3 is nowhere dense. Let $(x_m) \subset [\inf I, x_0) \cap G_3$ be a sequence convergent to x_0. Then

$$|\varphi(x_m)| \le \operatorname{dist}(K, x_m)/2 < x_0 - x_m \to 0 = \varphi(x_0).$$

So, x_0 is a left-hand side point of continuity of $\varphi\restriction G_3$. It follows that φ is continuous from the left at x_0.

Similarly we can prove that the function φ is continuous from the right at each point $x_0 \in K \setminus \{\sup I\}$. So, $K \subset \mathcal{C}(\varphi)$ and in particular $\operatorname{bd} I \subset \mathcal{C}(\varphi)$.

Now we will prove that

$$\forall_{x\in G_3\setminus\{\sup J : J\in\mathcal{J}\}} \forall_{\delta>0} \;\; \varphi[(x-\delta, x) \cap \mathcal{C}(\varphi)] \;\supset\; \mathrm{I}(0, \varphi(x)]. \tag{15}$$

Indeed, fix an $x \in G_3 \setminus \{\sup J : J \in \mathcal{J}\}$ and a $\delta > 0$. Assume that $\varphi(x) > 0$. (The case $\varphi(x) < 0$ is analogous.) Then $x \notin K$, so $d \overset{\text{df}}{=} \operatorname{dist}(K, x) > 0$. Notice that $\varphi(x) \le \min\{1, d/2\}$, and define $\delta_1 = \min\{\varphi(x), \delta\}$. There is a $J \in \mathcal{J}$ such that $\operatorname{cl} J \subset I \cap (x - \delta_1, x)$. For each $z \in I \cap (x - \delta_1, x)$ we have

$$\operatorname{dist}(K, z) \ge \operatorname{dist}(K, x) - |z - x| > d - \delta_1 \ge d/2 > 0$$

(in particular, $\operatorname{dist}(K, \inf J) > d/2$) and

$$(x - \delta_1, x) \cap K = (x - \delta_1, x) \cap Y_f \cap I = \emptyset.$$

Consequently, the function f does not change its sign in the interval $I \cap (x - \delta_1, x)$. But $\operatorname{sgn} \circ \varphi = \operatorname{sgn} \circ f\restriction \operatorname{cl} I$, so $\varphi \ge 0$ on $I \cap (x - \delta_1, x)$. Finally since

$$c_J \ge \min\bigl\{1, \operatorname{dist}(K, \inf J)\bigr\} \ge \min\bigl\{1, d/2\bigr\} \ge \varphi(x),$$

we obtain

$$\varphi[(x - \delta, x) \cap \mathcal{C}(\varphi)] \supset \varphi[J \cap \mathcal{C}(\varphi)] \supset (0, c_J] \supset (0, \varphi(x)].$$

Similarly we can prove that

$$\forall_{x\in G_3\setminus\{\inf J : J\in\mathcal{J}\}} \forall_{\delta>0}\; \varphi[(x, x+\delta) \cap \mathcal{C}(\varphi)] \supset \mathrm{I}(0, \varphi(x)].$$

To complete the proof we will show that $\varphi \in \acute{\mathcal{S}}_s$. Let $c, d \in \operatorname{cl} I$, $c < d$, and $y \in \mathrm{I}(\varphi(c), \varphi(d))$. Assume that $\varphi(c) < \varphi(d)$. (The case $\varphi(c) > \varphi(d)$ is similar).

If there is a $J \in \mathcal{J}$ such that $c, d \in \operatorname{cl} J$, then since $\psi_J \in \acute{\mathcal{S}}_s$, there is an $x_0 \in (c, d) \cap \mathcal{C}(\varphi)$ such that $\varphi(x_0) = y$. So assume that this case does not hold.

If $y = 0$, then put $x_0 = \sup\bigl\{x \in [c, d] : \varphi(x) \le 0\bigr\}$. It suffices to show that $x_0 \in K \cap (c, d)$.

Indeed, suppose that $x_0 = d$. Then $d \in \operatorname{cl}[f \le 0]$. Since $\operatorname{sgn} f(d) = \operatorname{sgn} \varphi(d) = 1$ and $I \cap \operatorname{int}[f = 0] = \emptyset$, we have $d \in K$ and $f(d) = 0$, an impossibility. Similarly we can proceed if $x_0 = c$. So, $x_0 \in (c, d)$. Now it is easy to see that $x_0 \in K$ (recall that $I \cap \operatorname{int}[f = 0] = \emptyset$).

Finally assume that $y > 0$. (The case $y < 0$ is analogous.) Then $\varphi(d) > 0$. We consider two cases.

Case 1. $d \notin G_3$ or $d \in \{\sup J : J \in \mathcal{J}\}$.

Then for some $J \in \mathcal{J}$ we have $d \in \operatorname{cl} J$ and $c \notin \operatorname{cl} J$. If $y \in \mathrm{I}(\varphi(\inf J), \varphi(d))$, then there is an $x_0 \in (\inf J, d) \cap \mathcal{C}(\psi_J) \subset (c, d) \cap \mathcal{C}(\varphi)$ with $\varphi(x_0) = y$. (Recall that $\psi_J \in \acute{\mathcal{S}}_s$).

If $y \in (0, \varphi(\inf J)]$, then $\varphi(\inf J) > 0$, whence $\inf J \ne \sup J'$ for each $J' \in \mathcal{J}$. By (15), there is an $x_0 \in (c, \inf J) \cap \mathcal{C}(\varphi) \subset (c, d) \cap \mathcal{C}(\varphi)$ such that $\varphi(x_0) = y$.

Case 2. $d \in G_3 \setminus \{\sup J : J \in \mathcal{J}\}$.

