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DARBOUX PROBLEM WITH A DISCONTINUOUS RIGHT-HAND SIDE

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Abstract. We prove an existence theorem for the Darboux Problem $u_{xy}(x,y) = g(u(x,y))$, $u(x,0) = u(0,y) = 0$, where g is bounded and measurable.

1. Introduction

In this paper we consider the Darboux Problem

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) = g(u(x,y)) \qquad \text{a.e. in } [0,1]\times[0,1], \tag{1}$$

$$u(x,0) = u(0,y) = 0, \tag{2}$$

where $g\colon \mathbb{R} \to [a,b]$, $0 < a < b < +\infty$ is supposed to be Lebesgue measurable. The problem arises as a natural extension of the Cauchy Problem for an autonomous equation $x'(t) = f(x(t))$ with a discontinuous right-hand side, see [1].

When considering Darboux Problem for equations $u_{xy} = f(x,y,u)$ or $u_{xy} = f(x,y,u,u_x,u_y)$, most authors assume Carathéodory-type conditions, i.e. f is measurable with respect to the first two variables, continuous

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or Lipschitz with respect to the others, and bounded by an integrable function $M(x,y)$, see e.g. [3], [4], [5], [6]. It should also be mentioned that there is a large number of articles on Darboux Problem in Banach spaces, see e.g. [9], where it is assumed that f is continuous and satisfies a condition expressed in terms of measure of noncompactness. Another approach to the Darboux Problem is to consider u_{xy} the mixed approximate derivative, see [2].

Although the main purpose of this paper is to establish an existence theorem for (1)–(2), the method used here, see also see [7] involves functional differential equations, namely, we deal with the problem

$$q'(t) = g\left(\int_0^t \frac{q(s)}{s} ds\right) \quad \text{a.e. in t} \in [0,1] \tag{3}$$

$$q(0) = 0 \tag{4}$$

where $g\colon \mathbb{R} \to [a,b]$, $0 < a < b < +\infty$, is Lebesgue measurable.

Throughout this paper the term *measure* instead of *Lebesgue measure* μ is used and also other concepts such as measurability and integrability are understood as Lebesgue measurability and Lebesgue integrability. We denote by $C[0,1]$ the normed linear space of all continuous functions $x\colon [0,1] \to \mathbb{R}$ with the norm $\|x\| = \sup_{t\in[0,1]} |x(t)|$.

2. Functional equation $q'(t) = g\left(\int_0^t \frac{q(s)}{s} ds\right)$

First we remind two facts which we need further in the proof of Theorem 2.1.

Lemma 2.1. *Assume that $f\colon [A,B] \to \mathbb{R}$, $-\infty < A < B < +\infty$ is continuous and has bounded variation in $[A,B]$. Then f is absolutely continuous if and only if $\mu(f(E)) = 0$ for every $E \subset [A,B]$ such that $\mu(E) = 0$ [8, Theorem 4, p. 314].*

Lemma 2.2. *Assume that $f\colon [A,B] \to \mathbb{R}$, $-\infty < A < B < +\infty$ is continuous. If $F \subset [A,B]$ is measurable and $\mu(f(E)) = 0$ for every $E \subset [A,B]$ such that $\mu(E) = 0$, then $f(F)$ is measurable* [8, Corollary 2, p. 219].

Theorem 2.1. *If $g\colon \mathbb{R} \to [a,b]$, $0 < a < b < +\infty$ is measurable, then the problem (3)–(4) has a solution.*

Proof. Proof will be divided into several steps.

1. Define

$$Z = \{x \in C[0,1] : x(0) = 0, a(t-\tau) \le x(t) - x(\tau) \le b(t-\tau),\\ 0 \le \tau < t \le 1\}.$$

For each $f, g \in Z$ and $\alpha \in [0,1]$ we have $\alpha f + (1-\alpha) g \in Z$. Thus Z is convex. Moreover $0 \le f(t) \le b$ and

$$f(t) - f(\tau) \le b(t-\tau) = b|t-\tau|$$
$$f(\tau) - f(t) \le a(\tau - t) \le a|t-\tau| \le b|t-\tau|$$

for all $t, \tau \in [0,1]$, $t > \tau$. Therefore $|f(t) - f(\tau)| \le b|t-\tau|$, $t, \tau \in [0,1]$ and Z is compact.

