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NOTES ON UNIQUENESS OF MEROMORPHIC FUNCTIONS

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Abstract. In this paper, we study the problem of uniqueness on meromorphic functions involving in differential polynomials and obtain some results which extend and improve the theorems of M. Fang and W. Hong et al.

1. Introduction and main results

In this paper, a meromorphic function means meromorphic in the open complex plane. We use the usual notations of Nevanlinna theory of meromorphic functions as defined in [3], [6]. By E we denote a set of finite linear measure, not necessary the same at each occurrence. For a nonconstant meromorphic function f and a complex number a, and by $E(a,f)$, $E_{k)}(a,f)$ we denote the set of zeros of $f-a$ (counting multiplicity), and the set of zeros of $f-a$ with multiplicity $\leq k$ (counting multiplicity); $N_{1)}(r,1/(f-a))$ stands for the counting function of simple a-points of f and $\overline{N}_{(k}(r,1/(f-a))$ stands for the counting function of a-points of f with multiplicity $\geq k$, where k is a positive integer and each a-points is

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counted only once. In addition, $S(r,f)$ stands for any quantity satisfying $S(r,f)=o(T(r,f))$ $(r\to\infty,\ r\notin E)$ (see [6]). Let

$$N_2\left(r,\frac{1}{f-a}\right)=\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}_{(2}\left(r,\frac{1}{f-a}\right).$$

W. Hayman in [2] obtained the following result.

Theorem A. *Let f be an entire function, $n\geq 1$ be a positive integer. If $f^nf'\neq 1$, then f is a constant.*

Recently, corresponding to Theorem A, M. Fang and W. Hong in [1] proved the following two theorems.

Theorem B. *Let f be an entire function, $n\geq 2$ be a positive integer. If $f^n(f-1)f'\neq 1$, then f is a constant.*

Theorem C. *Let f and g be transcendental entire functions, $n\geq 11$ be a positive integer. If $E(1,f^n(f-1)f')=E(1,g^n(g-1)g')$, then $f\equiv g$.*

Motivated by Theorem C, we obtain the more general results which improve Theorem C.

Theorem 1.1. *Let f and g be distinct nonconstant meromorphic functions, $n\geq 10$ be a positive integer. If $E(\infty,f)=E(\infty,g)$ and $E_{3)}(1,f^n(f-1)f')=E_{3)}(1,g^n(g-1)g')$, then*

$$\begin{aligned} f&=\frac{(n+2)(1+h+\ldots+h^n)h}{(n+1)(1+h+\ldots+h^{n+1})},\\ g&=\frac{(n+2)(1+h+\ldots+h^n)}{(n+1)(1+h+\ldots+h^{n+1})}, \end{aligned} \tag{1.1}$$

where h is a nonconstant meromorphic function.

Corollary. *Let f and g be nonconstant entire functions, $n\geq 7$ be a positive integer. If $E_{3)}(1,f^n(f-1)f')=E_{3)}(1,g^n(g-1)g')$, then $f\equiv g$.*

Clearly, Corollary is an improvement of Theorem C.

2. Some lemmas

Let us introduce some lemmas which will be used to prove Theorem 1.1.

Lemma 2.1 ([5]). *Let f be a meromorphic function and $p = a_0 f^n + a_1 f^{n-1} + \ldots + a_n$, where $a_0(\not\equiv 0), a_1, \ldots, a_n$ are small functions related to f, that is $T(r, a_j) = S(r, f)$, $j = 0, 1, \ldots, n$. Then*

$$T(r, p) = nT(r, f) + S(r, f).$$

Lemma 2.2. *Let F, G be two meromorphic functions such that $E(\infty, F) = E(\infty, G)$ and $E_{3)}(1, F) = E_{3)}(1, G)$, then one of the following cases must occur:*

(i) $$\frac{1}{2}T(r, F) + \frac{1}{2}T(r, G) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + 2\overline{N}(r, F) + S(r, F) + S(r, G);$$

(ii) $F \cdot G \equiv 1$;

(iii) $F \equiv G$.