Then $y \in (0, \varphi(d))$, so by (15), there is an $x_0 \in (c, d) \cap \mathcal{C}(\varphi)$ with $\varphi(x_0) = y$. $\square$

Remark 3.1. It is quite evident that if $f = 0$ on $\operatorname{cl} I$, then there is a function $\varphi\colon \operatorname{cl} I \to [-1, 1]$, which satisfies all the requirements of Lemma 3.6.

Lemma 3.7. *Let $I = (a, b) \subset E$ be an interval such that $f = 0$ on $(Y_f \cap I) \cup \{b\}$. Assume that $I \subset [f = 0]$ or $I \cap \operatorname{int}[f = 0] = \emptyset$ and $a \in \operatorname{cl}(I \cap [f = 0])$. There are strong Świątkowski functions $g_1, g_2\colon \operatorname{cl} I \to [-1, 1]$ with $\operatorname{sgn} \circ (g_1 g_2) = \operatorname{sgn} \circ f \restriction \operatorname{cl} I$ such that for $i \in \{1, 2\}$, we have: $g_i(a) = (\operatorname{sgn} f(a))^{i+1}/2$, $g_i(b) = 0$, $b \in \mathcal{C}(g_i)$, and $g_i[(a, c) \cap \mathcal{C}(g_i)] = [-1, 1]$ for each $c \in (a, b)$.*

Proof. Choose a sequence $(x_n) \subset I \cap [f = 0]$ such that $x_n \searrow a$. Put $x_0 = b$. Define the function $h\colon (a, b] \to [-1, 1]$ by

$$h(x) = \begin{cases} 0 & \text{if } x = x_{n-1},\ n \in \mathbb{N}, \\ (-1)^{n-1} & \text{if } x = (x_{n-1} + x_n)/2,\ n \in \mathbb{N}, \\ \text{linear} & \text{in each interval of the form } [x_n, (x_{n-1} + x_n)/2] \\ & \text{or } [(x_{n-1} + x_n)/2, x_{n-1}],\ n \in \mathbb{N}. \end{cases}$$

Then h is continuous on $(a,b]$. For each $n \in \mathbb{N}$, use Lemma 3.6 to construct a strong Świątkowski function $\varphi_n \colon [x_{2n}, x_{2n-2}] \to [-1,1]$ with $\operatorname{sgn} \circ \varphi_n = \operatorname{sgn} \circ f \restriction [x_{2n}, x_{2n-2}]$ such that $x_{2n}, x_{2n-2} \in \mathcal{C}(\varphi_n)$.

Fix an $i \in \{1,2\}$. Define the function $g_i \colon \operatorname{cl} I \to [-1,1]$ by

$$g_i(x) = \begin{cases} \quad h(x) & \text{if } x \in \bigcup_{n=1}^{\infty} [x_{4n-2i+2}, x_{4n-2i}], \\ \varphi_{2n+i-2}(x) \operatorname{sgn} h(x) & \text{if } x \in \bigcup_{n=1}^{\infty} [x_{4n+2i-4}, x_{4n+2i-6}], \\ (\operatorname{sgn} f(x))^{i+1}/2 & \text{if } x = a. \end{cases}$$

Since $h(x_n) = 0$ for each $n \in \mathbb{N} \cup \{0\}$, the function g_i is well defined and $g_i(b) = 0$. We can easily see that $g_i(a) = (\operatorname{sgn} f(a))^{i+1}/2$, $b \in \mathcal{C}(g_i)$ and

$$g_i[(a,c) \cap \mathcal{C}(g_i)] = [-1,1] \quad \text{for each } c \in (a,b). \tag{16}$$

By Lemma 3.1, $g_i \restriction (a,b] \in \acute{\mathcal{S}}_s$. Hence by (16) and Lemma 3.2, $g_i \in \acute{\mathcal{S}}_s$.

Clearly $\operatorname{sgn} \circ (g_1 g_2) = \operatorname{sgn} \circ f \restriction \operatorname{cl} I$. This completes the proof. □

Lemma 3.8. *Let $I \subset E$ be an open interval such that $I \cap [f = 0] = \emptyset$. There are strong Świątkowski functions $g_1, g_2 \colon \operatorname{cl} I \to [-1,1]$ with $\operatorname{sgn} \circ (g_1 g_2) = \operatorname{sgn} \circ f \restriction \operatorname{cl} I$ such that $g_1[I \cap \mathcal{C}(g_1)] \supset (0,1]$ and for $i \in \{1,2\}$, we have: $g_i(x) = (\operatorname{sgn} f(x))^{i+1}/2$ for $x \in \operatorname{bd} I$ and $\operatorname{bd} I \subset \mathcal{C}(g_i)$.*

Proof. Use Lemma 3.5 to construct a strong Świątkowski function $\psi \colon \operatorname{cl} I \to [0,1]$ such that $\psi[I] = \psi[I \cap \mathcal{C}(\psi)] = (0,1]$, $\operatorname{bd} I \subset \mathcal{C}(\psi)$ and $\psi(x) = |\operatorname{sgn} f(x)|/2$ for $x \in \operatorname{bd} I$. For $i \in \{1,2\}$, define $g_i = \psi \cdot (\operatorname{sgn} \circ f)^{i+1}$. It is not hard to see that the functions g_1 and g_2 possess all the required properties. □

Lemma 3.9. *Let $I = (a,b) \subset E$ be an interval such that $f = 0$ on $(Y_f \cap I) \cup \{b\}$. Assume that $I \cap \operatorname{int}[f = 0] = \emptyset$ and $a \notin \operatorname{cl}(I \cap [f = 0])$. Then there are strong Świątkowski functions $g_1, g_2 \colon \operatorname{cl} I \to [-1,1]$ with $\operatorname{sgn} \circ (g_1 g_2) = \operatorname{sgn} \circ f \restriction \operatorname{cl} I$ such that $g_1[I \cap \mathcal{C}(g_1)] \supset (0,1]$ and for $i \in \{1,2\}$, we have: $g_i(a) = (\operatorname{sgn} f(a))^{i+1}/2$, $g_i(b) = 0$, and $\operatorname{bd} I \subset \mathcal{C}(g_i)$.*