2. We claim that $A\colon Z \to Z$, defined by

$$(Aq)(t) = \int_0^t g\left(\int_0^z \frac{q(s)}{s} ds\right) dz, \quad t \in [0,1],$$

is continuous.

2a. For $q \in Z$ define $h\colon [0,1] \to \mathbb{R}$,

$$h(z) = \int_0^z \frac{q(s)}{s} ds.$$

The function h is continuous, strictly increasing and for each $t, \tau \in [0,1]$, $t > \tau$, satisfies

$$a(t-\tau) \le h(t) - h(\tau) = \int_\tau^t \frac{q(s)}{s} ds \le b(t-\tau).$$

Thus $h \in Z$ and, for each $u, v \in h([0,1])$, we have

$$\left|h^{-1}(u) - h^{-1}(v)\right| \le \frac{1}{a}\left|h\left(h^{-1}(u)\right) - h\left(h^{-1}(v)\right)\right| = \frac{1}{a}|u-v|, \qquad (5)$$

so h^{-1} is absolutely continuous.

Because $h \in Z$ is continuous and strictly monotonic on $[0,1]$, h^{-1} is continuous and strictly monotonic on a closed interval $h([0,1])$. Thus h^{-1} is of bounded variation on $h([0,1])$. By Lemma 2.1 and Lemma 2.2, $(g \circ h)^{-1}(P) = h^{-1}\left(g^{-1}(P)\right)$ is measurable for every open interval $P \subset h([0,1])$. Hence, $g(h(\cdot))$ is measurable and Aq is well defined.

Observe that $Aq \in Z$, because $(Aq)(0) = 0$ and for all $\tau, t \in [0,1]$, $t > \tau$,

$$a(t-\tau) \le (Aq)(t) - (Aq)(\tau) = \int_\tau^t g\left(\int_0^z \frac{q(s)}{s} ds\right) dz \le b(t-\tau).$$

2b. Take $\varepsilon > 0$ and any sequence $q_n \in Z$, $n \in \mathbb{N}$, convergent (uniformly) to $q \in Z$. Define $h_n\colon [0,1] \to \mathbb{R}$,

$$h_n(z) = \int_0^z \frac{q_n(s)}{s} ds, \quad n \in \mathbb{N}.$$

By Lusin's theorem there exists a compact set $K \subset [0,b]$ such that $g_{|K}\colon K \to [a,b]$ is continuous and

$$\mu([0,b] \setminus K) < \frac{a\varepsilon}{8b}.$$

Since $g_{|K}$ is uniformly continuous, there exists $\delta > 0$ such that $|u - v| < \delta$, $u, v \in K$ implies $|g(u) - g(v)| < \varepsilon/2$.

For $z \le \delta/(2b+1)$ we have

$$\begin{aligned} |h_n(z) - h(z)| &\le \int_0^z \frac{|q_n(s) - q(s)|}{s} ds \\ &\le \int_0^{\delta/(2b+1)} \frac{|q_n(s) - q(s)|}{s} ds \\ &\le \frac{2b\delta}{2b+1} < \delta. \end{aligned}$$

If $z > \delta/(2b+1)$, then

$$\begin{aligned} |h_n(z) - h(z)| &\le \int_0^z \frac{|q_n(s) - q(s)|}{s} ds \\ &= \int_0^{\delta/(2b+1)} \frac{|q_n(s) - q(s)|}{s} ds + \int_{\delta/(2b+1)}^z \frac{|q_n(s) - q(s)|}{s} ds \\ &\le \frac{2b\delta}{2b+1} + \int_{\delta/(2b+1)}^z \frac{\|q_n - q\|}{s} ds \\ &\le \frac{2b\delta}{2b+1} + \|q_n - q\| \int_{\delta/(2b+1)}^1 \frac{ds}{s} \\ &\le \frac{2b\delta}{2b+1} + \|q_n - q\| \ln \frac{2b+1}{\delta}. \end{aligned}$$