Proof. Let

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.1}$$

If $H \not\equiv 0$, since $E_{3)}(1, F) = E_{3)}(1, G)$, then $E_{1)}(1, F) = E_{1)}(1, G)$, a simple computation on local expansion shows that $H(z_0) = 0$ if z_0 is a simple zero of $F - 1$ and $G - 1$. Hence, we get

$$N_{1)}\left(r, \frac{1}{F-1}\right) \leq N\left(r, \frac{1}{H}\right). \tag{2.2}$$

By the first fundamental theorem, we have

$$N\left(r, \frac{1}{H}\right) \leq T(r, H) + O(1). \tag{2.3}$$

From (2.1), we get $m(r, H) = S(r, F) + S(r, G)$, thus by (2.2) and (2.3), we get

$$N_{1)}\left(r, \frac{1}{F-1}\right) \leq N(r, H) + S(r, F) + S(r, G). \tag{2.4}$$

Noticing that $E(\infty, F) = E(\infty, G)$, we say that any pole of F, G cannot be a pole of H. In addition, since $E_{3)}(1, F) = E_{3)}(1, G)$, we can also easily see from (2.1) that any k-fold ($k \leq 3$) zero-point of $F - 1$ and $G - 1$ is not a pole of H. Hence

$$N_{1)}\left(r, \frac{1}{F-1}\right) \leq \overline{N}_{(4}\left(r, \frac{1}{F-1}\right) + \overline{N}_{(2}\left(r, \frac{1}{F}\right) + \overline{N}_{(2}\left(r, \frac{1}{G}\right)$$

$$+\overline{N}_{(4}\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,F)+S(r,G), \quad (2.5)$$

where $N_0(r,1/F')$ is the counting function corresponding to the zeros of F' which are not zeros of $F(F-1)$, and $N_0(r,1/G')$ is the counting function corresponding to the zeros of G' which are not zeros of $G(G-1)$.

Noticing

$$\overline{N}\left(r,\frac{1}{F-1}\right)-\frac{1}{2}N_{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(4}\left(r,\frac{1}{F-1}\right)$$
$$\leq\frac{1}{2}N\left(r,\frac{1}{F-1}\right), \quad (2.6)$$

and

$$\overline{N}\left(r,\frac{1}{G-1}\right)-\frac{1}{2}N_{1)}\left(r,\frac{1}{G-1}\right)+\overline{N}_{(4}\left(r,\frac{1}{G-1}\right)$$
$$\leq\frac{1}{2}N\left(r,\frac{1}{G-1}\right). \quad (2.7)$$

Combining (2.5), (2.6), (2.7) with the second fundamental theorem, we can prove Lemma 2.2 by imitating the proof as did in [4], [7]. So we omit the details here. □

Lemma 2.3. *Let f_1, f_2 be nonconstant meromorphic functions and c_1, c_2, c_3 be nonzero constants. If $c_1f_1+c_2f_2\equiv c_3$, then*

$$T(r,f_1)<\overline{N}\left(r,\frac{1}{f_1}\right)+\overline{N}\left(r,\frac{1}{f_2}\right)+\overline{N}(r,f_1)+S(r,f_1).$$

Proof. By the second fundamental theorem, we have

$$T(r,f_1)<\overline{N}\left(r,\frac{1}{f_1}\right)+\overline{N}\left(r,\frac{1}{f_1-\frac{c_3}{c_1}}\right)+\overline{N}(r,f_1)+S(r,f_1). \quad (2.8)$$

Noticing that

$$\overline{N}\left(r,\frac{1}{f_1-\frac{c_3}{c_1}}\right)=\overline{N}\left(r,\frac{1}{f_2}\right). \quad (2.9)$$

Thus Lemma 2.3 follows from (2.8) and (2.9). □

Lemma 2.4 ([6])**.** *Let f be a meromorphic function, then*

$$N\left(r,\frac{1}{f'}\right)\leq N\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+S(r,f).$$

3. Proof of Theorem 1.1 and Corollary

3.1. Proof of Theorem 1.1. In order to prove Theorem 1.1, we set

$$F = f^n(f-1)f', \quad G = g^n(g-1)g', \tag{3.1}$$

and

$$F_1 = \frac{1}{n+2}f^{n+2} - \frac{1}{n+1}f^{n+1}, \quad G_1 = \frac{1}{n+2}g^{n+2} - \frac{1}{n+1}g^{n+1}. \tag{3.2}$$

Thus, $E(\infty, F) = E(\infty, G)$ and $E_{3)}(1, F) = E_{3)}(1, G)$. So, by Lemma 2.2, we consider the following three cases.