Proof. Put

$$z = \sup\big\{x \in [a,b] : (a,x) \cap [f = 0] = \emptyset\big\}.$$

Use Lemma 3.8 to construct strong Świątkowski functions $\psi_1, \psi_2 \colon [a,z] \to [0,1]$ with $\operatorname{sgn} \circ (\psi_1 \psi_2) = \operatorname{sgn} \circ f \restriction [a,z]$ such that $\psi_1[(a,z) \cap \mathcal{C}(\psi_1)] \supset (0,1]$, and for $i \in \{1,2\}$, we have $\psi_i(a) = (\operatorname{sgn} f(a))^{i+1}/2$, $\psi_i(z) = (\operatorname{sgn} f(z))^{i+1}/2$ and

$$a, z \in \mathcal{C}(\psi_i). \tag{17}$$

If $z \neq b$ and $z \notin \operatorname{cl}\big((z,b) \cap [f=0]\big)$, then $f(z)=0$. Construct a continuous function $\varphi_1\colon [z,b] \to [0,1]$ such that $\varphi_1[(z,b)] = (0,1]$ and $\varphi_1(z) = \varphi_1(b) = 0$, and use Lemma 3.6 to construct a strong Świątkowski function $\varphi_2\colon [z,b] \to [-1,1]$ with $\operatorname{sgn} \circ \varphi_2 = \operatorname{sgn} \circ f{\restriction}[z,b]$ such that $z, b \in \mathcal{C}(\varphi_2)$.

If $z \in \operatorname{cl}\big((z,b) \cap [f=0]\big)$, then use Lemma 3.7 to construct strong Świątkowski functions $\bar{g}_1, \bar{g}_2\colon [z,b] \to [-1,1]$ with $\operatorname{sgn} \circ (\bar{g}_1\bar{g}_2) = \operatorname{sgn} \circ f{\restriction}(z,b]$ such that for $i \in \{1,2\}$, we have $\bar{g}_i(b) = 0$, $b \in \mathcal{C}(\bar{g}_i)$, and

$$\forall_{c\in(z,b)}\ \bar{g}_i[(z,c) \cap \mathcal{C}(\bar{g}_i)] = [-1,1]. \tag{18}$$

Fix an $i \in \{1,2\}$. Define the function $g_i\colon \operatorname{cl} I \to [-1,1]$ by

$$g_i(x) = \begin{cases} \psi_i(x) & \text{if } x \in [a,z], \\ \varphi_i(x) & \text{if } x \in (z,b] \text{ and } z \notin \operatorname{cl}((z,b) \cap [f=0]), \\ \bar{g}_i(x) & \text{if } x \in (z,b] \text{ and } z \in \operatorname{cl}((z,b) \cap [f=0]). \end{cases}$$

We will show that $g_i \in \acute{\mathcal{S}}_s$. Clearly $g_i{\restriction}[a,z] = \psi_i \in \acute{\mathcal{S}}_s$.

If $z \neq b$ and $z \notin \operatorname{cl}\big((z,b)\cap[f=0]\big)$, then $f(z)=0$ and $g_i{\restriction}(z,b] = \varphi_i \in \acute{\mathcal{S}}_s$. So, $g_i(z) = \psi_i(z) = \varphi_i(z) = 0$ and by (17), z is a left-hand side point of continuity of g_i. Moreover, since $z \in \mathcal{C}(\varphi_i)$, z is a right-hand side point of continuity of g_i. Hence by Lemma 3.1, $g_i \in \acute{\mathcal{S}}_s$.

If $z \in \operatorname{cl}\big((z,b) \cap [f=0]\big)$, then $g_i{\restriction}(z,b] = \bar{g}_i$. In this case by (18) and Lemma 3.2, we obtain $g_i \in \acute{\mathcal{S}}_s$.

The other requirements of the lemma are evident. $\square$

Lemma 3.10. *Let $I = (a,b) \subset E$ be an open interval such that $f = 0$ on $Y_f \cap I$. Assume that $I \subset [f=0]$ or $I \cap \operatorname{int}[f=0] = \emptyset$. There are strong Świątkowski functions $g_1, g_2\colon \operatorname{cl} I \to [-1,1]$ with $\operatorname{sgn} \circ (g_1g_2) = \operatorname{sgn} \circ f{\restriction}\operatorname{cl} I$ such that*

$$g_1[I \cap \mathcal{C}(g_1)] \supset (0,1] \tag{19}$$

and for $i \in \{1,2\}$, we have:

i) $g_i(a) = (\operatorname{sgn} f(a))^{i+1}/2$, $\quad g_i(b) = (\operatorname{sgn} f(b))^{i+1}/2$,
ii) *if $a \in \operatorname{cl}(I \cap [f=0])$, then $g_i[(a,c) \cap \mathcal{C}(g_i)] = [-1,1]$ for each $c \in (a,b)$,*
iii) *if $a \notin \operatorname{cl}(I \cap [f=0])$, then $a \in \mathcal{C}(g_i)$,*
iv) *if $b \in \operatorname{cl}(I \cap [f=0])$, then $g_i[(c,b) \cap \mathcal{C}(g_i)] = [-1,1]$ for each $c \in (a,b)$,*
v) *if $b \notin \operatorname{cl}(I \cap [f=0])$, then $b \in \mathcal{C}(g_i)$,*
vi) *if $I \cap [f=0] \neq \emptyset$, then $I \cap [g_i = 0] \cap \mathcal{C}(g_i) \neq \emptyset$.*

Proof. We consider two cases.

Case 1. $I \cap [f=0] = \emptyset$.