Since $\|q_n - q\| \to 0$, $n \to \infty$, there exists n_0 such that

$$\|q_n - q\| < \frac{\delta}{2b+1} \left(\ln \frac{2b+1}{\delta} \right)^{-1}$$

for $n > n_0$. Therefore for $n > n_0$ and each $z \in [0,1]$, we have

$$|h_n(z) - h(z)| \le \sup_{z \in [0,1]} \left| \int_0^z \frac{q_n(s)}{s} ds - \int_0^z \frac{q(s)}{s} ds \right| < \delta.$$

2c. Fix $n > n_0$ and define $F = h^{-1}(K) \cap h_n^{-1}(K)$. We have

$$\begin{aligned}[0,1] \setminus F &= [0,1] \setminus \left(h^{-1}(K) \cap h_n^{-1}(K)\right)\\ &= \left([0,1] \setminus h^{-1}(K)\right) \cup \left([0,1] \setminus h_n^{-1}(K)\right)\\ &= \left(h^{-1}([0,b]) \setminus h^{-1}(K)\right) \cup \left(h_n^{-1}([0,b]) \setminus h_n^{-1}(K)\right)\\ &= h^{-1}([0,b] \setminus K) \cup h_n^{-1}([0,b] \setminus K)\end{aligned}$$

and using (5) we get

$$\begin{aligned}\mu([0,1] \setminus F) &\le \mu\left(h^{-1}([0,b] \setminus K)\right) + \mu\left(h_n^{-1}([0,b] \setminus K)\right)\\ &= \int_{h^{-1}([0,b]\setminus K)} dz + \int_{h_n^{-1}([0,b]\setminus K)} dz\\ &= \int_{[0,b]\setminus K} \left(h^{-1}\right)'(u)\, du + \int_{[0,b]\setminus K} \left(h_n^{-1}\right)'(u)\, du\\ &\le \frac{\mu([0,b] \setminus K)}{a} + \frac{\mu([0,b] \setminus K)}{a} \le \frac{\varepsilon}{4b}.\end{aligned}$$

Applying **2b**, we obtain

$$\begin{aligned}\|Aq_n - Aq\| &= \sup_{t\in[0,1]} \left| \int_0^t g(h_n(z))\, dz - \int_0^t g(h(z))\, dz \right|\\ &\le \sup_{t\in[0,1]} \int_0^t |g(h_n(z)) - g(h(z))|\, dz\\ &= \int_0^1 |g(h_n(z)) - g(h(z))|\, dz\\ &= \int_F |g(h_n(z)) - g(h(z))|\, dz\\ &\quad + \int_{[0,1]\setminus F} |g(h_n(z)) - g(h(z))|\, dz\\ &\le \mu(F) \cdot \frac{\varepsilon}{2} + \frac{\varepsilon}{4b} \cdot 2b \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus $A\colon Z \to Z$ is continuous.

3. It follows from Schauder's fixed point theorem that A has a fixed point in Z. Thus the problem (3)–(4) has a solution. □

3. Darboux Problem

Definition 3.1. We say that a continuous function $u\colon [0,1] \times [0,1] \to \mathbb{R}$ is a solution to the Darboux Problem (1)–(2) if u satisfies the equation (1) a.e. in $[0,1] \times [0,1]$ and the initial condition (2) for $x, y \in [0,1]$.

Theorem 3.1. *If $g\colon \mathbb{R} \to [a,b]$, $0 < a < b < +\infty$, is measurable, then the problem (1)–(2) has a solution.*

Proof. Let q be a solution to the problem (3)–(4). Define $v\colon\ [0,1] \to \mathbb{R}$,

$$v(t) = \int_0^t \frac{q(s)}{s} ds, \quad t \in [0,1],$$

and $u\colon\ [0,1] \times [0,1] \to \mathbb{R}$,

$$u(x,y) = v(xy), \quad (x,y) \in [0,1] \times [0,1].$$

We have

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) = v'(xy) + xy \cdot v''(xy) = q'(xy) = g(u(x,y))$$

a.e. in $(x,y) \in [0,1] \times [0,1]$. Obviously, $u(x,0) = u(0,y) = v(0) = 0$. Thus u is a solution to the problem (1)–(2). $\square$

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