Case 1. Suppose that F and G satisfy (i) in Lemma 2.2:

$$\begin{aligned}\frac{1}{2}T(r,F) + \frac{1}{2}T(r,G) \leq & N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + 2\overline{N}(r,F) \\ & + S(r,F) + S(r,G).\end{aligned} \tag{3.3}$$

From (3.1), we get

$$\begin{aligned}& N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) \leq 2N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-1}\right) + N\left(r, \frac{1}{f'}\right) \\ & + 2N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g-1}\right) + N\left(r, \frac{1}{g'}\right).\end{aligned} \tag{3.4}$$

By Lemma 2.1, the first main theorem and (3.2), we have

$$\begin{aligned}T(r,F_1) = & (n+2)T(r,f) + S(r,f) \\ \leq & T(r,F) + N\left(r, \frac{1}{F_1}\right) - N\left(r, \frac{1}{F}\right) + S(r,f) \\ \leq & T(r,F) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f - \frac{n+2}{n+1}}\right) - N\left(r, \frac{1}{f-1}\right) \\ & - N\left(r, \frac{1}{f'}\right) + S(r,f),\end{aligned}$$

combining with (3.3), (3.4), it will yield

$$\begin{aligned}(n+2)T(r,f) \leq & 3N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f - \frac{n+2}{n+1}}\right) + 2\overline{N}(r,f) + 2N\left(r, \frac{1}{g}\right) \\ & + N\left(r, \frac{1}{g-1}\right) + N\left(r, \frac{1}{g'}\right) + S(r,f) + S(r,g).\end{aligned} \tag{3.5}$$

It follows from $E(\infty, f) = E(\infty, g)$, (3.5) and Lemma 2.4 that

$$(n-2)T(r,f) \leq 7T(r,g) + S(r,f) + S(r,g). \tag{3.6}$$

Similarly, we have

$$(n-2)T(r,g) \le 7T(r,f) + S(r,f) + S(r,g). \tag{3.7}$$

By (3.6) and (3.7), we find $(n-9)T(r,f) + (n-9)T(r,g) \le S(r,f) + S(r,g)$, which contradicts $n \ge 10$.

Case 2. Suppose that F and G satisfy (ii) in Lemma 2.2: $f^n(f-1)f'g^n(g-1)g' \equiv 1$. From $f^n(f-1)f'g^n(g-1)g' \equiv 1$ and $E(\infty, f) = E(\infty, g)$, it follows that f and g are entire functions. By the Picard theorem, we deduce f may possess two values in C : either 0 or 1. Contradiction.

Case 3. Suppose that F and G satisfy (iii) in Lemma 2.2: $F \equiv G$, which implies that $F_1 \equiv G_1 + c$, where c is a constant.

First, by Lemma 2.1, we obtain

$$T(r,f) = T(r,g) + S(r,f) + S(r,g). \tag{3.8}$$

We claim that $c = 0$. Otherwise, if $c \ne 0$, Lemma 2.3, (3.2) and (3.8) will have

$$\begin{aligned} T(r,F_1) &= (n+2)T(r,f) + S(r,f) \\ &\le \overline{N}(r,\frac{1}{F_1}) + \overline{N}(r,\frac{1}{G_1}) + \overline{N}(r,f) + S(r,f) \\ &\le 5T(r,f) + S(r,f), \end{aligned}$$

which also contradicts $n \ge 10$, and hence $c = 0$, $F_1 \equiv G_1$. Let $f = hg$, which leads from $f \not\equiv g$ that

$$f = \frac{(n+2)(1+h+\ldots+h^n)h}{(n+1)(1+h+\ldots+h^{n+1})}, \quad g = \frac{(n+2)(1+h+\ldots+h^n)}{(n+1)(1+h+\ldots+h^{n+1})},$$

where $n(\ge 10)$ is a positive integer, and h is a meromorphic function.

This proves Theorem 1.1. □

3.2. Proof of Corollary. First, for entire functions f and g, from (3.5) we obtain (3.6) and (3.7) with "7" replaced by "4". That is, we can get

$$(n-2)T(r,f) \le 4T(r,g) + S(r,f) + S(r,g). \tag{3.6$'$}$$

and

$$(n-2)T(r,g) \le 4T(r,f) + S(r,f) + S(r,g). \tag{3.7$'$}$$

Thus $n \ge 7$ and either (1.1) is true or $f \equiv g$. But (1.1) is not fulfilled since $h \ne$ const is entire and

$$g = \frac{n+2}{n+1} \cdot \frac{1-h^{n+1}}{1-h^{n+2}}$$

has poles (by the Picard theorem).

Therefore $f \equiv g$, which completes the proof of Corollary. □

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