This case follows by Lemma 3.8.

Case 2. $I \cap [f=0] \neq \emptyset$.

Fix a $z \in I \cap [f=0]$. If $a \in \operatorname{cl}(I \cap [f=0])$, then use Lemma 3.7 to construct strong Świątkowski functions $g_1, g_2 \colon [a,z] \to [-1,1]$ with $\operatorname{sgn} \circ (g_1 g_2) = \operatorname{sgn} \circ f \restriction [a,z]$ such that for $i \in \{1,2\}$, we have $g_i(a) = (\operatorname{sgn} f(a))^{i+1}/2$, $g_i(z) = 0$, and

$$z \in \mathcal{C}(g_i), \tag{20}$$

$$\forall_{c \in (a,z)}\ g_i[(a,c) \cap \mathcal{C}(g_i)] = [-1,1]. \tag{21}$$

If $a \notin \operatorname{cl}(I \cap [f=0])$, then use Lemma 3.9 to construct strong Świątkowski functions $g_1, g_2 \colon [a,z] \to [-1,1]$ with $\operatorname{sgn} \circ (g_1 g_2) = \operatorname{sgn} \circ f \restriction [a,z]$ such that $g_1[(a,z) \cap \mathcal{C}(g_1)] \supset (0,1]$ and for $i \in \{1,2\}$, we have $g_i(a) = (\operatorname{sgn} f(a))^{i+1}/2$, $g_i(z) = 0$, and

$$a, z \in \mathcal{C}(g_i). \tag{22}$$

Proceed similarly, using Lemma 3.7 (if $b \in \operatorname{cl}(I \cap [f=0])$) or Lemma 3.9 (if $b \notin \operatorname{cl}(I \cap [f=0])$), to extend the functions g_1 and g_2 to whole the interval $\operatorname{cl} I$. Then clearly $\operatorname{sgn} \circ (g_1 g_2) = \operatorname{sgn} \circ f \restriction \operatorname{cl} I$ and the conditions (19) and i) hold.

Fix an $i \in \{1,2\}$. Then $g_i(z) = 0$. By (22) or (20), z is a left-hand side point of continuity of g_i. Analogously, z is a right-hand side point of continuity of g_i. So, the condition vi) is fulfilled and by Lemma 3.1, $g_i \in \acute{\mathcal{S}}_s$.

If $a \in \operatorname{cl}(I \cap [f=0])$, then by (21), $g_i[(a,c) \cap \mathcal{C}(g_i)] = [-1,1]$ for each $c \in (a,b)$, and the condition ii) holds. If $a \notin \operatorname{cl}(I \cap [f=0])$, then by (22), $a \in \mathcal{C}(g_i)$, and the condition iii) is fulfilled. Similarly we can prove the conditions iv) and v). $\square$

Lemma 3.11. *Let $I \subset E$ be an open interval. Assume that there is a G_δ-set $A \subset [f=0]$ such that*

$$\forall_{a,b \in I}\ f(a)f(b) < 0 \Rightarrow A \cap \mathrm{I}(a,b) \neq \emptyset$$

and $\operatorname{card}\big(I \cap A \setminus \operatorname{int}[f=0]\big) \leq \omega$. *Then there are strong Świątkowski functions* $g_1, g_2 \colon \operatorname{cl} I \to [-1,1]$ *with* $\operatorname{sgn} \circ (g_1 g_2) = \operatorname{sgn} \circ f \restriction \operatorname{cl} I$ *such that* $g_1[I \cap \mathcal{C}(g_1)] \supset (0,1]$ *and* $|g_i(x)| \leq 2^{-1}$ *for* $i \in \{1,2\}$ *and* $x \in \operatorname{bd} I$.

Proof. Define

$$X = \big\{x \in Y_f \cap \operatorname{cl} I : f(x) \neq 0\big\} \cup \big(\operatorname{bd} \operatorname{int}[f=0] \cap \operatorname{cl} I\big),$$

and let $C = \operatorname{cl} X$. We can easily see that C is nowhere dense. Write the set C as the disjoint union $C = C_1 \cup C_2$, where C_1 is countable and C_2 is perfect. Let $\mathcal{P}$ be the family of all components of $I \setminus C_2$. Fix a $P \in \mathcal{P}$.

Let $\mathcal{J}_P$ be the family of all components of $P \setminus C_1$. Since C_1 is nowhere dense, $\operatorname{cl} P = \operatorname{cl} \bigcup \mathcal{J}_P$. For each $J \in \mathcal{J}_P$ construct functions $g_{1,J}, g_{2,J} \colon \operatorname{cl} J \to [-1,1]$, which fulfill the requirements of Lemma 3.10.

Fix an $i \in \{1,2\}$. Define the function $g_{i,P}\colon \operatorname{cl} P \to [-1,1]$ by

$$g_{i,P}(x) = \begin{cases} g_{i,J}(x) & \text{if } x \in J,\ J \in \mathcal{J}_P, \\ (\operatorname{sgn} f(x))^{i+1}/2 & \text{otherwise.} \end{cases}$$

Observe that $g_{i,P}\restriction \operatorname{cl} J = g_{i,J} \in \acute{\mathcal{S}}_s$ for each $J \in \mathcal{J}_P$ and

$$\begin{aligned} &\text{if } x \in \operatorname{bd} J \cap \operatorname{bd} J' \text{ for some } J, J' \in \mathcal{J}_P,\ J \neq J', \\ &\text{then } x \in \operatorname{cl}(J \cap [f=0]) \text{ or } x \in \operatorname{cl}(J' \cap [f=0]) \end{aligned} \tag{23}$$

(since the above assumption yields $x \in X \subset \operatorname{cl}[f=0]$).

Now we will show that $g_{i,P} \in \acute{\mathcal{S}}_s$. Let $c, d \in \operatorname{cl} P$, $c < d$, and $y \in \mathrm{I}(g_{i,P}(c), g_{i,P}(d))$. If $c, d \in \operatorname{cl} J$ for some $J \in \mathcal{J}_P$, then, since $g_{i,J} \in \acute{\mathcal{S}}_s$, there is an $x_0 \in (c,d) \cap \mathcal{C}(g_{i,P})$ such that $g_{i,P}(x_0) = y$.

In the opposite case there is an $x_1 \in C_1 \cap (c,d)$ which is isolated in C_1; i.e., $x_1 \in \operatorname{bd} J \cap \operatorname{bd} J' \cap (c,d)$ for some $J, J' \in \mathcal{J}_P$, $J \neq J'$. By (23) and conditions ii) or iv) of Lemma 3.10, we obtain $g_{i,P}[(c,d) \cap \mathcal{C}(g_{i,P})] = [-1,1]$. Hence there is an $x_0 \in (c,d) \cap \mathcal{C}(g_{i,P})$ such that $g_{i,P}(x_0) = y$.

Observe that by condition (19) of Lemma 3.10,

$$g_{1,P}[P \cap \mathcal{C}(g_{1,P})] \supset (0,1]. \tag{24}$$

Moreover

$$\text{if } P \cap [f=0] \neq \emptyset, \text{ then } P \cap [g_{i,P}=0] \cap \mathcal{C}(g_{i,P}) \neq \emptyset \text{ for } i \in \{1,2\}. \tag{25}$$

Indeed, choose an $x_0 \in P \cap [f=0]$. If $x_0 \in J$ for some $J \in \mathcal{J}_P$, then by condition vi) of Lemma 3.10, we obtain (25). So, assume $x_0 \in C_1$. Then $P \cap C_1 \neq \emptyset$ and there is an $x_1 \in P \cap C_1$ which is isolated in C_1. By (23), there is a $J \in \mathcal{J}_P$ such that $x_1 \in \operatorname{cl}(J \cap [f=0])$. Hence by conditions ii) or iv) of Lemma 3.10, we obtain (25).

If $C_2 = \emptyset$, then $\mathcal{P} = \{I\}$ and $g_i = g_{i,I} \in \acute{\mathcal{S}}_s$ for $i \in \{1,2\}$. One can easily show that the other requirements of the lemma are also fulfilled.

So, assume that $C_2 \neq \emptyset$. Let $\mathcal{P}_1, \ldots, \mathcal{P}_4$ be pairwise disjoint families of components of $I \setminus C_2$ such that $I \setminus C_2 = \bigcup_{j=1}^4 \bigcup \mathcal{P}_j$ and $C_2 \subset \operatorname{cl} \bigcup \mathcal{P}_j$ for $j \in \{1, \ldots, 4\}$. For $i \in \{1,2\}$ define the function $g_i\colon \operatorname{cl} I \to [-1,1]$ by

$$g_i(x) = \begin{cases} (-1)^j g_{i,P}(x) & \text{if } x \in \operatorname{cl} P \text{ and } P \in \mathcal{P}_j,\ j \in \{1,2\}, \\ (-1)^j g_{3-i,P}(x) & \text{if } x \in \operatorname{cl} P \text{ and } P \in \mathcal{P}_j,\ j \in \{3,4\}, \\ (\operatorname{sgn} f(x))^{i+1}/2 & \text{otherwise.} \end{cases}$$

Then clearly $\operatorname{sgn} \circ (g_1 g_2) = \operatorname{sgn} \circ f \restriction \operatorname{cl} I$ and by (24), $g_1[I \cap \mathcal{C}(g_1)] \supset (0,1]$.

Fix an $i \in \{1,2\}$. We can easily see that $|g_i(x)| \le 2^{-1}$ for $x \in \operatorname{bd} I$. So, to complete the proof we should show that $g_i \in \acute{\mathcal{S}}_s$.

Let $c, d \in \operatorname{cl} I$, $c < d$, and $y \in \mathrm{I}(g_i(c), g_i(d))$. We consider two cases.

Case 1. $(c,d) \cap C_2 = \emptyset$.

Then there is a $P \in \mathcal{P}$ such that $[c,d] \subset \operatorname{cl} P$. Since $g_{i,P} \in \acute{\mathcal{S}}_s$, there is an $x_0 \in (c,d) \cap \mathcal{C}(g_i)$ with $g_i(x_0) = y$.

Case 2. $(c,d) \cap C_2 \neq \emptyset$.

Then for each $j \in \{1,\dots,4\}$, since $C_2 \subset \operatorname{cl} \bigcup \mathcal{P}_j$, there is an interval $P_j \in \mathcal{P}_j$ with $P_j \subset (c,d)$. Hence by (24), we obtain the following inclusion

$$g_i[(c,d) \cap \mathcal{C}(g_i)] \supset g_i\Big[\bigcup_{j=1}^{4} P_j \cap \mathcal{C}(g_i)\Big] \supset [-1,1] \setminus \{0\}.$$

Since $A \cap C_2 \subset \operatorname{cl} I \cap A \setminus \operatorname{int}[f=0]$ is a countable G_δ-set, it is nowhere dense in C_2. So, there is an interval $(c',d') \subset (c,d)$ with $(c',d') \cap C_2 \neq \emptyset$ such that $(c',d') \cap A \cap C_2 = \emptyset$. Hence the function f changes its sign in (c',d') or $(c',d') \cap \operatorname{int}[f=0] \neq \emptyset$. In both cases there is a $P \in \mathcal{P}$ such that $P \subset (c',d') \subset (c,d)$ and $P \cap [f=0] \neq \emptyset$. Then by (25), $g_i[(c,d) \cap \mathcal{C}(g_i)] = [-1,1]$, whence $g_i(x_0) = y$ for some $x_0 \in (c,d) \cap \mathcal{C}(g_i)$. □

4. Main results

Proposition 4.1. *Assume that there is a G_δ-set $A \subset [f=0]$ such that*

$$\forall_{a,b \in E}\ f(a)f(b) < 0 \Rightarrow A \cap \mathrm{I}(a,b) \neq \emptyset.$$

Then there are functions $g_1,\dots,g_4 \in \acute{\mathcal{S}}_s$ such that $f = g_1 \dots g_4$.

Proof. Define

$$C = E \setminus \bigcup \big\{(a,b) : \operatorname{card}\big((a,b) \cap A \setminus \operatorname{int}[f=0]\big) \leq \omega\big\}.$$

First we will show that

$$\begin{aligned}&\text{there are strong Świątkowski functions } g_1, g_2\\ &\text{with } \operatorname{sgn} \circ (g_1 g_2) = \operatorname{sgn} \circ f.\end{aligned} \tag{26}$$

If $C = \emptyset$, then (26) follows by Lemma 3.11.

So, assume that $C \neq \emptyset$. Then C is perfect and nowhere dense. Let $\mathcal{I}_1,\dots,\mathcal{I}_4$ be pairwise disjoint families of components of $E \setminus C$ such that $E \setminus C = \bigcup_{j=1}^4 \bigcup \mathcal{I}_j$ and $C \subset \operatorname{cl} \bigcup \mathcal{I}_j$ for $j \in \{1,\dots,4\}$. Put $\mathcal{I} = \mathcal{I}_1 \cup \dots \cup \mathcal{I}_4$ and

$$A_1 = A \cap C \setminus \bigcup_{I \in \mathcal{I}} \operatorname{bd} I. \tag{27}$$

Since A is a G_δ-set, A_1 is a G_δ-set, too. Then $C \setminus A_1$ is an F_σ-set, whence there is a sequence (F_n) consisting of closed sets such that

$$C \setminus A_1 = \bigcup_{n \in \mathbb{N}} F_n. \tag{28}$$

Define $F'_0 = \emptyset$. For each $n \in \mathbb{N}$, use four times Lemma 3.4 to construct a sequence of sets (F'_n) and a sequence of families of intervals $(\mathcal{J}'_n)$ such that

$$\mathcal{J}'_n = \bigcup_{j=1}^{4} \mathcal{J}'_{j,n}, \tag{29}$$

$$F'_n = F_n \cup \bigcup_{k<n} \big(F'_k \cup \bigcup_{I \in \mathcal{J}'_k} \operatorname{bd} I\big) \tag{30}$$

and for $j \in \{1, \dots, 4\}$,

$$\mathcal{J}'_{j,n} \subset \mathcal{I}_j, \tag{31}$$

$$\text{for each } I \in \mathcal{I}_j, \text{ if } F'_n \cap \operatorname{bd} I \neq \emptyset, \text{ then } I \in \mathcal{J}'_{j,n}, \tag{32}$$

for each $c \in F'_n$, if c is a right-hand (left-hand) limit point of C, then c is a right-hand (left-hand) limit point of the union $\bigcup \mathcal{J}'_{j,n}$, (33)

$$\operatorname{cl} \bigcup \mathcal{J}'_{j,n} \subset F'_n \cup \bigcup_{J \in \mathcal{J}'_{j,n}} \operatorname{cl} J. \tag{34}$$

(Observe that by (34), for each $k < n$, the set $F'_k \cup \bigcup_{I \in \mathcal{J}'_k} \operatorname{bd} I$ is closed. So by (30), the set F'_n is also closed and $F'_n \subset C \setminus A_1$.)

Fix an $I \in \mathcal{I}$. Construct strong Świątkowski functions $g_{1,I}, g_{2,I} \colon \operatorname{cl} I \to [-1, 1]$ such that $\operatorname{sgn} \circ (g_{1,I} g_{2,I}) = \operatorname{sgn} \circ f \restriction \operatorname{cl} I$, $g_{1,I}[I \cap \mathcal{C}(g_{1,I})] \supset (0, 1]$, and $|g_{i,I}(x)| \le 2^{-1}$ for $i \in \{1, 2\}$ and $x \in \operatorname{bd} I$. (We use Lemma 3.11.) Put

$$n_I = \min\{n \in \mathbb{N} : I \in \mathcal{J}'_n\},$$

and observe that by (32), $\bigcup_{n \in \mathbb{N}} \bigcup \mathcal{J}'_n = E \setminus C$, whence n_I is well defined.

Fix an $i \in \{1, 2\}$. Define the function $g_i \colon E \to [-2^{-1}, 2^{-1}]$ by

$$g_i(x) = \begin{cases} 0 & \text{if } x \in A_1, \\ (-1)^j 2^{-n_I} g_{i,I}(x) & \text{if } x \in \operatorname{cl} I \text{ and } I \in \mathcal{I}_j,\ j \in \{1, 2\}, \\ (-1)^j 2^{-n_I} g_{3-i,I}(x) & \text{if } x \in \operatorname{cl} I \text{ and } I \in \mathcal{I}_j,\ j \in \{3, 4\}, \\ 2^{-n} (\operatorname{sgn} f(x))^{i+1} & \text{if } x \in F'_n \setminus \big(\bigcup_{I \in \mathcal{I}} \operatorname{bd} I \cup F'_{n-1}\big),\ n \in \mathbb{N}. \end{cases}$$

First we will show that $A_1 \subset \mathcal{C}(g_i)$.

Take an $x_0 \in A_1$ and let $\varepsilon > 0$. Choose $n_0 \in \mathbb{N}$ such that $2^{-n_0} < \varepsilon$ and put $\delta \overset{\text{df}}{=} \operatorname{dist}\big(\operatorname{cl} \bigcup \mathcal{J}'_{n_0}, x_0\big)$. Since by (34), (30), (27), and (28),

$$\begin{aligned} A_1 \cap \operatorname{cl} \bigcup \mathcal{J}'_{j,n_0} &\subset (A_1 \cap F'_{n_0}) \cup \big(A_1 \cap \bigcup_{J \in \mathcal{J}'_{j,n_0}} \operatorname{cl} J\big) \\ &\subset \big(A_1 \cap \bigcup_{n \le n_0} F_n\big) \cup \big(\big(C \setminus \bigcup_{I \in \mathcal{I}} \operatorname{bd} I\big) \cap \bigcup_{J \in \mathcal{I}} \operatorname{cl} J\big) = \emptyset, \end{aligned}$$

we have $x \notin \mathrm{cl} \bigcup \mathcal{J}'_{j,n_0}$ and $\delta > 0$.

Observe that by (33), $F'_{n_0} \subset \mathrm{cl} \bigcup \mathcal{J}'_{n_0}$. If $|x - x_0| < \delta$, then $x \notin \mathrm{cl} \bigcup \mathcal{J}'_{n_0}$, whence

$$|g_i(x) - g_i(x_0)| = |g_i(x)| \le 2^{-n_0} < \varepsilon.$$

So, $x_0 \in \mathcal{C}(g_i)$.

Now we will prove that

$$\begin{aligned} \forall_{n\in\mathbb{N}} \forall_{\delta>0} \bigl(&x \in F'_n \setminus \{\sup I : I \in \mathcal{I}\} \\ &\Rightarrow g_i[(x-\delta, x) \cap \mathcal{C}(g_i)] \supset [-2^{-n}, 2^{-n}]\bigr). \end{aligned} \tag{35}$$

Let $n \in \mathbb{N}$, $\delta > 0$ and $x \in F'_n \setminus \{\sup I : I \in \mathcal{I}\}$. Then for $j \in \{1, \ldots, 4\}$, by (33), there is an $I_j \in \mathcal{J}'_{j,n}$ with $I_j \subset (x - \delta, x)$. Notice that $\max\bigl\{n_{I_j} : j \in \{1, \ldots, 4\}\bigr\} \le n$. So,

$$g_i[(x-\delta, x) \cap \mathcal{C}(g_i)] \supset \bigcup_{j=1}^{4} g_i[I_j \cap \mathcal{C}(g_i)] \supset [-2^{-n}, 2^{-n}] \setminus \{0\}.$$

Since $x \notin \{\sup I : I \in \mathcal{I}\}$, we have $\mathrm{card}\bigl((x-\delta, x) \cap A_1\bigr) > \omega$. Hence

$$\emptyset \ne (x-\delta, x) \cap A_1 \subset (x-\delta, x) \cap \mathcal{C}(g_i) \cap [g_i = 0]$$

and finally

$$g_i[(x-\delta, x) \cap \mathcal{C}(g_i)] \supset [-2^{-n}, 2^{-n}].$$

Similarly we can prove that

$$\forall_{n\in\mathbb{N}} \forall_{\delta>0} \bigl(x \in F'_n \setminus \{\inf I : I \in \mathcal{I}\} \Rightarrow g_i[(x, x+\delta) \cap \mathcal{C}(g_i)] \supset [-2^{-n}, 2^{-n}]\bigr).$$

Now we will show that $g_i \in \acute{\mathcal{S}}_s$. Let $c, d \in E$, $c < d$, and $y \in \mathrm{I}(g_i(c), g_i(d))$. Assume that $g_i(c) < g_i(d)$. (The other case is similar.) If $c, d \in \mathrm{cl}\, I$ for some $I \in \mathcal{I}$, then since $g_{1,I}, g_{2,I} \in \acute{\mathcal{S}}_s$, there is an $x_0 \in (c, d) \cap \mathcal{C}(g_i)$ with $g_i(x_0) = y$. So, assume that the opposite case holds.

Assume that $y \ge 0$. (The case $y < 0$ is analogous.) Then $g_i(d) > 0$, whence $d \notin A_1$. We consider two cases.

Case 1. $d \notin \bigcup_{n\in\mathbb{N}} F'_n$ or $d \in \{\sup I : I \in \mathcal{I}\}$.

Then there is an $I \in \mathcal{I}$ such that $d \in \mathrm{cl}\, I$ and $c \notin \mathrm{cl}\, I$. If $y \in \mathrm{I}(g_i(\inf I), g_i(d))$, then, since $g_{1,I}, g_{2,I} \in \acute{\mathcal{S}}_s$, there is an $x_0 \in (\inf I, d) \cap \mathcal{C}(g_i) \subset (c, d) \cap \mathcal{C}(g_i)$ with $g_i(x_0) = y$.

Now let $y \in [0, g_i(\inf I)]$. By (30), since $I \in \mathcal{J}'_{n_I}$, we have $\inf I \in F'_{n_I+1}$. By (35),

$$y \in [0, g_i(\inf I)] \subset [-2^{-n_I-1}, 2^{-n_I-1}] \subset g_i[(c, \inf I) \cap \mathcal{C}(g_i)].$$

So, there is an $x_0 \in (c, \inf I) \cap \mathcal{C}(g_i) \subset (c, d) \cap \mathcal{C}(g_i)$ with $g_i(x_0) = y$.

Case 2. $d \in \bigcup_{n\in\mathbb{N}} F'_n \setminus \{\sup I : I \in \mathcal{I}\}$.

Then $d \in F'_n \setminus F'_{n-1}$ for some $n \in \mathbb{N}$. By (35),

$$y \in [0, g_i(d)) \subset [-2^{-n}, 2^{-n}] \subset g_i[(c,d) \cap \mathcal{C}(g_i)].$$

Consequently, there is an $x_0 \in (c,d) \cap \mathcal{C}(g_i)$ with $g_i(x_0) = y$. It follows that $g_i \in \acute{\mathcal{S}}_s$.

It is not hard to see that $\operatorname{sgn} \circ (g_1 g_2) = \operatorname{sgn} \circ f$. This completes the proof of (26).

Now define the function $\tilde{f}\colon E \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} \dfrac{f}{g_1 g_2}(x) & \text{if } f(x) \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

Notice that $\tilde{f}$ is cliquish. Indeed, it is obvious that

$$\mathcal{C}(\tilde{f}) \supset \mathcal{C}(f) \cap \mathcal{C}(g_1) \cap \mathcal{C}(g_2) \cap U,$$

where

$$U \stackrel{\text{df}}{=} \operatorname{int}[f = 0] \cup \operatorname{int}[f \neq 0].$$

By assumption, the set $[f = 0]$ is simply open. So, $\mathbb{R} \setminus U = \operatorname{bd}[f = 0]$ is nowhere dense and U is residual. Since the sets $\mathcal{C}(f)$, $\mathcal{C}(g_1)$, and $\mathcal{C}(g_2)$ are also residual, so is the set $\mathcal{C}(\tilde{f})$.

Clearly $\tilde{f} > 0$ on E. So, the function $\ln \circ \tilde{f}\colon E \to \mathbb{R}$ is cliquish. By [7, Corollary II.3.4], there are functions $h_1, h_2 \in \acute{\mathcal{S}}_s$ such that $\ln \circ \tilde{f} = h_1 + h_2$.

Define $g_3 = \exp \circ h_1$ and $g_4 = \exp \circ h_2$. By Lemma 3.3, $g_3, g_4 \in \acute{\mathcal{S}}_s$. Clearly

$$f = g_1 g_2 \cdot \tilde{f} = g_1 g_2 (\exp \circ h_1)(\exp \circ h_2) = g_1 \dots g_4,$$

which completes the proof. $\square$

Remark 4.1. In Theorem 4.2, we do not assume that conditions (7)–(9) are fulfilled.

Theorem 4.2. *Let $f\colon \mathbb{R} \to \mathbb{R}$. The following conditions are equivalent:*

i) there are functions $g_1, \dots, g_4 \in \acute{\mathcal{S}}_s$ such that $f = g_1 \dots g_4$,
ii) there is a $k \in \mathbb{N}$ and functions $g_1, \dots, g_k \in \acute{\mathcal{S}}_s$ such that $f = g_1 \dots g_k$,
iii) the function f is cliquish, the set $[f = 0]$ is simply open, and there is a G_δ-set $A \subset [f = 0]$ such that for all $a, b \in \mathbb{R}$, if $f(a)f(b) < 0$, then $A \cap \mathrm{I}(a,b) \neq \emptyset$.

Proof. The implication i) $\Rightarrow$ ii) is evident.

ii) $\Rightarrow$ iii). Since $f \in \acute{\mathcal{S}}_s$, it is cliquish as well. By [3], the set $[f = 0]$ is simply open. (Cf. also [8, Theorem] or [2].)

For $i \in \{1, \dots, k\}$, define

$$\tilde{g}_i = \min\bigl\{\max\{g_i, -1\}, 1\bigr\}.$$

Then the function $\tilde{g}_i$ is bounded. One can readily verify that $\tilde{g}_i \in \acute{\mathcal{S}}_s$.

Put $\tilde{f} = \tilde{g}_1 \dots \tilde{g}_k$. Observe that $\operatorname{sgn} \circ \tilde{f} = \operatorname{sgn} \circ f$. Define a G_δ-set

$$A \stackrel{\text{df}}{=} \bigcap_{n=1}^{\infty} \operatorname{int} \tilde{f}^{-1}\bigl((-n^{-1}, n^{-1})\bigr) = [\tilde{f} = 0] \cap \mathcal{C}(\tilde{f}) \subset [f = 0].$$

Let $I \subset \mathbb{R}$ be an interval in which f changes its sign. Then at least one of the functions $\tilde{g}_1, \dots, \tilde{g}_k$, say $\tilde{g}_1$, changes its sign in I, too. Since $\tilde{g}_1 \in \acute{\mathcal{S}}_s$, there is an $x_0 \in I \cap \mathcal{C}(\tilde{g}_1)$ such that $\tilde{g}_1(x_0) = 0$. The functions $\tilde{g}_1, \dots, \tilde{g}_k$ are bounded, so $x_0 \in \mathcal{C}(\tilde{f})$, and finally $x_0 \in I \cap [\tilde{f} = 0] \cap \mathcal{C}(\tilde{f}) = I \cap A$.

iii) $\Rightarrow$ i). Put $E = [-\pi/2, \pi/2]$. Define the function $\tilde{f} \colon E \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} (f \circ \tan)(x) & \text{if } x \in (-\pi/2, \pi/2), \\ 0 & \text{if } x \in \{-\pi/2, \pi/2\}. \end{cases}$$

Then clearly $\tilde{A} \stackrel{\text{df}}{=} \arctan[A] \subset [\tilde{f} = 0]$ is a G_δ-set, and for each interval $I \subset E$, if the function $\tilde{f}$ changes its sign in I, then $I \cap \tilde{A} \neq \emptyset$. So, by Proposition 4.1, there are functions $\tilde{g}_1, \dots, \tilde{g}_4 \in \acute{\mathcal{S}}_s$ such that $\tilde{f} = \tilde{g}_1 \dots \tilde{g}_4$. For $i \in \{1, \dots, 4\}$ define $g_i = \tilde{g}_i \circ \arctan$ and notice that by Lemma 3.3, $g_i \in \acute{\mathcal{S}}_s$. Clearly

$$f = \tilde{f} \circ \arctan = (\tilde{g}_1 \circ \arctan) \dots (\tilde{g}_4 \circ \arctan) = g_1 \dots g_4,$$

which completes the proof. $\square$

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